Chapter 3: Static Failure Theories

What is a static load

A static load is a stationary force or couple applied to a member.

To be stationary, the force or couple must be:

- Unchanging in magnitude and direction
- Unchanging in its point of application

A static load can produce:

- an axial tension or compression,
- a shear load,
- a bending load,
- a torsional load,
- any combination of the above

What is meant by “Failure”?

Failure can mean:

- A part has separated into two ore more pieces
- Has become permanently distorted – ruining its geometry
- Has had its reliability downgraded
- Has had its function compromised
- Any combination of the above

Failure due to *Static Loading*

fundamentally different from

Failure due to *Dynamic Loading*

We'll study Failure due to Dynamic Loading in the next chapter.

Experimental testing
vs.
“Theoretical” Design

“Theoretical” Design

Compute stresses → Compare with strength

Strength:

- Yield strength
- Ultimate strength

But ... something important

Geometric discontinuities → *Stress concentrations*

Geometric discontinuity examples (stress raisers):

- Shoulders on shafts for bearings
- Key slots on shafts for pulleys and gears
- Screw threads on bolts
- Holes, oil grooves, notches of various kinds

Stress concentration factor

A *theoretical, or geometric, stress-concentration factor* relates the actual maximum stress at a discontinuity to the *nominal stress.*

Normal stress: $\sigma_{\max} = K_t \sigma_0$
Shear stress: $\tau_{\max} = K_{ts} \tau_0$

Nominal stress σ_0 or τ_0 calculated from elementary stress equations.

K_t or K_{ts} depends *only* on the geometry, *not* on the material.

Warning: But nature of material does matter
➥ Ductile or Brittle

Stress concentration: Ductile vs Brittle

In **static loading:**

Brittle materials ($\epsilon_f < 0.05$):

$$\text{Computed stress} \xrightarrow{\text{use } K_t \text{ or } K_{ts}} \text{Compare with strength}$$

Ductile materials ($\epsilon_f \geq 0.05$):

$$\text{Computed stress} \xrightarrow{\text{do NOT use } K_t \text{ or } K_s} \text{Compare with strength}$$

In **dynamic loading:**

Stress concentration effect is significant for both ductile and brittle materials.

Static Failure Theories

General idea: Compare against a failure mechanism in a simple test.

"Failure theory" $\leftarrow$ Accepted practice

Different kinds of failure theories: Need to choose judiciously

Different failure theories

Ductile materials (yield criteria)

- Maximum shear stress
- Distortion energy
- Ductile Coulomb-Mohr

Brittle materials (fracture criteria)

- Maximum normal stress
- Brittle Coulomb-Mohr
- Modified Mohr

Failure Theories for Ductile Materials

Maximum Shear Stress Theory

This theory predicts:

Yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress stress in a tension-test specimen of the same material when that specimen begins to yield.

Other names:

- Tresca theory
- Guest theory

Maximum Shear Stress Theory ... contd.$_2$

For a general state of stress, order the three principal stresses:

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

Maximum shear stress is then: $\tau_{\max} = \dfrac{\sigma_1 - \sigma_3}{2}$

For a simple tensile test, we have a uniaxial stress: σ.

Corresponding maximum shear stress stress will be: $\tau_{\max,\text{ten}} = \dfrac{\sigma}{2}$

At yielding, $\sigma = S_y$. So, $\tau_{\max,\text{ten}} = \dfrac{S_y}{2}$

Maximum Shear Stress Theory ... contd.$_3$

Therefore, the maximum shear stress theory predicts yielding when:

$$\tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \qquad \text{or,} \qquad \sigma_1 - \sigma_3 \geq S_y$$

Implication: Yield strength in shear is given by: $S_{sy} = 0.5 S_y$

For design: Incorporate a factor of safety, n:

$$\tau_{\text{max,ten}} = \frac{S_y}{2n} \qquad \text{or,} \qquad \sigma_1 - \sigma_3 = \frac{S_y}{n}$$

Maximum Shear Stress Theory ... contd.4

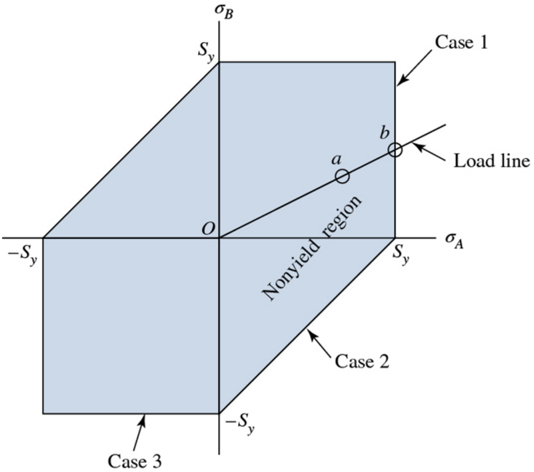
Consider a *plane stress* state, with principal stresses: σ_A and σ_B. The third principal stress is 0.

Case 1: $\sigma_A \geq \sigma_B \geq 0$.
Yield condition:
$\sigma_A \geq S_y$

Case 2: $\sigma_A \geq 0 \geq \sigma_B$.
Yield condition:
$\sigma_A - \sigma_B \geq S_y$

Case 3: $0 \geq \sigma_A \geq \sigma_B$.
Yield condition:
$\sigma_B \leq -S_y$

Other 3 lines for
$\sigma_B \geq \sigma_A$



Maximum Shear Stress Theory ... contd.$_5$

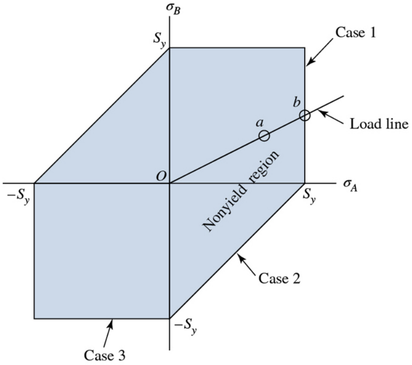
Point *a* represents a state of stress.

If load is increased, it is typical to assume that the principal stresses will increase proportionally along the line from the origin through point *a*.

➥ This is the *load line*.

➥ At point *b*, yielding occurs.

➥ Factor of safety, $n = \dfrac{Ob}{Oa}$



Distortion Energy Theory

This theory predicts:

Yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.

Other names:

- von Mises or von Mises-Hencky theory
- Shear energy theory
- Octahedral shear stress theory

Distortion Energy Theory ... contd.$_2$

Fundamentally:

➥ Yielding is related to *shape change*

➥ *Not* to volumetric changes

A state of stress can be decomposed into *hydrostatic* and *deviatoric* components.

→ Hydrostatic component: volume change

→ Deviatoric component: shape change (distortion)

In terms of principal stresses:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \sigma_{\rm avg} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_{\rm avg} & 0 & 0 \\ 0 & \sigma_2 - \sigma_{\rm avg} & 0 \\ 0 & 0 & \sigma_3 - \sigma_{\rm avg} \end{bmatrix}$$

Distortion Energy Theory ... contd.$_3$

Strain energy per unit volume: $u = \frac{1}{2}\left[\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3\right]$

Substitute the strain-stress relations:

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$$
$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_3 + \sigma_1)\right]$$
$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right]$$

$$\rightarrow u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

Distortion Energy Theory ... contd.$_4$

For strain energy producing *only* volumetric change, u_v:

➥ Replace σ_1, σ_2, σ_3 by σ_{avg}

$$u_v = \frac{3\sigma_{\text{avg}}^2}{2E}(1 - 2\nu)$$

Substitute $\sigma_{\text{avg}} = \dfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ and simplify:

$$\rightarrow u_v = \frac{1 - 2\nu}{6E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1)$$

Distortion Energy Theory ... contd.$_5$

Distortion energy = Strain Energy − Strain Energy due to Volumetric Change

$$u_d = u - u_v = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]$$

For simple tensile test at yield: $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$.

$$u_{d,\mathrm{ten}} = \frac{1+\nu}{3E}S_y^2$$

Distortion Energy Theory ... contd.$_6$

Therefore, by Distortion Energy Theory, yielding is predicted when:

$$u_d \geq u_{d,\text{ten}}$$

$$\Rightarrow \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \geq S_y$$

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \rightarrow \sigma'$$

σ' : *Single, equivalent, or effective stress*

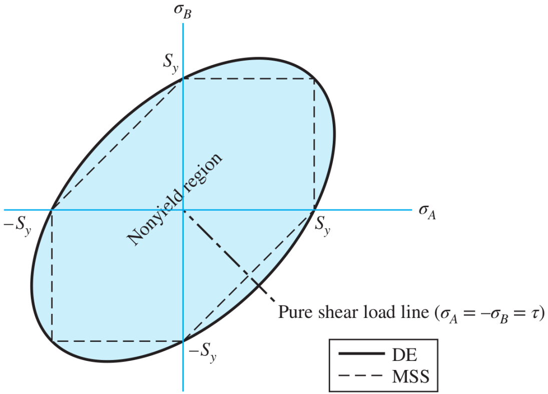
This effective stress is usually called the **von Mises stress.**

For design, incorporate a factor of safety: $\sigma' = \dfrac{S_y}{n}$

Distortion Energy Theory ... contd.$_7$

For a *plane stress* state with principal stresses: σ_A, σ_B, and 0,

➥ von Mises stress, $\sigma' = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2}$



Distortion Energy (DE) Theory is *less conservative* than Maximum Shear Stress (MSS) Theory

Distortion Energy Theory ... contd.$_8$

Relation to Octahedral Shear Stress: (Refer Q7 and Q8 in TS4 of Mechanics of Solids)

In a coordinate system with axes oriented along principal stress directions, planes that are equally inclined to the axes are octahedral planes. → *8 such planes*

- ➥ Normal stress on each plane: Octahedral normal stress (σ_{oct})
- ➥ Shear stress on each plane: Octahedral shear stress (τ_{oct})

$$\sigma_{\text{oct}} = \sigma_{\text{avg}}$$

$$\tau_{\text{oct}} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

For a simple tensile test: $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$. Thus:

$$\tau_{\text{oct,ten}} = \frac{\sqrt{2}}{3} S_y$$

Distortion Energy Theory ... contd.$_9$

The yielding criterion can be set as: $\tau_{\mathrm{oct}} \geq \tau_{\mathrm{oct,ten}}$.

$$\Rightarrow \left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y$$

This is the same criterion derived earlier.

That is why: *Another name* of the Distortion Energy Theory is the **Octahedral Shear Stress Theory**.

Distortion Energy Theory ... contd.$_{10}$

Using *xyz* components of 3D stress, the von Mises stress can be written as:

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2}$$

For plane stress: $\sigma' = (\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2)^{1/2}$

For a case of pure shear τ_{xy} under plane stress with $\sigma_x = \sigma_y = 0$:

Distortion energy criterion: $(3\tau_{xy})^{1/2} \geq S_y$.

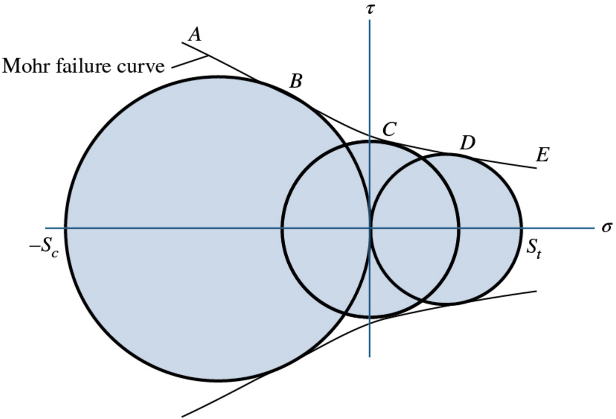
Thus, the shear yield strength predicted by this theory is:

$$S_{sy} = \frac{S_y}{\sqrt{3}} = 0.577 S_y.$$

Note: This S_{sy} is about 15% greater than the S_{sy} predicted by the Maximum Shear Stress Theory.

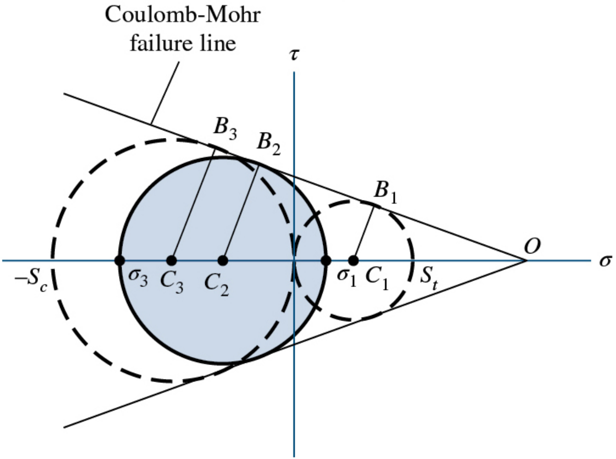
Mohr Theory

- Some materials have compressive strengths different from tensile strengths
- Mohr theory is based on three simple tests: tension, compression, and shear
- Plotting Mohr's circle for each, bounding curve defines failure envelope



Coulomb-Mohr Theory

- Curved failure curve is difficult to determine analytically
- Coulomb-Mohr theory simplifies to linear failure envelope using only tension and compression tests (dashed circles)



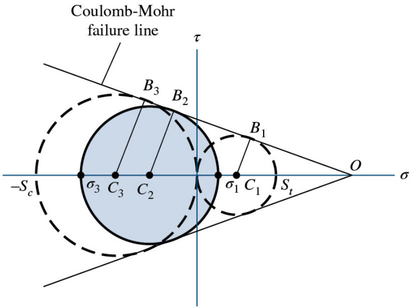
Coulomb-Mohr Theory ... contd.$_2$

Triangles OB_iC_i are similar.
$(i = 1, 2, 3)$

$$\frac{B_1C_1}{OC_1} = \frac{B_2C_2}{OC_2} = \frac{B_3C_3}{OC_3}$$

$$\Rightarrow \frac{B_2C_2 - B_1C_1}{OC_2 - OC_1} = \frac{B_3C_3 - B_1C_1}{OC_3 - OC_1}$$

$$\Rightarrow \frac{B_2C_2 - B_1C_1}{C_1C_2} = \frac{B_3C_3 - B_1C_1}{C_1C_3}$$



Here, $B_1C_1 = S_t/2$, $B_2C_2 = (\sigma_1 - \sigma_3)/2$, $B_3C_3 = S_c/2$ are the three circle radii. The distance from origin to C_1 is $S_t/2$, to C_3 is $S_c/2$, and to C_2 is $(\sigma_1 + \sigma_3)/2$. Thus, we have:

$$\frac{\dfrac{\sigma_1 - \sigma_3}{2} - \dfrac{S_t}{2}}{\dfrac{S_t}{2} - \dfrac{\sigma_1 + \sigma_3}{2}} = \frac{\dfrac{S_c}{2} - \dfrac{S_t}{2}}{\dfrac{S_t}{2} + \dfrac{S_c}{2}} \implies \frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

Coulomb-Mohr Theory ... contd.$_3$

Consider plane stress case with principal stresses: σ_A, σ_B, and 0.

Case 1: $\sigma_A \geq \sigma_B \geq 0$

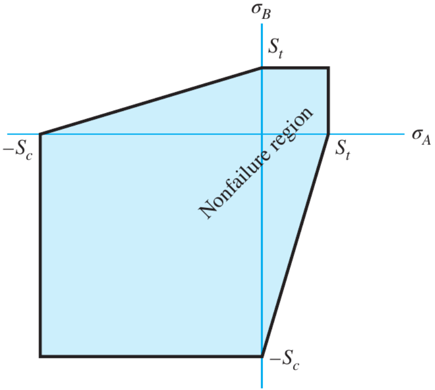
$\Rightarrow \sigma_A = S_t$

Case 2: $\sigma_A \geq 0 \geq \sigma_B$

$$\Rightarrow \frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} = 1$$

Case 3: $0 \geq \sigma_A \geq \sigma_B$

$\Rightarrow \sigma_B = -S_c$



Coulomb-Mohr Theory ... contd.$_4$

For design, incorporate the factor of safety n by dividing all strengths by it. Thus, we have:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n}$$

Coulomb-Mohr Theory ... contd.$_5$

For the case of pure shear, we have $\tau = \sigma_1 = -\sigma_3$.

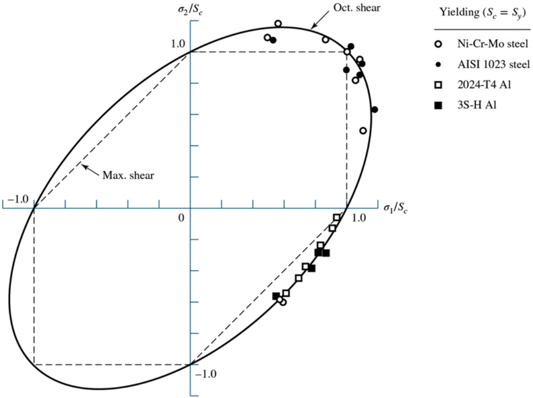
Also, $\tau_{\max} = S_{sy}$.

Substitute $\sigma_1 = -\sigma_3 = S_{sy}$ in the Coulomb-Mohr criterion:

$$\frac{S_{sy}}{S_t} - \frac{-S_{sy}}{S_c} = 1$$

$$\implies S_{sy} = \frac{S_t S_c}{S_t + S_c}$$

MSS and DE Theories Comparison



Maximum Shear Stress (MSS) Theory is more conservative than Distortion Energy (DE) Theory.

Failure Theories for Brittle Materials

Maximum Normal Stress Theory

This theory states: *That failure occurs whenever one of the three principal stresses equal or exceeds the strength.* For principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$, failure occurs whenever

$$\sigma_1 \geq S_{ut} \qquad \text{or} \qquad \sigma_3 \leq -S_{uc}$$

Here, S_{ut} and S_{uc} are the ultimate tnesile and compressive strengths.

For a plane stress case with non-zero principal stresses $\sigma_A \geq \sigma_B$:

$$\sigma_A \geq S_{ut} \qquad \text{or} \qquad \sigma_B \leq -S_{uc}$$

For design with a factor of safety n: $\sigma_A = \dfrac{S_{ut}}{n}$ or $\sigma_B = -\dfrac{S_{uc}}{n}$

Warning: This theory is not recommended for use.

Brittle Coulomb-Mohr Theory

This theory is exactly like the Ductile Coulomb-Mohr Theory discussed earlier, except that the yield strengths are replaced by ultimate strengths.

For a plane stress case with non-zero principal stresses σ_A and σ_B and incorporating a factor of safety, n for design:

Case 1: $\sigma_A \geq \sigma_B \geq 0$

$$\Rightarrow \sigma_A = \frac{S_{ut}}{n}$$

Case 2: $\sigma_A \geq 0 \geq \sigma_B$

$$\Rightarrow \frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n}$$

Case 3: $0 \geq \sigma_A \geq \sigma_B$

$$\Rightarrow \sigma_B = -\frac{S_{uc}}{n}$$

Modified Mohr Theory

Case 1: $\sigma_A \geq \sigma_B \geq 0$

$$\Rightarrow \sigma_A = \frac{S_{ut}}{n}$$

Case 2: $\sigma_A \geq 0 \geq \sigma_B$ and $\left|\frac{\sigma_B}{\sigma_A}\right| \leq 1$

$$\Rightarrow \sigma_A = \frac{S_{ut}}{n}$$

Case 3: $\sigma_A \geq 0 \geq \sigma_B$ and $\left|\frac{\sigma_B}{\sigma_A}\right| > 1$

$$\frac{(S_{uc} - S_{ut})\sigma_A}{S_{uc}S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n}$$

Case 4: $0 \geq \sigma_A \geq \sigma_B$

$$\sigma_B = -\frac{S_{uc}}{n}$$

For **Case 3**, start with:

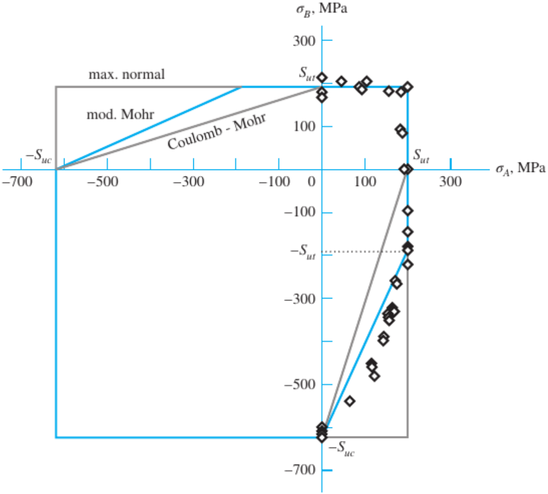
$$\frac{\sigma_A}{k} - \frac{\sigma_B}{S_{uc}/n} = 1$$

To find k, note that when $\sigma_A = \frac{S_{ut}}{n}$, we have

$$\sigma_B = -\frac{S_{ut}}{n}$$

(Refer the figure in the next slide.)

Depiction of Brittle Coulomb-Mohr and Modified Mohr Theories



Fracture data of gray cast iron