

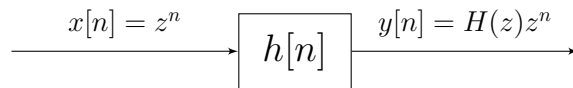
Z-Transform

- Recall that eigenfunction property was satisfied for general complex exponential signals, rather than exponential signals with an imaginary exponent.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] z^n z^{-k}$$

$$= z^n \left(\underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}_{H(z)} \right)$$



$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \quad \checkmark$$

For any DT signal $x[n]$, its z -transform is the function:

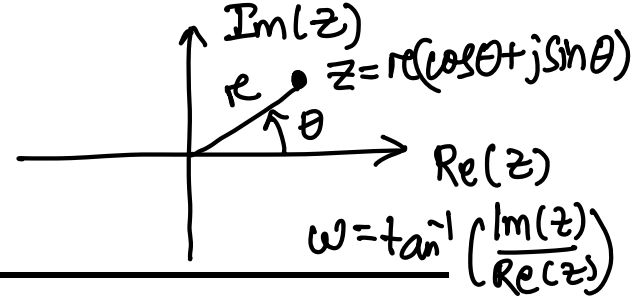
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

The Region of Convergence ^(ROC) of $x[n]$ is defined to be $\{z \mid X(z) \text{ exists, i.e., is finite}\}$ which is a subset of the complex plane.

$$X: \mathbb{C} \rightarrow \mathbb{C}$$

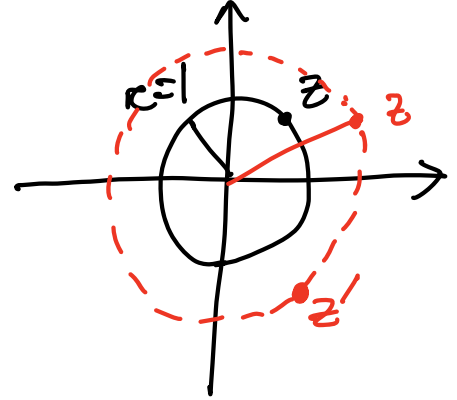
$\mathbb{C}$: set of complex numbers

z-Transform



- For a complex number z , let $z = re^{j\omega}$. We have

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}. \end{aligned}$$



- Therefore, $X(z) = X(re^{j\omega}) = \text{DTFT}(x[n]r^{-n})$.
- In particular, for $|z| = r = 1$, $z = e^{j\omega}$, which implies $X(z) = \text{DTFT}(x[n])$.
- Thus, when $r = 1$ and we vary ω , z moves on the unit circle in the complex plane. The unit circle plays a critical role in z -transform of discrete-time signals.
- Since $X(z) = \text{DTFT}(x[n]r^{-n})$, then a point in the complex plane $\bar{z}$ lies in the ROC if its magnitude $\bar{r}$ is such that the $x[n]\bar{r}^{-n}$ is summable.
- Thus, the ROC is determined by the magnitude $\bar{r}$ rather than the phase. Hence, the ROC mainly consists of circular regions and/or rings in the complex plane.
- DTFT of $x[n]$ exists only when the ROC of the z -transform of $x[n]$ contains the unit circle.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Examples

Example 1

- What is the z -transform of $x[n] = \delta[n] + \delta[n-1]$? What is its ROC?

$$\sum_{n=-\infty}^{\infty} (\delta[n] + \delta[n-1]) z^{-n} = 1 \cdot z^{-0} + 1 \cdot z^{-1} = 1 + \frac{1}{z}$$

ROC: $\{z \in \mathbb{C} \mid z \neq 0\}$

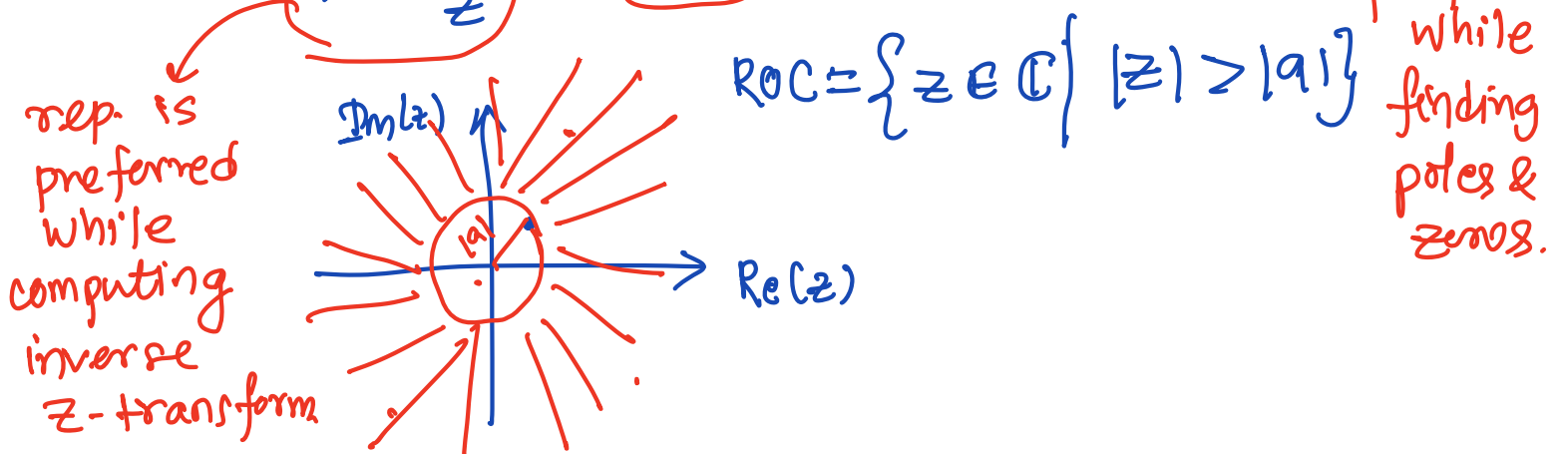
Example 2

- What is the z -transform of $x[n] = a^n u[n]$ where a is a complex number? What is its ROC?

$$X(z) = \sum_{n=-\infty}^{\infty} (a^n u[n]) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \text{ is finite when } \frac{|a|}{|z|} < 1$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

$\Rightarrow |z| > |a|$
 zero: $z=0$
 pole: $z=a$



Example 3

- What is the z -transform of $x[n] = -a^n u[-n-1]$ where a is a complex number? What is its ROC?

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} \\ &= \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{m=1}^{\infty} -a^{-m} z^m \quad (m \triangleq -n) \\ &= \sum_{m=1}^{\infty} -\left(\frac{z}{a}\right)^m = -\left(\frac{z}{a}\right) \left[\sum_{m=0}^{\infty} \left(\frac{z}{a}\right)^m \right] \\ &= -\left(\frac{z}{a}\right) \cdot \frac{1}{1 - \frac{z}{a}}, \quad \text{if } |z| < |a| \\ &= -\frac{z}{a} \cdot \frac{a}{a - z} \quad \text{ROC} = \{z \in \mathbb{C} \mid |z| < |a|\} \\ &= \frac{z}{z - a}. \end{aligned}$$

Note: Two distinct discrete-time signals can have the same algebraic expression of their z -transform. Their ROCs are different in that case.

z -transform without ROC is meaningless

- What is the $X(z)$ for $x[n] = -a^n u[-n - 1]$?
- Let's investigate:

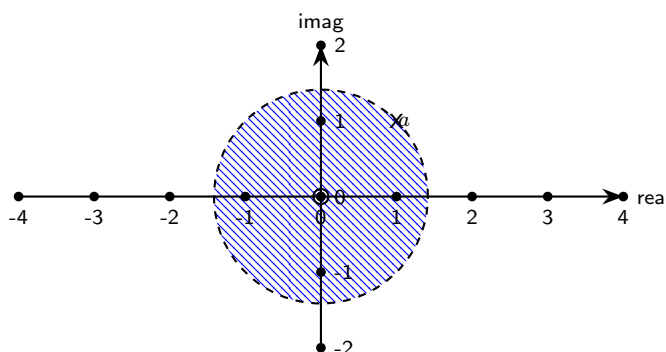
$$\begin{aligned}
 X(z) &= - \sum_{n=-\infty}^{\infty} x[n] z^{-n} = - \sum_{n=-\infty}^{-1} a^n z^{-n} \\
 &= - \sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n = - \sum_{k=1}^{\infty} \left(\frac{z}{a}\right)^k \\
 &= - \frac{z/a}{1 - \frac{z}{a}},
 \end{aligned}$$

where the last equality holds for $|\frac{z}{a}| < 1$. The ROC is sketched below.

- Therefore,

$$-a^n u[-n - 1] \longleftrightarrow \frac{z}{z - a}.$$

- But $X(z) = \frac{z}{z-a}$ for $x[n] = a^n u[n]$!
- Therefore: **z -transform without ROC is meaningless!**
- Since it is possible that z -transforms of distinct signals would have the same algebraic expression, it is unable to determine the inverse z -transform without the ROC.



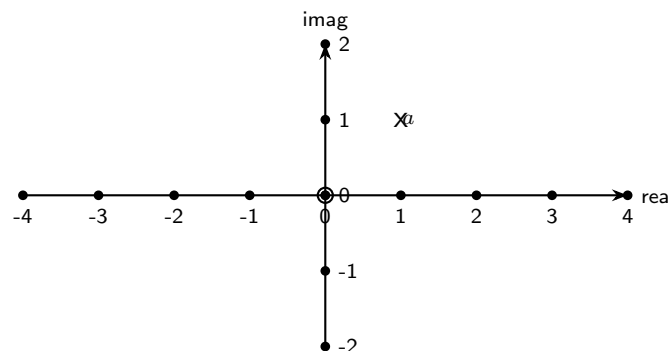
ROC and Pole-Zero

- Most of the time, we will encounter z -transforms that are rational functions of z . For a z -transform of the form:

$$H(z) = a \frac{(z - p_1) \cdots (z - p_m)}{(z - q_1) \cdots (z - q_n)},$$

the roots of the numerator $p_1, \dots, p_m$ are called the **zeros** and the roots of the denominator $q_1, \dots, q_n$ are called the **poles** of $H(z)$.

- On the complex plane, the poles are marked by (x) and zeros are marked by (o).
- It is more convenient to express $X(z)$ as polynomials in z^{-1} rather than z .



- Determine the poles and zeros of $X_1(z) = \frac{z}{z-a}$. zero: 0
pole: a
- Determine the poles and zeros of $X_2(z) = 1 + \frac{1}{z}$. → zero: -1
pole: 0
- Determine the poles and zeros of $X_3(z) = \frac{1-3z^{-1}}{(1-5z^{-1})(1-0.5z^{-1})}$.

$$\downarrow$$

$$\frac{1 - \frac{3}{z}}{\left(1 - \frac{5}{z}\right)\left(1 - \frac{0.5}{z}\right)} = \frac{z(z-3)}{(z-5)(z-0.5)}$$

Practice Problem

- a. What is the ROC for the z -transform of $x[n] = (\frac{1}{2})^n u[n] + 2^n u[n]$?

$$\underbrace{|z| > \frac{1}{2}} \quad \underbrace{|z| > 2}$$

$$X(z) = \frac{2z}{2z-1} + \frac{z}{z-2}, \quad \text{ROC} = \{z \in \mathbb{C} \mid |z| > 2\}$$

ROC of $x[n]$: intersection of ROCs of its constituent signals.

It does not admit DTFT.

- b. Determine the z -transform with ROC as well as the location of poles and zeros of the signal

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n].$$

Does the above signal admit DTFT?

$$x[n] = \left(\frac{1}{3}\right)^n \left[\frac{1}{2j} \left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n} \right) \right]$$

$$= \frac{1}{2j} \left[\left(\frac{e^{j\frac{\pi}{4}}}{3} \right)^n u[n] - \left(\frac{e^{-j\frac{\pi}{4}}}{3} \right)^n u[n] \right]$$

$$X(z) = \frac{1}{2j} \left[\frac{z}{z - \left(\frac{e^{j\frac{\pi}{4}}}{3}\right)} - \frac{z}{z - \frac{1}{3}e^{-j\frac{\pi}{4}}} \right]$$

$$\text{ROC} = \{z \in \mathbb{C} \mid |z| > \frac{1}{3}\}.$$

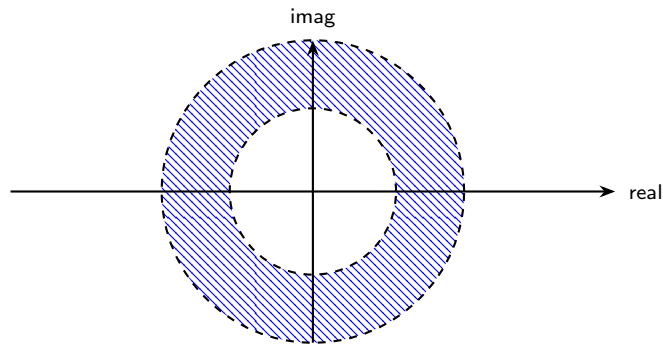
DTFT exists.

$$\text{poles: } \frac{1}{3}e^{j\frac{\pi}{4}}, \frac{1}{3}e^{-j\frac{\pi}{4}}$$

$$\text{zeros: } 0$$

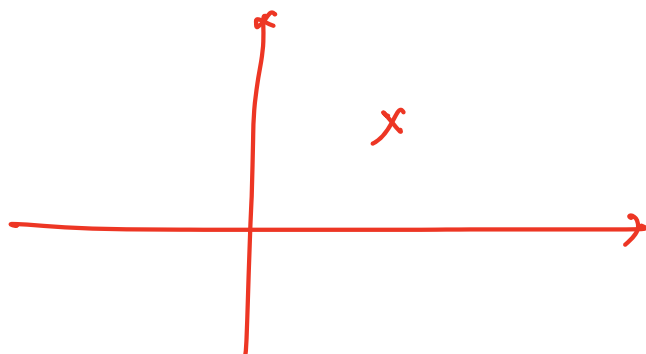
Properties of ROC

- Property 1: ROC is a ring on the complex plain.



- The inner circle's radius can be 0 (e.g., $x[n] = -a^n u[-n - 1]$)
- The outer circle's radius can be ∞ (e.g., $x[n] = a^n u[n]$)

- Property 2: ROC does not contain any pole.
Simply because at a pole, the value of $X(z)$ would be infinity!



Properties of ROC

- **Property 3:** If $x[n]$ is finite duration, then ROC is the whole complex plane, except possibly $z = 0$ and $z = \infty$.

– If $x[n]$ is only non-zero for $n = N_1, \dots, N_2$ for $N_1 \leq N_2$. Then:

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}.$$

$$\begin{aligned} x[n] &= a_1 \delta[n - N_1] && \rightsquigarrow a_1 z^{-N_1} \\ &+ a_2 \delta[n - N_1 - 1] && \rightsquigarrow + a_2 z^{-N_1 - 1} \\ &\vdots && \vdots \\ &+ a_M \delta[n - N_2] && \rightsquigarrow + a_M z^{-N_2} \end{aligned}$$

- Determine whether 0 and ∞ belong to the ROC in the following cases.

- $N_1 < 0, N_2 > 0$: $\rightarrow$ neither belong to ROC
- $N_1 \geq 0$: $\rightarrow z = \infty$ belongs to ROC, but $z = 0$ does not
- $N_2 \leq 0$: $\rightarrow z = 0$ belongs to ROC, $z = \infty$ does not

15th October

$$X(z) = \sum_{n=-N}^{-1} x[n] z^{-n} + \sum_{n=0}^{\infty} x[n] z^{-n}$$

finite

Properties of ROC

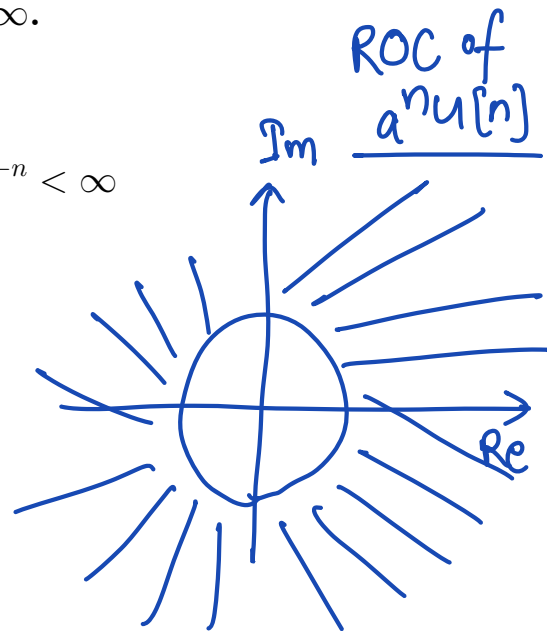
e.g. $u[n]$, unit step signal, $N=0$, $a^n u[n]$

- Definition: We say that a signal $x[n]$ is right-sided if $x[n] = 0$ for all $n < N$ and some N .
- Property 4: If $x[n]$ is right-sided and z_0 is in ROC, then z is in ROC for all $|z| > |z_0|$, possibly excluding ∞ .
- The main reason is:

$$\sum_{n=0}^{\infty} |x[n]| |z|^{-n} \leq \sum_{n=0}^{\infty} |x[n]| |z_0|^{-n} < \infty$$

and if $N < 0$, then $\sum_{n=N}^{-1} |x[n]| |z|^{-n} < \infty$.

If $N \geq 0$, ROC may include ∞ .



- Definition: We say that a signal $x[n]$ is left-sided if $x[n] = 0$ for all $n > N$ and some N .
- Property 5: If $x[n]$ is left-sided and z_0 is in ROC, then z is in ROC for all $0 < |z| < |z_0|$.
- The main reason is:

$$\sum_{n=-\infty}^0 |x[n]| |z|^{-n} \leq \sum_{n=-\infty}^0 |x[n]| |z_0|^{-n} < \infty$$

and if $N > 0$, then $\sum_{n=1}^N |x[n]| |z|^{-n} < \infty$.

In this case $z=0$ is not part of ROC.

Properties of ROC

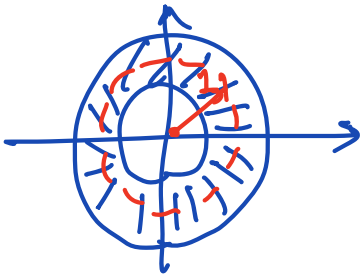
- Definition: We say that a signal $x[n]$ is double-sided if $x[n] \neq 0$ for arbitrarily large and small n .
- Property 6: If $x[n]$ is double-sided and z_0 is in ROC, then ROC is a ring containing z_0 .
- Example: What is the z -transform of $x[n] = b^{|n|}$ for $|b| < 1$?

$$X(z) = \sum_{n=-\infty}^{-1} b^{-n} z^{-n} + \sum_{n=0}^{\infty} b^n z^{-n} = \sum_{n=0}^{\infty} (bz)^m + \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n - 1$$

$\underbrace{\hspace{10em}}_{\text{converges when } |bz| < 1} \quad \underbrace{\hspace{10em}}_{\text{converges when } \left|\frac{b}{z}\right| < 1}$

$\Rightarrow |z| < \frac{1}{|b|} \quad \Rightarrow |z| > |b|$

$1 \in (|b|, \frac{1}{|b|})$
 DFT exists.



– We have:

$$x[n] = b^n u[n] + b^{-n} u[-n-1].$$

- $b^n u[n] \longleftrightarrow \frac{1}{1-bz^{-1}}$ with ROC $|z| > |b|$
- $b^{-n} u[-n-1] \longleftrightarrow -\frac{1}{1-b^{-1}z^{-1}}$ with ROC $|z| < \frac{1}{|b|}$
- Combining the two:

$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}} = \frac{b^2-1}{b} \frac{z}{(z-b)(z-b^{-1})} = \underline{\underline{X\left(\frac{1}{z}\right)}}$$

for $|b| < |z| < \frac{1}{|b|}$.

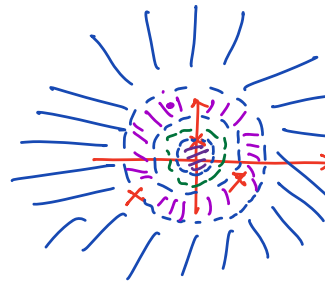
- If $|b| > 1$, there is no overlap in the ROCs of two parts, hence the z -transform is not defined.

$$= \frac{\frac{1}{z}}{\left(\frac{1}{z}-b\right)\left(\frac{1}{z}-b^{-1}\right)} \cdot \frac{z^2}{z^2} = \frac{z}{(z-b)(z-b^{-1})}$$

Properties of ROC

- **Property 7:** If $X(z)$ is rational, then ROC is always bounded by poles or extend to infinity.

- **Definition:** We say that $x[n]$ is a **causal signal** if $x[n] = 0$ for $n < 0$.
- **Property 8:** If $X(z)$ is rational and $x[n]$ is right-sided, then ROC is the region from the outermost pole to infinity. Furthermore, if $x[n]$ is causal, $X(\infty)$ exists.



- **Definition:** We say that $x[n]$ is an **anti-causal signal** if $x[n] = 0$ for $n > 0$.
- **Property 9:** If $X(z)$ is rational and $x[n]$ is left-sided, then ROC is the region from the innermost pole to zero. Furthermore, if $x[n]$ is anti-causal ROC includes $z = 0$.

Recall: $a^n u[n] \longleftrightarrow \frac{1}{1 - \frac{a}{z}}$, ROC: $\{z \mid |z| > |a|\}$
 $-a^n u[-n-1] \longleftrightarrow \frac{1}{1 - a/z}$, Question ROC: $\{z \mid |z| < |a|\}$

Suppose that we know that a signal is causal and its z -transform is:

$$X(z) = \frac{2z}{(z+j)(z-j)(z-(3-4j))}.$$

Determine the ROC of $X(z)$.

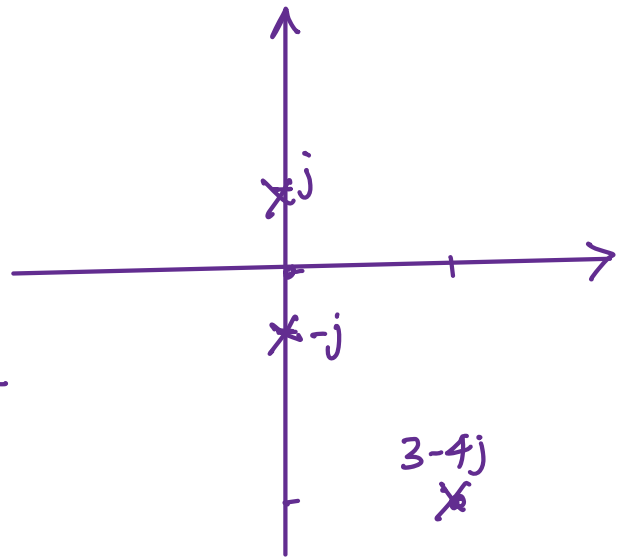
outermost pole / pole with largest magnitude is $3-4j$.

$$|3-4j| = 5$$

Since signal is causal,

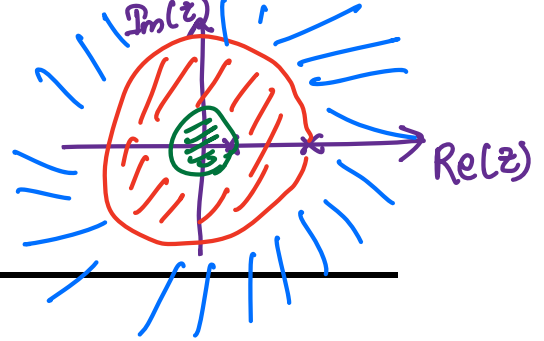
ROC extends from outermost pole to $+\infty$.

$$\{z \in \mathbb{C} \mid |z| > 5\}.$$



Solution: $\{z \mid |z| > 5\}$

Question



For the following z -transform

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})} = \frac{z^2}{(z - \frac{1}{3})(z - 2)}$$

Determine all possible ROCs of $X(z)$ and find the corresponding $x[n]$ for each of those cases.

$$= \frac{A}{(1 - \frac{1}{3}z^{-1})} + \frac{B}{(1 - 2z^{-1})}$$

$$A + B = 1 \Rightarrow A - 6A = 1 \Rightarrow A = -\frac{1}{5}$$

$$-2A - \frac{1}{3}B = 0 \Rightarrow 2A = -\frac{1}{3}B$$

$$X(z) = -\frac{1}{5} \cdot \frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{6}{5} \cdot \frac{1}{1 - 2z^{-1}} \Rightarrow B = -6A = \frac{6}{5}$$

case 1: ROC $\{z \in \mathbb{C} \mid |z| > 2\}$

$$x[n] = -\frac{1}{5} \left(\frac{1}{3}\right)^n u[n] + \frac{6}{5} 2^n u[n]$$

case 2: ROC $\{z \in \mathbb{C} \mid \frac{1}{3} < |z| < 2\}$

$$x[n] = -\frac{1}{5} \left(\frac{1}{3}\right)^n u[n] + \frac{6}{5} \underline{\underline{-2^n u[-n-1]}}$$

case 3: ROC $\{z \in \mathbb{C} \mid |z| < \frac{1}{3}\}$

$$x[n] = \left(-\frac{1}{5}\right) - \left(\frac{1}{3}\right)^n u[-n-1] + \frac{6}{5} - 2^n u[-n-1]$$

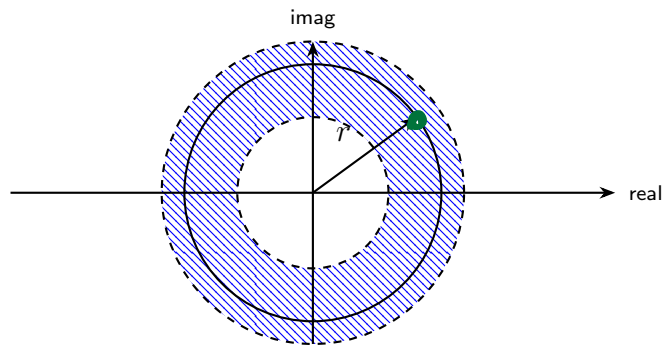
z -transform, Fourier Transform, and Inverse z -transform

- We know that if $z_0 = re^{j\omega}$ is in the ROC of the z -transform of $x[n]$, then

$$X(re^{j\omega}) = \text{DTFT}(r^{-n}x[n]).$$

- Therefore, if the z -transform exists on a circle of radius r , then:

$$r^{-n}x[n] = \text{DTFT}^{-1}(X(re^{j\omega})).$$



- Therefore, $x[n] = r^n \text{DTFT}^{-1}(X(re^{j\omega}))$

z -transform, Fourier Transform, and Inverse z -transform

- Example: We know that the z -transform of a signal $x[n]$ is $X(z) = \frac{1}{1-z^{-1}}$ and the ROC is $\{z \mid |z| > 1\}$. What is $x[n]$?
- Here are the steps to find $x[n]$:

1. Pick a circle of radius r in ROC. We pick $r = 2$.
2. Calculate $\text{DTFT}^{-1}(X(re^{j\omega}))$: $X(2e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$, but:

$$a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}},$$

therefore: $\text{DTFT}^{-1}(X(re^{j\omega})) = (\frac{1}{2})^n u[n]$

3. Finally: $x[n] = r^n \text{DTFT}^{-1}(X(re^{j\omega})) = 2^n (\frac{1}{2})^n u[n] = u[n]$

Inverse z -transform: Partial Fraction Expansion

- Try to write $X(z) = \sum_{i=1}^n \frac{b_i}{1-a_i z^{-1}}$
- If ROC is outside the circle with radius $|a_i|$, then use the fact that

$$a_i^n u[n] \longleftrightarrow \frac{1}{1 - a_i z^{-1}}, \quad \text{for } |z| > |a_i|$$

- If ROC is inside the circle with radius $|a_i|$, then use the fact that

$$-a_i^n u[-n-1] \longleftrightarrow \frac{1}{1 - a_i z^{-1}}$$

poles: $\frac{1}{4}, \frac{1}{3}$

- Example 1: What is the inverse z -transform of $X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$ with ROC $\{z \mid |z| > \frac{1}{3}\}$?

$$\begin{aligned} A+B &= 3, \quad \underline{A=1, B=2} \\ \frac{1}{3}A + \frac{1}{4}B &= \frac{5}{6} \\ \frac{1}{3} + \frac{1}{2} &= \frac{5}{6} \end{aligned}$$

$$\begin{aligned} &= \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}} \\ &= \frac{(A+B) - (\frac{1}{3}A + \frac{1}{4}B)z^{-1}}{() ()} \end{aligned}$$

- Example 2: What is the inverse z -transform of the above $X(z)$ if ROC is $\{z \mid \frac{1}{4} < |z| < \frac{1}{3}\}$?

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}, \quad \text{For example 1, } x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n].$$

$$\text{For example 2, } x[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[-n-1].$$

Inverse z -transform: Power Series Expansion

- If we can express $X(z) = \sum_k a_k z^{-k}$, then we can express $x[n] = \sum_k a_k \delta[n-k]$.

$$\frac{4z^2 + 2 + \frac{3}{z}}{(z)}$$

- Example 1: What is the inverse z -transform of $X(z) = \frac{4z^2 + 2 + 3z^{-1}}{(z)}$ with ROC $\{z \mid 0 < |z| < \infty\}$?

$$\begin{aligned} x[n] &= 4\delta[n+2] + 2\delta[n] + 3\delta[n-1] \\ X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 4 \delta[n+2] z^{-n} + 2 \delta[n] z^{-n} + 3 \delta[n-1] z^{-n} \\ &= \underline{4z^2 + 2 + 3z^{-1}} \end{aligned}$$

- Example 2: Recall that $\log(1 + \nu) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \nu^n}{n}$ for $|\nu| < 1$. Using this fact, determine the inverse z -transform of $X(z) = \log(1 + az^{-1})$ with ROC $|z| > |a|$.

$$\begin{aligned} x(z) &= \log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (az^{-1})^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n z^{-n} \end{aligned}$$

$$x[n] = \begin{cases} \frac{(-1)^{n+1}}{n} a^n, & \text{if } n \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\underline{x[n] = \begin{cases} \frac{(-a)^n}{n}, & n \geq 1 \\ 0, & \text{otherwise} \end{cases}}$$

Properties of z -transform

Suppose that $\underline{x_1[n]} \longleftrightarrow \underline{X_1(z)}$ with ROC $\underline{R_1}$ and $\underline{x_2[n]} \longleftrightarrow \underline{X_2(z)}$ with ROC $\underline{R_2}$

- Linearity: $a_1 x_1[n] + a_2 x_2[n] \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$ with ROC containing $R_1 \cap R_2$

$$\sum_{n=-\infty}^{\infty} (a_1 x_1[n] + a_2 x_2[n]) z^{-n} = a_1 \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} = a_1 X_1(z) + a_2 X_2(z).$$

- Time-shifting: $x_1[n - n_0] \longleftrightarrow z^{-n_0} X_1(z)$ with ROC R_1 except possible addition or removal of 0 or ∞

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x_1[n - n_0] z^{-n} &= z^{-n_0} \sum_{n=-\infty}^{\infty} x_1[n - n_0] z^{-(n - n_0)} \\ &= z^{-n_0} \sum_{m=-\infty}^{\infty} x_1[m] z^{-m} = \underline{\underline{z^{-n_0} X_1(z)}} \end{aligned}$$

- Frequency Scaling: $z_0^n x_1[n] \longleftrightarrow X_1\left(\frac{z}{z_0}\right)$ with ROC $|z_0| R_1$.

$$X_1\left(\frac{z}{z_0}\right) = \sum_{n=-\infty}^{\infty} x_1[n] \left(\frac{z}{z_0}\right)^{-n} = \sum_{n=-\infty}^{\infty} \underbrace{(x_1[n] z_0^n)}_{\text{z-tr. of } y[n] := x_1[n] z_0^n} z^{-n}$$

- Time-reversal: $x_1[-n] \longleftrightarrow X_1\left(\frac{1}{z}\right)$ with ROC $\frac{1}{R_1} \triangleq \{z \in \mathbb{C} \mid \frac{1}{z} \in R_1\}$

$$\sum_{n=-\infty}^{\infty} x_1[-n] z^{-n} = \sum_{m=-\infty}^{\infty} x_1[m] z^m = \sum_{m=-\infty}^{\infty} x_1[m] \left(\frac{1}{z}\right)^{-m} = \underline{\underline{X_1\left(\frac{1}{z}\right)}}.$$

- Conjugation: $x_1^*[n] \longleftrightarrow X_1^*(z^*)$ with ROC R_1

$$(X_1(z))^* = \sum_{n=-\infty}^{\infty} (x_1[n] z^{-n})^* = \sum_{n=-\infty}^{\infty} (x_1[n])^* (z^*)^{-n}$$

$$(X_1(z^*))^* = \sum_{n=-\infty}^{\infty} (x_1[n])^* z^{-n} \xrightarrow{10} \text{which is the } z\text{-transform expression.}$$

ROC for frequency scaling

if $\bar{z} \in R_1$, then $\sum_{n=-\infty}^{\infty} |x[n]| |\bar{z}|^n < \infty$

$$\begin{aligned} \text{at } z = \bar{z} |z_0|, \quad \sum_{n=-\infty}^{\infty} |x[n]| \left| \frac{z}{\bar{z}_0} \right|^n &= \sum_{n=-\infty}^{\infty} |x[n]| \frac{|\bar{z}| |\bar{z}_0|^{-n}}{|\bar{z}_0|^n} \\ &= \sum_{n=-\infty}^{\infty} |x[n]| |\bar{z}|^n < \infty \end{aligned}$$

Suppose $x[n]$ is real and even. What can we say about its z -transform?

$$\begin{aligned} x[n] = x[-n] &\Rightarrow X(z) = X\left(\frac{1}{z}\right) \\ x[n] = x[n]^* &\Rightarrow X(z) = (X(z^*))^* \end{aligned} \quad \Bigg\}$$

Hypothesis: If a is a pole, $\frac{1}{a}$ is a pole.

eg: $x[n] = b^{|n|}$

$$\text{ROC} = \{z \mid \frac{1}{2} < |z| < 2\}$$

is of this form

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

$$\frac{d}{dz} X(z) = \frac{1}{(1 + az^{-1})} a \frac{d}{dz} (z^{-1}) = \frac{-a}{z^2} \frac{1}{1 + az^{-1}}$$

$$-z \frac{d}{dz} X(z) = \frac{a}{z} \cdot \frac{1}{1 + az^{-1}} = \frac{a}{z + a}$$

$$\text{ROC} = \{z \mid |z| > |a|\}$$

$$nx[n] = \delta[n] - (a^n)u[n] = 1 - \frac{z}{z+a} = 1 - \frac{1}{1+az^{-1}}$$

Properties of z-transform

$$x[1] = a, 2 \cdot x[2] = -a^2, \dots$$

Suppose that $x_1[n] \longleftrightarrow X_1(z)$ with ROC R_1 and $x_2[n] \longleftrightarrow X_2(z)$ with ROC R_2

- **Convolution:** $x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$ with ROC containing $R_1 \cap R_2$.

Let $h[n] \longleftrightarrow H(z)$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$y[n] = x[n] * h[n] \Leftrightarrow Y(z) = X(z)H(z)$$

- Example: For the DT integrator system, $Y(z) = \frac{1}{1-z^{-1}}X(z)$

$$y[n] = \sum_{k=-\infty}^n x[k], \quad h[n] = \sum_{k=-\infty}^n \delta[k] = u[n]$$

$$H(z) = \frac{1}{1-z^{-1}}$$

- **Differentiation in z-domain:** $nx_1[n] \longleftrightarrow -z \frac{d}{dz} X_1(z)$ with ROC R_1 .

$$\frac{d}{dz} X_1(z) = \sum_{n=-\infty}^{\infty} x[n] (z^{-n})' = \sum_{n=-\infty}^{\infty} x[n] (-n) z^{-n-1} (z)^{-1}$$

- Example: If $X(z) = \log(1 + az^{-1})$ with $|z| > |a|$, then determine $x[n]$.

$$= -(z)^{-1} \sum_{n=-\infty}^{\infty} (nx[n]) z^{-n}$$

$$\Rightarrow -z \frac{d}{dz} X_1(z) = \sum_{n=-\infty}^{\infty} (nx[n]) z^{-n}$$

using this approach, we won't be able to determine $x[0]$.

Properties of z -transform

- **Initial value theorem:** For a causal signal, $x[n] = 0$ for $n < 0$, we have
 $x[0] = \lim_{z \rightarrow \infty} X(z)$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \dots$$

- For this class of signals, the number of finite zeros cannot be greater than the number of finite poles.
- For this class of signals, degree of numerator of $X(z)$ is at most the degree of denominator of $x(z)$.

$$\begin{aligned}
 X(z) &= \frac{(z - c_1)(z - c_2) \dots (z - c_m) N}{(z - p_1)(z - p_2) \dots (z - p_r) D} \\
 &= \frac{z^m (1 - \frac{c_1}{z})(1 - \frac{c_2}{z}) \dots (1 - \frac{c_m}{z})}{z^r (1 - \frac{p_1}{z})(1 - \frac{p_2}{z}) \dots (1 - \frac{p_r}{z})} \\
 \text{at } z \rightarrow \infty &= \frac{N}{D} z^{m-r}, \quad \underline{m > r \text{ is not possible}}
 \end{aligned}$$

z-transform properties

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R , except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $ a R$ = the set of points $\{ a z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} , where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $ z > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $ z > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \rightarrow \infty} X(z)$			

z-transform pairs

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$ z < \alpha $
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z > \alpha $
8. $-n\alpha^n u[-n - 1]$	$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$ z < \alpha $
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$

Homework

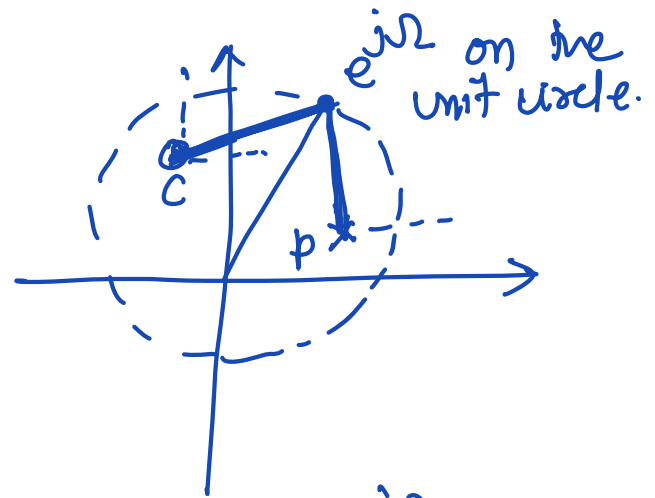
Evaluation of DTFT from Pole-Zero Plots

- From the pole-zero plot of the z -transform, we can graphically evaluate the DTFT when the unit circle is contained in the ROC.
- Let $h[n] = a^n u[n]$ with $|a| < 1$.

$$H(z) = \frac{N(z-c)}{(z-p)}$$

$$\underline{H(e^{j\Omega}) = \frac{N(e^{j\Omega}-c)}{(e^{j\Omega}-p)}}$$

$$|H(e^{j\Omega})| = |N| \frac{|e^{j\Omega}-c|}{|e^{j\Omega}-p|}$$



$$\angle H(e^{j\Omega}) = \angle(e^{j\Omega}-c) - \angle(e^{j\Omega}-p)$$

- What happens when there are multiple zeros and poles?