Lecture Notes on Torsion

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1 Variational formulation

Consider a shaft with a cross-section of arbitrary shape as shown in Fig. 1. The rate of twist along the length is given by $\alpha = \frac{\theta}{\mathrm{d}z}$, where θ is the angular displacement of a material point on a cross-section. Then, taking the shaft to be fixed to a wall at $z = 0$, we have $\theta = 0$ at $z = 0$ so that $\theta = \alpha z$.

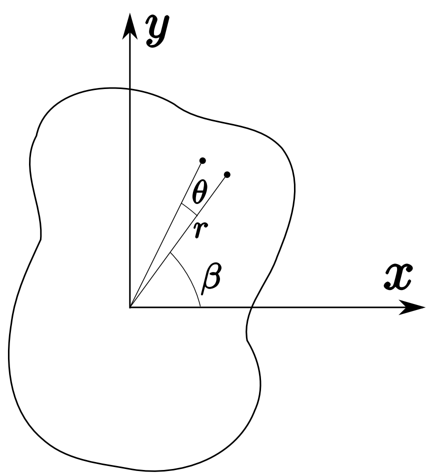

Figure 1: Cross-section with arbitrary shape

For a cross-section with arbitrary shape, the assumption that plane sections remain plane is no longer true (unlike a circular cross-section). We use the following kinematical hypothesis:

$$u = -(r\theta)\sin\beta = -y\theta = -\alpha yz, \tag{1a}$$

$$v = (r\theta)\cos\beta = x\theta = \alpha xz, \tag{1b}$$

$$w = \kappa(x, y), \tag{1c}$$

where $\kappa(x, y)$ denotes that the plane sections do not remain plane under torsion. For a circular cross-section, $\kappa = 0$.

Then the strains are given by:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0, \tag{2a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \tag{2b}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0, \tag{2c}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(-\alpha z + \alpha z) = 0, \tag{2d}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\alpha x + \frac{\partial \kappa}{\partial y}\right) \tag{2e}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right) \tag{2f}$$

Now, consider the virtual work equation: $\int_V \sigma_{ij}\delta\varepsilon_{ij}\ \mathrm{d}V = \int_A t_i \delta u_i\ \mathrm{d}A$.

Proceeding first with the lhs of the virtual work equation:

$$\begin{aligned}
\text{LHS} &= \int_V \left(2\sigma_{xz}\delta\varepsilon_{xz} + 2\sigma_{yz}\delta\varepsilon_{yz}\right)\ \mathrm{d}V \\
&= \int_V 4G\left(\varepsilon_{xz}\delta\varepsilon_{xz} + \varepsilon_{yz}\delta\varepsilon_{yz}\right)\ \mathrm{d}V \\
&= \int_V 4G\left[\frac{1}{2}\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)\frac{1}{2}\left(-y\delta\alpha + \frac{\partial \delta\kappa}{\partial x}\right) + \frac{1}{2}\left(\alpha x + \frac{\partial \kappa}{\partial y}\right)\frac{1}{2}\left(x\delta\alpha + \frac{\partial \delta\kappa}{\partial y}\right)\right]\ \mathrm{d}V \\
&= GL\int_A \left\{\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)(-y) + \left(\alpha x + \frac{\partial \kappa}{\partial y}x\right)\right\}\delta\alpha\ \mathrm{d}A + GL\int_A \left\{\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)\frac{\partial \delta\kappa}{\partial x} + \left(\alpha x + \frac{\partial \kappa}{\partial y}\right)\frac{\partial \delta\kappa}{\partial y}\right\}\ \mathrm{d}A \\
&= GL\int_A \left\{\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)(-y) + \left(\alpha x + \frac{\partial \kappa}{\partial y}x\right)\right\}\delta\alpha\ \mathrm{d}A \\
&\qquad + GL\int_A \left[\frac{\partial}{\partial x}\left\{\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)\delta\kappa\right\} - \frac{\partial}{\partial x}\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)\delta\kappa\right. \\
&\qquad \left. + \frac{\partial}{\partial y}\left\{\left(\alpha x + \frac{\partial \kappa}{\partial y}\right)\kappa\right\} - \frac{\partial}{\partial y}\left(\alpha x + \frac{\partial \kappa}{\partial y}\right)\delta\kappa\right]\ \mathrm{d}A
\end{aligned}$$

Next, proceeding with the rhs of the virtual work equation, we note that in $t_i\delta u_i\ \mathrm{d}A$ the shearing force due to traction on the surface of the beam can be written in terms of the externally applied torque as $\frac{T}{R}$ while the displacement can be written as $R\theta$ so that the rhs becomes:

$$\begin{aligned}
\text{RHS} &= T\delta\theta|_L - T\delta\theta|_0, \\
&= [T\delta\theta]_0^L \\
&= \int_0^L T\delta\frac{\mathrm{d}\theta}{\mathrm{d}z}\ \mathrm{d}z \\
&= \int_0^L T\delta\alpha \mathrm{d}z \\
&= TL\delta\alpha
\end{aligned}$$

Setting LHS = RHS, we have

$$
\begin{aligned}
GL \int_A \left\{ \left(-\alpha y + \frac{\partial \kappa}{\partial x} \right) (-y) + \left(\alpha x + \frac{\partial \kappa}{\partial y} \right) x \right\} \delta \alpha \, \mathrm{d}A \\
- GL \int_A \left[\frac{\partial}{\partial x} \left(-\alpha y + \frac{\partial \kappa}{\partial x} \right) \delta \kappa + \frac{\partial}{\partial y} \left(\alpha x + \frac{\partial \kappa}{\partial y} \right) \delta \kappa \right] \mathrm{d}A \\
+ GL \oint \left[\left(-\alpha y + \frac{\partial \kappa}{\partial x} \right) \delta \kappa \, n_x + \left(\alpha x + \frac{\partial \kappa}{\partial y} \right) \delta \kappa \, n_y \right] \mathrm{d}s = TL\delta\alpha
\end{aligned}
$$

Therefore, we must have the following:

$$
TL = GL \int_A \left\{ \left(-\alpha y + \frac{\partial \kappa}{\partial x} \right) (-y) + \left(\alpha x + \frac{\partial \kappa}{\partial y} \right) x \right\} \delta \alpha \, \mathrm{d}A, \implies T = G \int \left\{ -y \frac{\partial \kappa}{\partial x} + x \frac{\partial \kappa}{\partial y} + \alpha \left(x^2 + y^2 \right) \right\} \mathrm{d}A \quad (3)
$$

$$
GL \frac{\partial}{\partial x} \left(-\alpha y + \frac{\partial \kappa}{\partial x} \right) + \frac{\partial}{\partial y} \left(\alpha x + \frac{\partial \kappa}{\partial y} \right) = 0 \implies \frac{\partial^2 \kappa}{\partial x^2} + \frac{\partial^2 \kappa}{\partial y^2} = 0, \quad (4)
$$

and

$$
\left(-\alpha y + \frac{\partial \kappa}{\partial x} \right) n_x + \left(\alpha x + \frac{\partial \kappa}{\partial y} \right) n_y = 0 \quad \text{on the boundary.} \quad (5)
$$

Since for $\alpha = 0$, we have $w = 0$ (basically no twisting case), it may be supposed that w is proportional to α as long as α is small. Thus, we let $\kappa = \alpha\varphi$ so that $w = \alpha\varphi$. Here, φ is referred to as the warping function.

Therefore, from (3), we have

$$
\begin{aligned}
T &= G \int \left\{ -y\alpha \frac{\partial \varphi}{\partial x} + x\alpha \frac{\partial \varphi}{\partial y} + \alpha \left(x^2 + y^2 \right) \right\} \mathrm{d}A \\
&= G\alpha \int \left\{ x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} + \left(x^2 + y^2 \right) \right\} \mathrm{d}A \\
&= G\alpha J,
\end{aligned}
$$

where

$$
J = \left\{ x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} + \left(x^2 + y^2 \right) \right\} \mathrm{d}A.
$$

The term GJ is referred to as the torsional rigidity.

From (4), we have

$$
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \quad (6)
$$

And, from (5), we have

$$
\left(-y + \frac{\partial \varphi}{\partial x} \right) n_x + \left(x + \frac{\partial \varphi}{\partial y} \right) n_y = 0 \quad \text{on the boundary.} \quad (7)
$$

Let us consider the simplest solution to the Laplace equation (6) as $\varphi = c$, a constant. Then from (7), we have

$$
\begin{aligned}
&(-y)\frac{\mathrm{d}y}{\mathrm{d}s} - (x)\frac{\mathrm{d}x}{\mathrm{d}s} = 0,\\
\text{or,}\quad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(x^2 + y^2\right) = 0,\\
\text{or,}\quad &x^2 + y^2 = \text{constant}
\end{aligned}
$$

So the boundary is a circle. Then we have

$$
\begin{aligned}
J &= \int \left[x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x} + \left(x^2 + y^2\right)\right]\,\mathrm{d}A\\
&= \int \left(x^2 + y^2\right)\,\mathrm{d}A
\end{aligned}
$$

which is the familiar polar moment of inertia encountered in first year mechanics.

Thus, we have

$$
T = G\alpha J \implies \alpha = \frac{T}{GJ}
$$

Note that with α a constant, and considering no twisting at one end for a shaft of length L, we have $\theta = \alpha L$ from $\alpha = \frac{\mathrm{d}\theta}{\mathrm{d}z}$ at the other end. Then, from the expression of α, we have

$$
\theta = \frac{TL}{GJ},
$$

which again is the familiar formula from first year mechanics for the twisting angle obtained for a shaft of circular cross-section.

2 Alternative formulation using Prandtl stress function

It is definitely good that this sophisticated theory is able to recover the formulae for the simplest case. However, there is a shortcoming. Note that warping function, φ had to be guessed to find something appropriate for a particular geometry. Basically, we considered it be a constant, and it turned out to be the solution corresponding to a circle. As further examples, we have $\varphi = Axy$ for a shaft of elliptical cross-section and $\varphi = A(y^3 - 3x^2y)$ for a shaft having a cross-section in the form of an equilateral triangle.

It would, however, be much better if we could proceed to deduce the solution in response to a given geometry. That necessitates reformulating the theory in terms of what is known as the Prandtl stress function.

Must like the Airy stress function formulation used in plane stress and plane strain problems, in the Prandtl stress function formulation for torsion problems, we start by assuming the following forms

$$
\sigma_{xz} = 2G\alpha\frac{\partial\psi}{\partial y} \quad \text{and} \quad \sigma_{yz} = -2G\alpha\frac{\partial\psi}{\partial x}. \tag{8}
$$

The motivation of assuming these forms is to identically satisfy

$$
\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} = 0. \qquad \text{(Note that } \sigma_{zz} \text{ is zero.)}
$$

Now, compare the expressions of σ_{xz} and σ_{yz} from (8) with those obtained earlier using Hooke's law to obtain

$$-2\frac{\partial \psi}{\partial x} = x + \frac{\partial \varphi}{\partial y}, \tag{9a}$$

$$2\frac{\partial \psi}{\partial y} = -y + \frac{\partial \varphi}{\partial x}. \tag{9b}$$

Eliminate φ by taking the derivative of (9a) with respect to x, taking the derivative of (9b) with respect to y, and then subtracting, to obtain:

$$\nabla^2 \psi = -1.$$

We now use the no traction boundary condition on the outer surface to obtain

$$\begin{aligned}
& \sigma_{xz} n_x + \sigma_{yz} n_y = 0 \\
\text{or,}\quad & 2G\alpha \frac{\partial \psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} - 2G\alpha \frac{\partial \psi}{\partial x}\left(-\frac{\mathrm{d}x}{\mathrm{d}s}\right) = 0 \\
\text{or,}\quad & \frac{\partial \psi}{\partial y}\mathrm{d}y + \frac{\partial \psi}{\partial x}\mathrm{d}x = 0 \\
\text{or,}\quad & \mathrm{d}\psi = 0 \\
\text{or,}\quad & \psi = \text{constant on the periphery}
\end{aligned}$$

Since only derivatives of ψ appear in the definitions of σ_{xz} and σ_{yz}, any constant may be added to ψ.

Now, the torsional rigidity is

$$\begin{aligned}
GJ &= G\int_A \left\{ x\frac{\partial \varphi}{\partial y} - y\frac{\partial \varphi}{\partial x} + \left(x^2 + y^2\right) \right\} \mathrm{d}A \\
&= G\int_A \left\{ x\left(-2\frac{\partial \psi}{\partial x} - x\right) - y\left(2\frac{\partial \psi}{\partial y} + y\right) + \left(x^2 + y^2\right) \right\} \mathrm{d}A \\
&= -2G\int_A \left(x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} \right) \mathrm{d}A \\
&= -2G\int_A \left\{ \frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi) - 2\psi \right\} \mathrm{d}A \\
&= -2G\oint (x\psi n_x + y\psi n_y)\,\mathrm{d}s + 4G\int_A \psi\,\mathrm{d}A \\
&= -2G\oint (x n_x + y n_y)\psi\,\mathrm{d}s + 4G\int_A \psi\,\mathrm{d}A
\end{aligned} \tag{10}$$

2.1 Simply-connected domain

For a cross-section that is simply-connected, there is only one contour (i.e. the periphery of the cross-section) and over it we take $\psi = 0$. This condition $\psi = 0$ becomes the boundary condition for $\nabla^2 \psi = -1$.

With $\psi = 0$ along the contour, the torsional rigidity for a simply connected-region becomes (see (10))

$$GJ = 4G\int_A \psi\,\mathrm{d}A. \tag{11}$$

For a circle, if ψ has to be zero on the periphery, it can be taken as $\psi = K\left\{R^2 - \left(x^2 + y^2\right)\right\}$, where K is as yet-unknown. We note that the torsional rigidity becomes

$$
\begin{aligned}
GJ &= 4G \int_A \psi \, \mathrm{d}A \\
&= 4GK \int_A \left\{ R^2 - \left(x^2 + y^2\right) \right\} \, \mathrm{d}x\mathrm{d}y \\
&\equiv 4GK \int_0^R (R^2 - r^2) 2\pi r \, \mathrm{d}r \\
&= 2KG\pi R^4.
\end{aligned}
$$

So we end up with $J = 2K\pi R^4$. But we know that $J = \frac{1}{2}\pi R^4$ for a circular cross-section. Therefore, $K = \frac{1}{4}$, so that $\psi = \frac{1}{4}\left\{ R^2 - \left(x^2 + y^2\right) \right\}$.

With $\psi = \frac{1}{4}\left\{ R^2 - \left(x^2 + y^2\right) \right\}$, check for φ. We have:

$$
\begin{aligned}
\frac{\partial \varphi}{\partial y} &= -x - 2\frac{\partial \psi}{\partial x} = 0, \\
\frac{\partial \varphi}{\partial x} &= y + 2\frac{\partial \psi}{\partial y} = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{d}\varphi &= \frac{\partial \varphi}{\partial x}\mathrm{d}x + \frac{\partial \varphi}{\partial y}\mathrm{d}y = 0, \\
\text{or} \quad \varphi &= \text{constant},
\end{aligned}
$$

which is exactly what we had obtained earlier for a circle.

2.2 Multiply-connected domain

We had shown earlier (see (10)) that the torsional rigidity is

$$
GJ = -2G \oint (x n_x + y n_y)\psi \, \mathrm{d}s + 4G \int_A \psi \, \mathrm{d}A.
$$

For a simply-connected domain, ψ was taken as zero on the periphery. So the torsional rigidity, GJ was $4G \int_A \psi \, \mathrm{d}A$.

However, for a multiply-connected domain, ψ is a different constant on different contours. So we can take ψ as zero on only one of them.

We use

$$
\begin{aligned}
x &= -2\frac{\partial \psi}{\partial x} - \frac{\partial \varphi}{\partial y} \implies x n_x = -2\frac{\partial \psi}{\partial x} n_x - \frac{\partial \varphi}{\partial y} n_x, \\
y &= -2\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \implies y n_y = -2\frac{\partial \psi}{\partial y} n_y + \frac{\partial \varphi}{\partial x} n_y.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& -2G \oint (x n_x + y n_y)\psi \,\mathrm{d}s \\
&= -2G \oint \left(-2\frac{\partial \psi}{\partial x} n_x - \frac{\partial \varphi}{\partial y} n_x - 2\frac{\partial \psi}{\partial y} n_y + \frac{\partial \varphi}{\partial x} n_y \right) \psi \,\mathrm{d}s \\
&= 4G \oint \left(\frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial y} n_y \right) \psi \,\mathrm{d}s + 2G \oint \left(\frac{\partial \varphi}{\partial y} n_x - \frac{\partial \varphi}{\partial x} n_y \right) \psi \,\mathrm{d}s \\
&= 4G \oint \frac{\partial \psi}{\partial n} \psi \,\mathrm{d}s + 2G \oint \mathrm{d}\varphi \psi \,\mathrm{d}s \\
&= 4G \sum_{C_i} \psi \oint_{C_i} \frac{\partial \psi}{\partial n} \,\mathrm{d}s + 2G \sum_{C_i} \psi \oint_{C_i} \mathrm{d}\varphi \,\mathrm{d}s \quad (\psi \text{ is a constant over any contour})
\end{aligned}
$$

Since the warping function, φ is single-valued, we must have that

$$
\begin{aligned}
& \oint \mathrm{d}\varphi = 0 \\
\text{or,}\quad & \oint \left(\frac{\partial \varphi}{\partial x} \mathrm{d}x + \frac{\partial \varphi}{\partial y} \mathrm{d}y \right) = 0 \\
\text{or,}\quad & \oint \left\{ \left(2\frac{\partial \psi}{\partial y} + y \right) \mathrm{d}x - \left(2\frac{\partial \psi}{\partial x} + x \right) \mathrm{d}y \right\} = 0 \\
\text{or,}\quad & -2 \oint \left(\frac{\partial \psi}{\partial x} \mathrm{d}y - \frac{\partial \psi}{\partial y} \mathrm{d}x \right) - \oint (x \mathrm{d}y - y \mathrm{d}x) = 0 \\
\text{or,}\quad & -2 \oint \frac{\partial \psi}{\partial n} \,\mathrm{d}s - \oint \boldsymbol{r} \cdot \boldsymbol{n} \,\mathrm{d}s = 0 \\
\text{or,}\quad & -2 \oint \frac{\partial \psi}{\partial n} \,\mathrm{d}s - \int \nabla \cdot \boldsymbol{r} \,\mathrm{d}A = 0 \\
\text{or,}\quad & 2 \oint \frac{\partial \psi}{\partial n} \,\mathrm{d}s = -\int \nabla \cdot \boldsymbol{r} \,\mathrm{d}A \\
\text{or,}\quad & 2 \oint \frac{\partial \psi}{\partial n} \,\mathrm{d}s = -2A \\
\text{or,}\quad & \oint \frac{\partial \psi}{\partial n} \,\mathrm{d}s = -A
\end{aligned}
$$

Therefore, the torsional rigidity is

$$
\begin{aligned}
GJ &= -2G \oint (x n_x + y n_y)\psi \,\mathrm{d}s + 4G \int_A \psi \,\mathrm{d}A \\
&= 4G\psi \oint \frac{\partial \psi}{\partial n} \,\mathrm{d}s + 4G \int_A \psi \,\mathrm{d}A.
\end{aligned}
$$