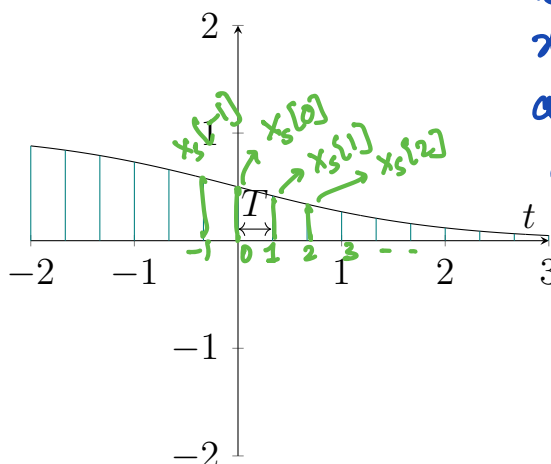


Sampling

- Question: Can we convert a **continuous-time** signal $x(t)$ to a discrete-time signal?
- One way to do this is to sample at regular intervals of time T and define the sampled signal $x_s[k] := x(kT)$.
- Question: Can we recover a **continuous-time** signal $x(t)$ from regular sampling $x_s[k] = x(kT)$?

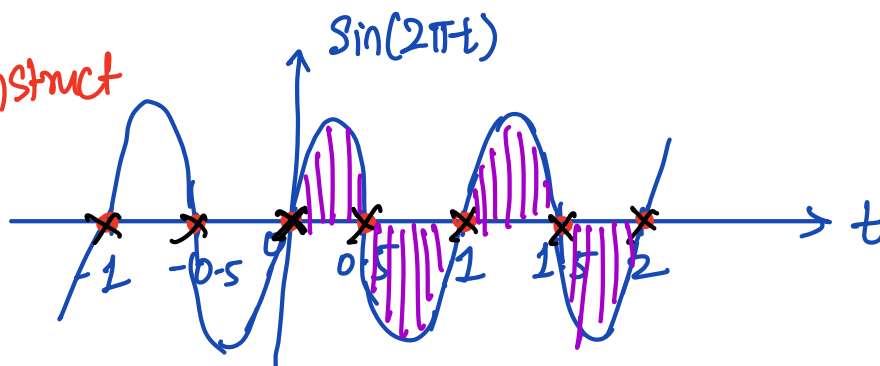


The time axis for $x(t)$ and $x_s[k]$ are different and not comparable.

- For example, consider $x(t) = \sin(2\pi t)$.
- Question: Is the recovered signal unique?
- Answer: No in general!
- But if we know the signal $x(t)$ is band-limited, then with sufficient samples, we can!

$T=1$, we cannot reconstruct

$T=0.5$: cannot reconstruct



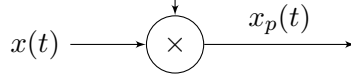
$p(t)$ is periodic with fundamental period T , $\omega_s = \frac{2\pi}{T}$.

$$p(t) = \sum_{k=-\infty}^{\infty} (a_k e^{+jk\omega_s t}), \text{ where } a_k = \frac{1}{T} \forall k \in \mathbb{Z}$$

Sampling by Impulse Train

$$\begin{aligned} \text{Let } x(t) &\leftrightarrow X(j\omega) \\ x_p(t) &\leftrightarrow X_p(j\omega) \\ p(t) &\leftrightarrow P(j\omega) \end{aligned}$$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



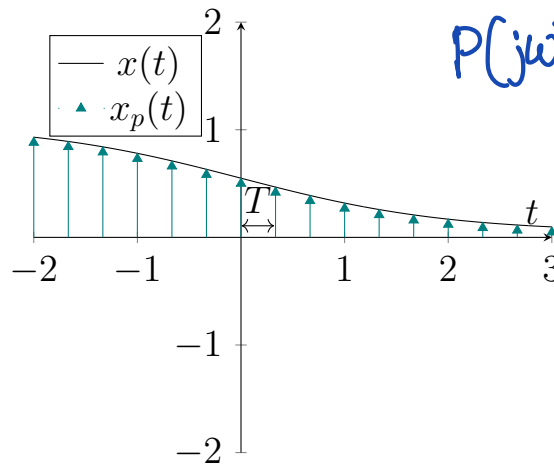
$$\begin{array}{c} \uparrow \uparrow \uparrow \uparrow \uparrow \\ \dots -T \quad 0 \quad T \quad 2T \quad 3T \dots \end{array}$$

Recall: $\mathcal{FT}[e^{-j\omega_0 t}]$

$$= 2\pi \delta(\omega + \omega_0)$$

Figure 1: Diagram of impulse train sampling system.

- The above diagram (system) described *impulse train sampling*.
- The time-domain representation of various signals:



$$P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_s)$$

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \otimes P(j\omega)$$

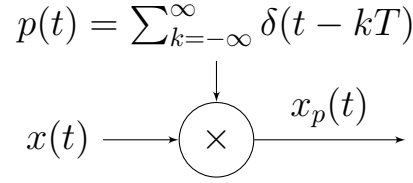
$$= \frac{1}{2\pi} X(j\omega) \otimes \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_s)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j\omega) \otimes \frac{1}{T} \delta(\omega - k\omega_s)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \left[X(j\omega) \otimes \delta(\omega - k\omega_s) \right]$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Impulse Train Sampling: Frequency Domain Analysis



- In frequency domain, we have:

$$X_p(j\omega) = \frac{1}{2\pi} P(j\omega) * X(j\omega) \quad (1)$$

- Note that $p(t)$ is periodic with period T .
- Therefore, $p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$, where

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} p(\tau) e^{-jk\omega_s \tau} d\tau = \frac{1}{T}.$$

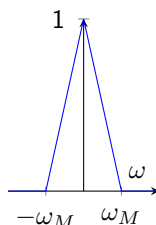
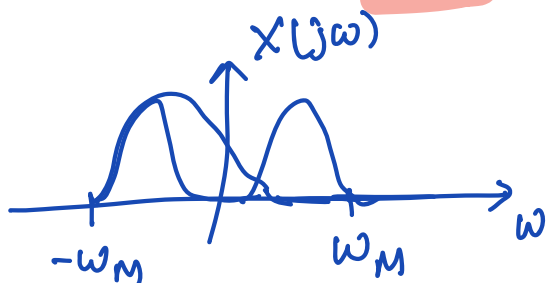
- As a result $P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$.
- Replacing this in (1), we get:

$$X_p(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(j(\omega - k\omega_s)).$$

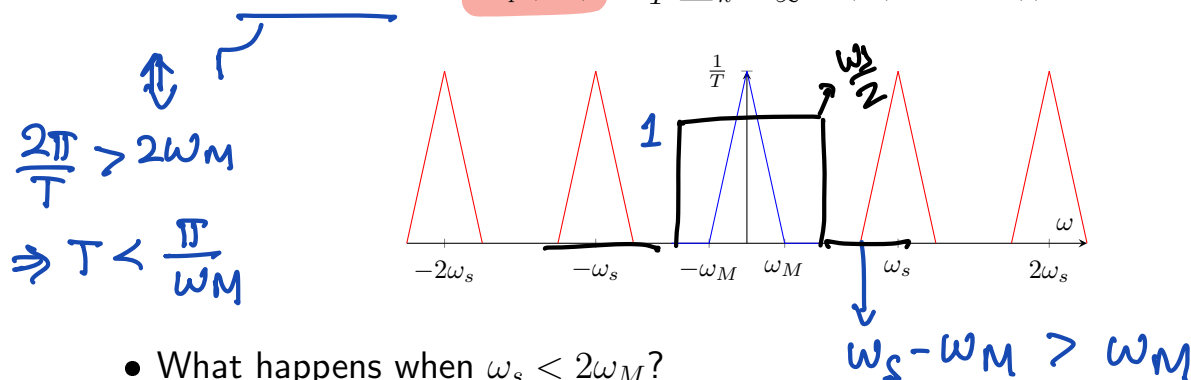
Bandlimited Signals

- Define the **bandwidth** of a signal $x(t)$ to be the smallest $\omega_M > 0$ such that:

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M.$$



- If $\omega_M < \infty$ we call this signal a *bandlimited signal*.
- So if $\omega_s > 2\omega_M$, for $X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ we get:



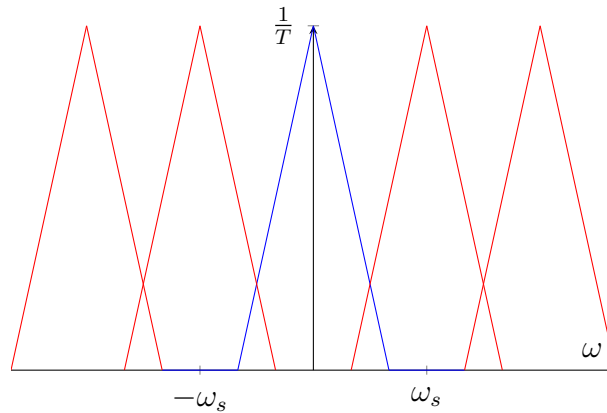
- What happens when $\omega_s < 2\omega_M$?

In order to ^{exact} reconstruct the original signal $x(t)$, the following must be true.

- $x(t)$ is bandlimited and its bandwidth ω_M is known.
- T must be chosen s.t. $\omega_s = \frac{2\pi}{T} > 2\omega_M$
- we have access to ideal low-pass filter with cut-off frequency $(\omega_M, \omega_s - \omega_M)$
usually $\frac{\omega_s}{2}$ is chosen.

Aliasing

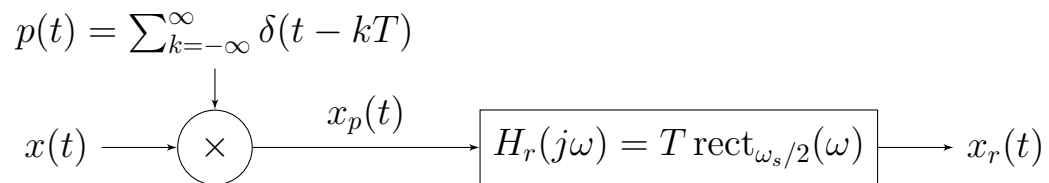
- Overlapping copies of $X(j(\omega - k\omega_s))$ is called *aliasing*.



- In this case, what would be $X_p(j\omega) = \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$?
- Therefore, in order not to have aliasing we need to have: $\omega_s > 2\omega_M$.

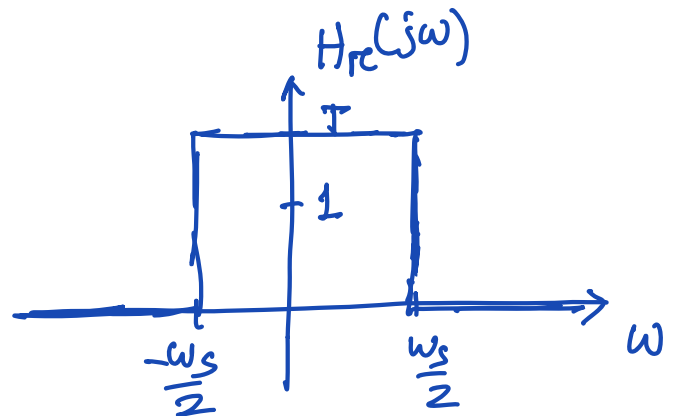
Optimal Recovery

- If $x(t)$ is a band-limited signal and $\omega_s > 2\omega_M$, then $x(t)$ can be recovered by (ideal) low-pass filtering of sampled signal $x_p(t)$.
- For perfect recovery, we need to have $H_r(j\omega) = T \text{rect}_{\omega_s/2}(\omega)$.



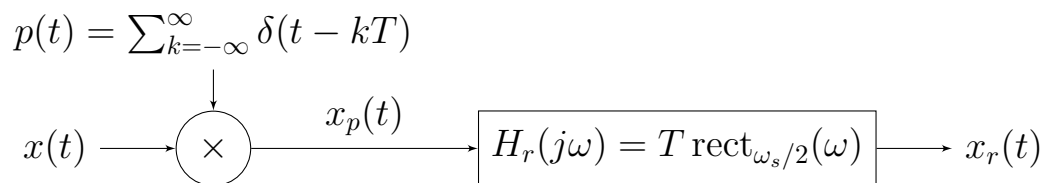
- $2\omega_M$ is called the *Nyquist-rate* (of signal $x(t)$).

$$X_p(j\omega) H_r(j\omega) = X(j\omega)$$



Sampling Theorem (Shannon-Nyquist)

Suppose that $x(t)$ is a band-limited signal with bandwidth ω_M . Then $x(t)$ can be recovered from the samples $x(kT)$ for $k = 0, \pm 1, \pm 2, \dots$ if $\frac{2\pi}{T} = \omega_s > 2\omega_M$. This can be achieved by impulse train sampling and recovery system with $H_r(j\omega) = T \text{rect}_{\omega_s/2}(\omega)$.



$$\sin(2\pi t)$$

$$1 \text{ Hz} = 2\pi \text{ rad/sec.}$$

Questions

1. Let $x(t)$ be a signal band-limited to the interval $[-20\text{kHz}, 20\text{kHz}]$. What condition should the sampling period T satisfy to avoid aliasing and allow reconstruction of the signal?

- a. $T < \frac{1}{20} \times 10^{-3} \text{ sec}$
 b. $T < \frac{1}{40} \times 10^{-3} \text{ sec}$
 c. $T < \frac{\pi}{20} \times 10^{-3} \text{ sec}$
 d. $T < \frac{\pi}{40} \times 10^{-3} \text{ sec}$

$$\omega_M = 2\pi \times 20000$$

$$\frac{2\pi}{T} = \omega_s > 4\pi \times 20000$$

$$\Rightarrow T < \frac{1}{40000}$$

$$\Rightarrow x(j\omega) = 0 \text{ for } |\omega| > \frac{\omega^*}{2}.$$

2. Let $x(t)$ be a signal with Nyquist rate given by ω^* . Determine Nyquist rate of the following signals.

- a. $x(t) + x(t-1) \leftrightarrow X(j\omega) + e^{-j\omega} X(j\omega) \rightarrow \text{same BW \& Nyquist rate}$
 b. $\frac{d}{dt}x(t)$
 c. $x(t) \cos(\omega_0 t)$

$$\begin{aligned} & \downarrow \\ & \frac{1}{2\pi} X(j\omega) \otimes [2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)] \text{ or } \omega = 0 \\ & = X(j(\omega-\omega_0)) + X(j(\omega+\omega_0)) \end{aligned}$$

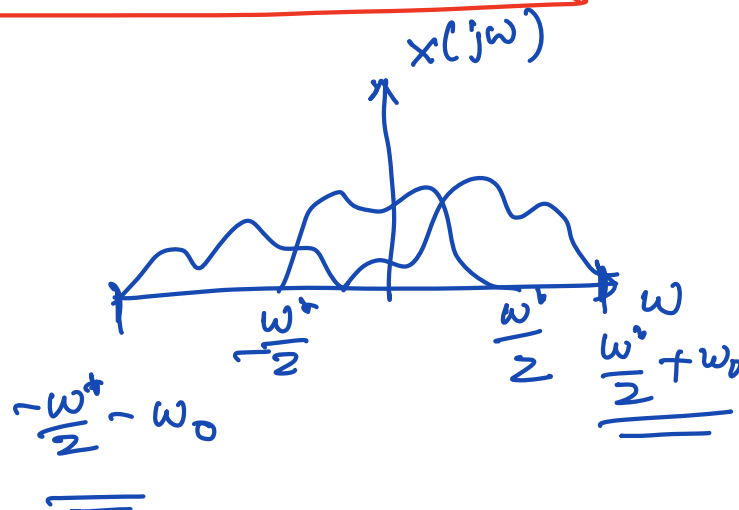
whenever $X(j\omega) \neq 0$ " " " "

3. Suppose signal $x_1(t)$ has bandwidth ω_1 and $x_2(t)$ has bandwidth ω_2 . Determine the Nyquist rate of $x_1(t) \times x_2(t)$.

$$\text{new BW: } \frac{\omega^*}{2} + \omega_0$$

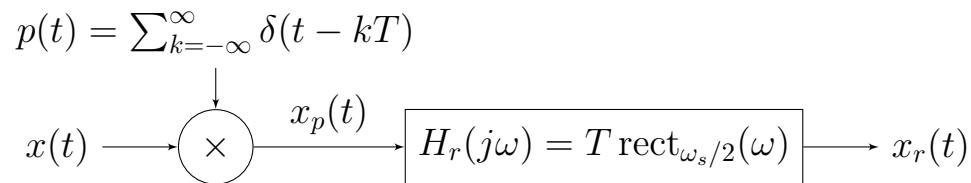
$$\text{new Nyquist rate:}$$

$$\omega^* + 2\omega_0$$



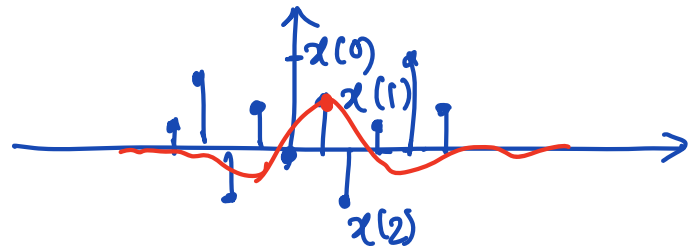
Reconstruction after Impulse Train Sampling

- A naive approach would be to do linear interpolation.



- In TD: $h_r(t) = \frac{T \sin(\frac{\pi}{T}t)}{\pi t}$

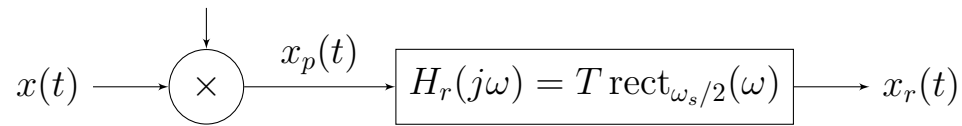
- Therefore,



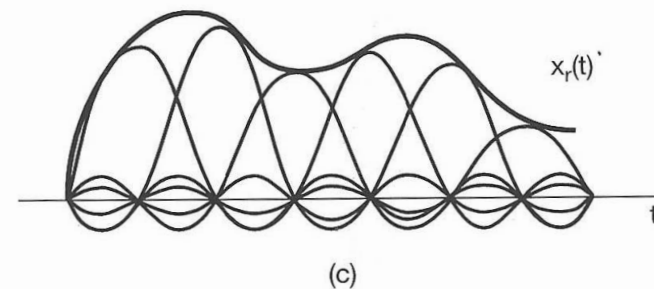
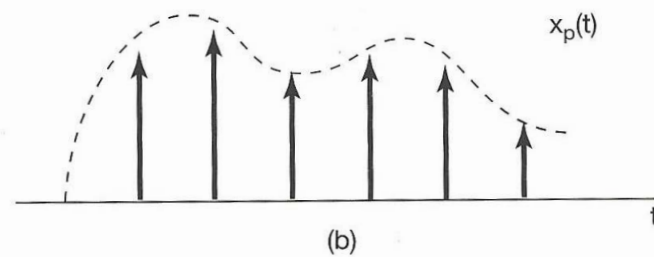
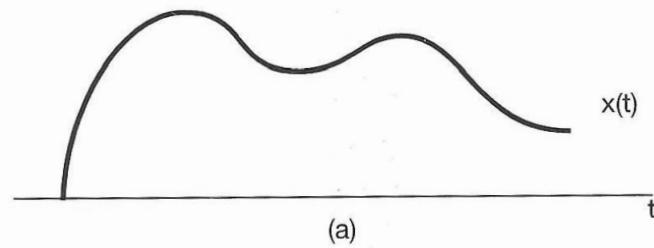
$$\begin{aligned}
 x_r(t) &= x_p(t) * h(t) \\
 &= \left[\sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \right] * h(t) \\
 &= \sum_{n=-\infty}^{\infty} x(nT) [\delta(t - nT) * h(t)] \\
 &= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \underbrace{\frac{T \sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)}}_{n=1}
 \end{aligned}$$

Time Domain View of Reconstruction Process

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



- $x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin(\frac{\pi}{T}(t-nT))}{\pi(t-nT)}$
- This is a linear combination of time-shifted sinc functions!



Exercise

$$x(t) =$$

- What is the bandwidth ω_M of ~~$\cos(\omega_0 t)$~~ $\cos(\omega_0 t)$

- Determine $X_p(j\omega)$, $X_r(j\omega)$, and $x_r(t)$ for $\omega_s = 2\omega_M$; $\omega_s > 2\omega_M$, $\omega_s < 2\omega_M$

$$X(j\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

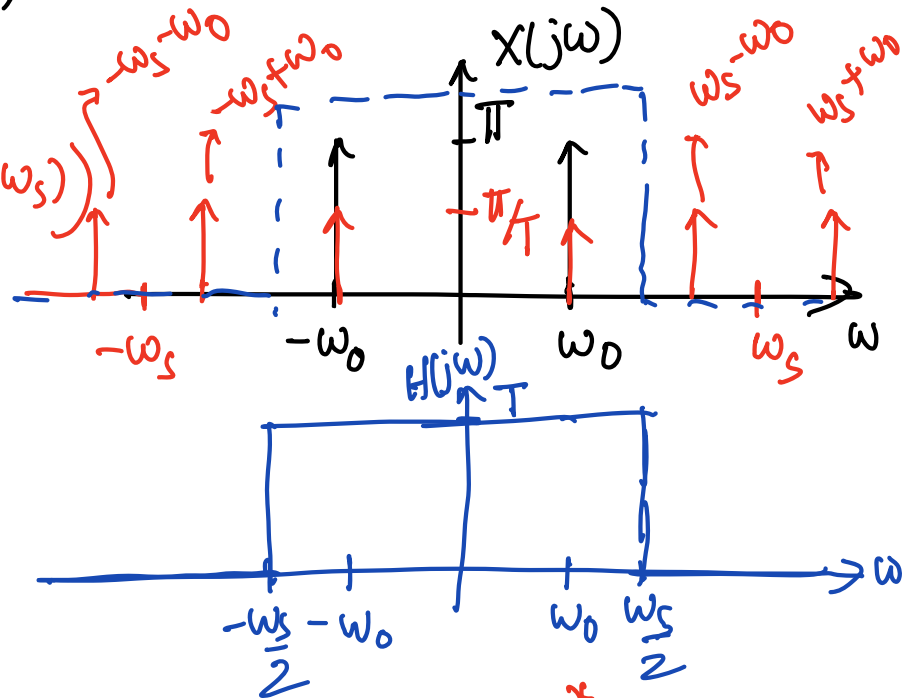
$\Rightarrow x(t)$ is band-limited, and bandwidth is ω_0

Case-1: $\omega_s > 2\omega_M = 2\omega_0$

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

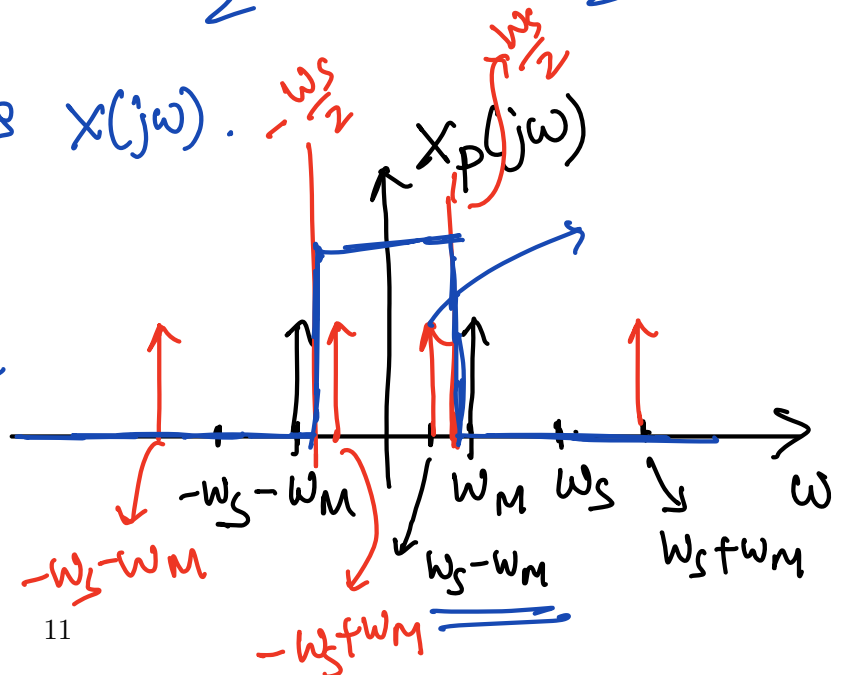
If $x_p(t)$ is fed to an ideal low pass filter with cutoff freq. $\frac{\omega_s}{2}$ and amplitude T ,

the output will have Fourier transform same as $X(j\omega)$.



Case-2: $\omega_s < 2\omega_M$

The output of the ideal low pass filter with cut-off frequency $\frac{\omega_s}{2}$ has FT given by

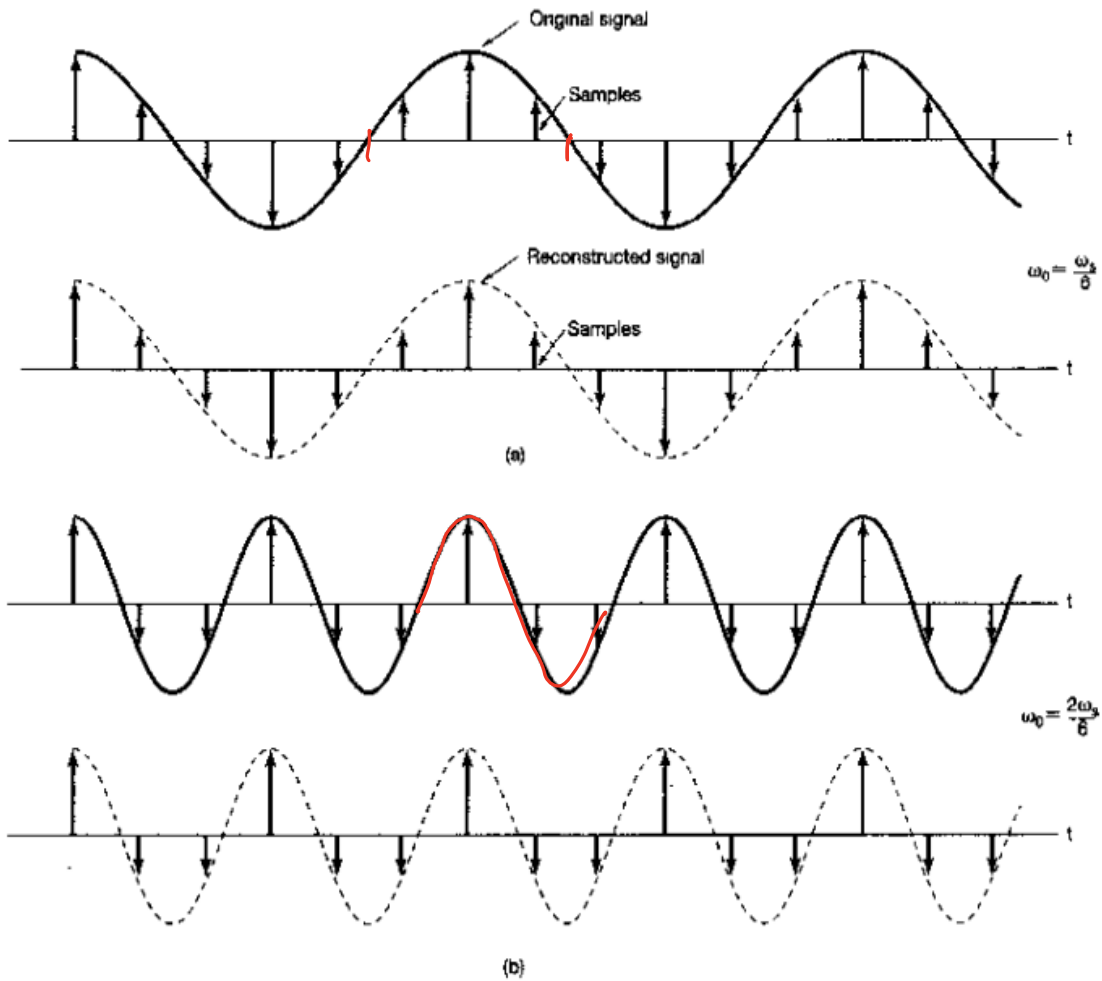


$$X_R(j\omega) = \pi \delta(\omega - (-\omega_s + \omega_M)) + \pi \delta(\omega - (\omega_s - \omega_M))$$

Under-Sampling and Aliasing Continued

$$x_r(t) = \cos((\omega_s - \omega_M)t), \quad \omega_M = \omega_0.$$

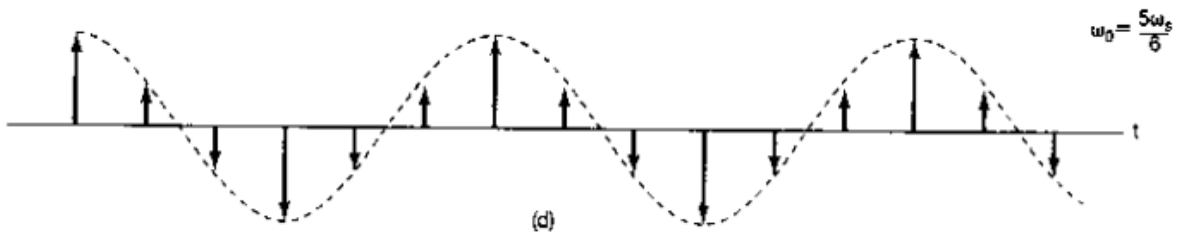
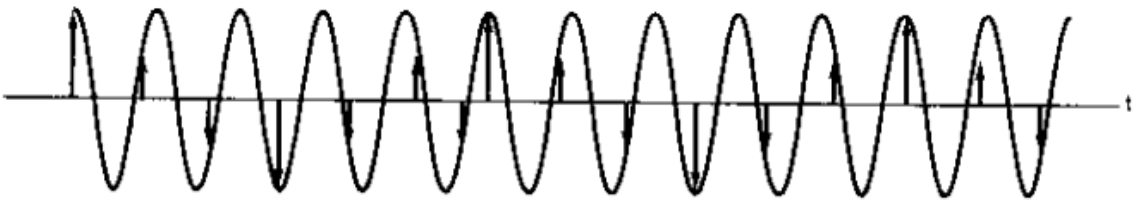
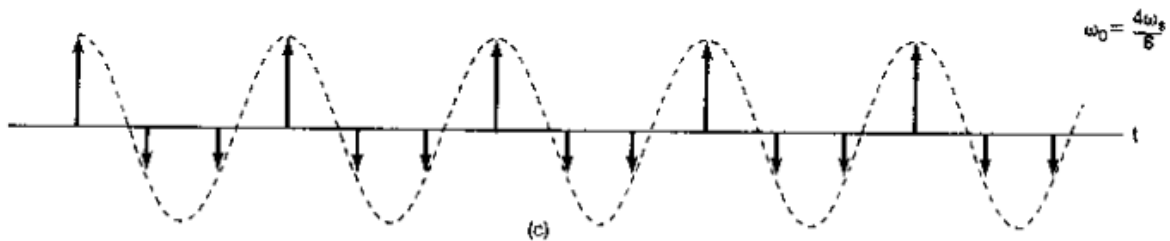
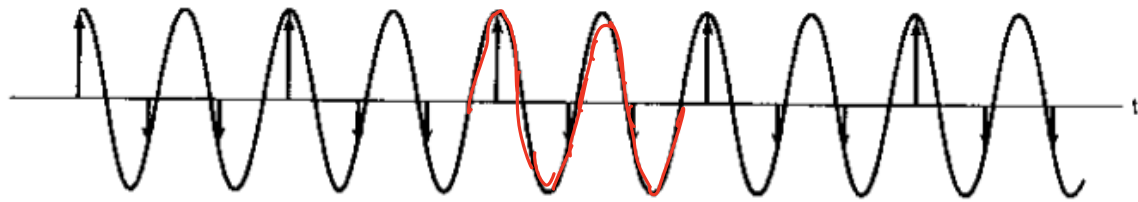
- When $\omega_s = 6\omega_0, 3\omega_0$, the reconstructed signal $x_r(t) = x(t)$.



Under-Sampling and Aliasing Continued

$$\cos(-\theta) = \cos\theta$$

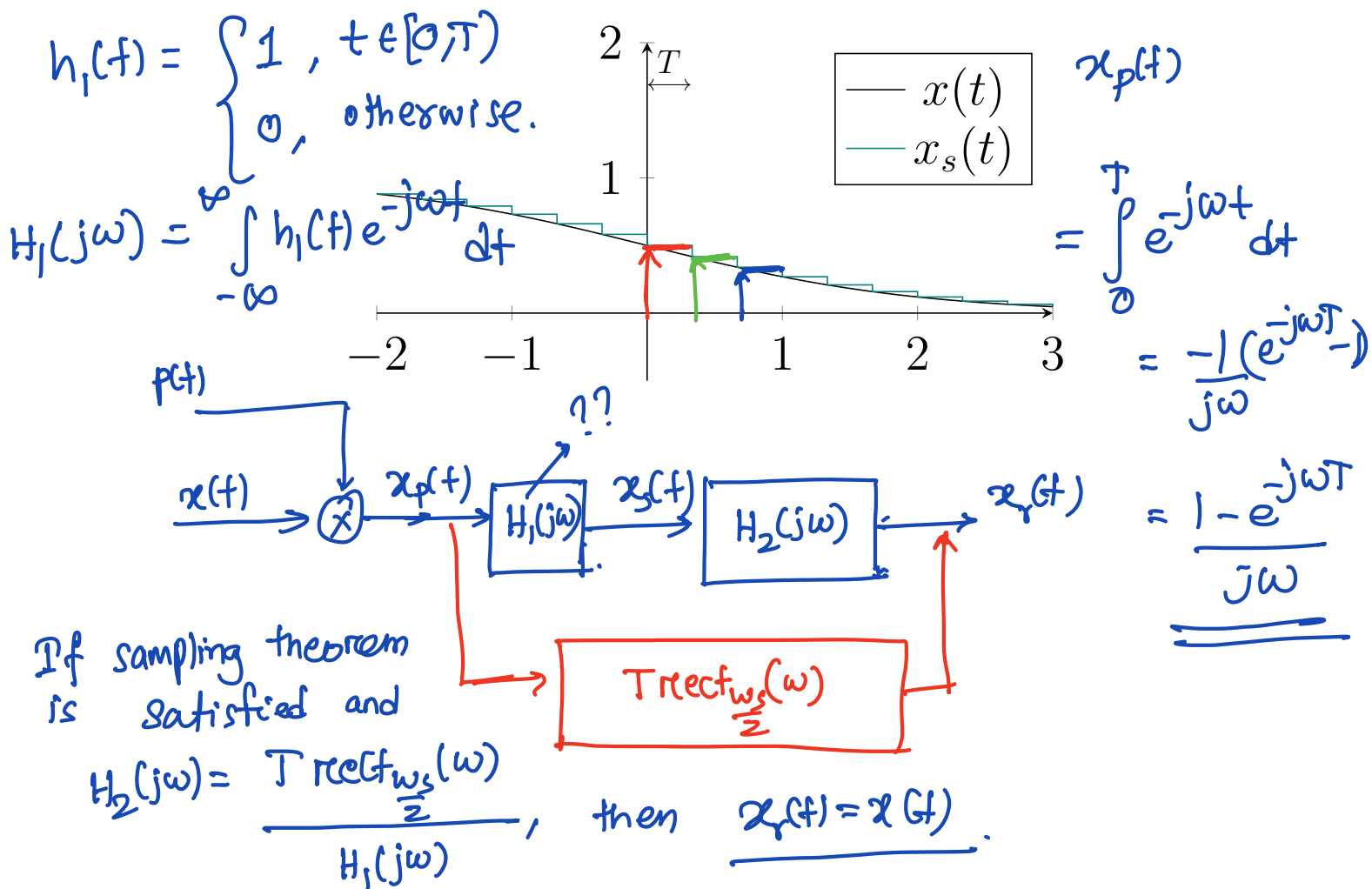
- When $\omega_s = 1.5\omega_0, 1.2\omega_0$, the reconstructed signal $x_r(t) = \cos((\omega_s - \omega_0)t) \neq x(t)$.
- As ω_0 increases relative to ω_s , the frequency of output decreases. Look up Wagon-Wheel Effect.
- When $\omega_s = \omega_0$, the output is constant.
- Phase reversal: If $x(t) = \cos(\omega_0 t + \phi)$, then $x_r(t) = \cos((\omega_s - \omega_0)t - \phi)$.



Zero-Order-Hold (ZOH) Sampling

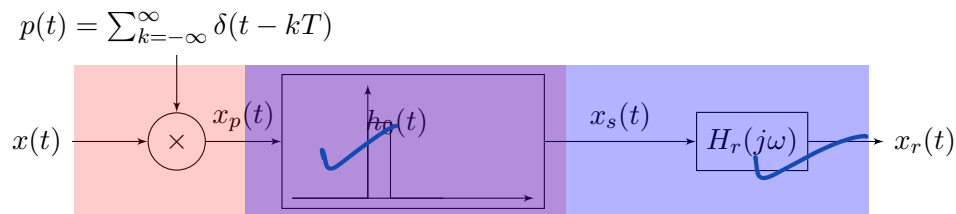
- Since creating an ideal impulse train and implementing an ideal low-pass filter is not possible, in practice, we often do *Zero-Order-Hold* (ZOH) sampling.
- In ZOH sampling, the sampled signal $x_s(t)$ holds the value of $x(t)$ at k th sampling time for T seconds, i.e.,

$$x_s(t) = x(kT) \quad \text{for } t \in [kT, (k+1)T)$$



ZOH Sampling and Perfect Recovery

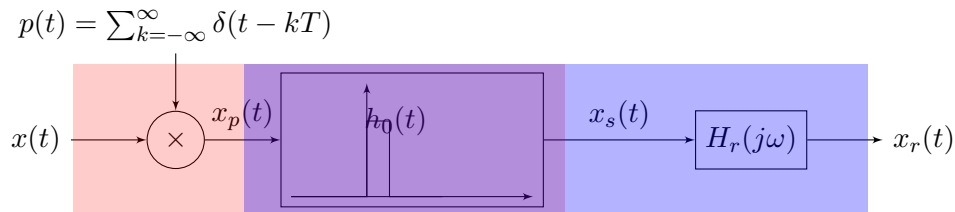
- ZOH sampling can be viewed as series interconnection of impulse train sampling and an LTI system with the impulse response $h_0(t) = \text{rect}_{T/2}(t - \frac{T}{2})$.



- The main question is that if $\omega_s > 2\omega_M$, then can we *find* a recovery system H_r so that $x_r(t) = x(t)$?
- For perfect recovery, the system in blue box should be the ideal low-pass filter $H_r(j\omega) = T \text{rect}_{\omega_s/2}(\omega)$.

ZOH Sampling and Perfect Recovery

- ZOH sampling can be viewed as series interconnection of impulse train sampling and an LTI system with the impulse response $h_0(t) = \text{rect}_{T/2}(t - \frac{T}{2})$.



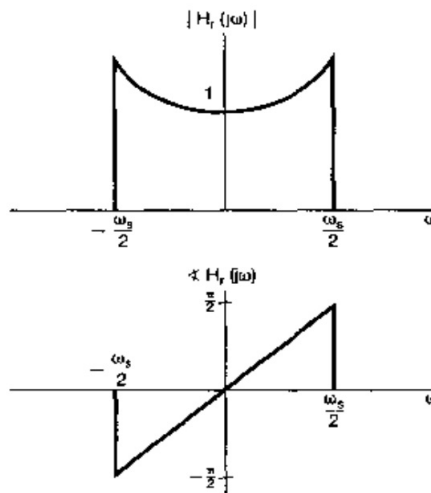
- For perfect recovery, the system in blue box should be the ideal low-pass filter $H_r(j\omega) = T \text{rect}_{\omega_s/2}(\omega)$.

- Therefore, for perfect recovery, we need to have:

$$H_r(j\omega) = e^{j\omega T/2} \frac{\omega T}{2 \sin(T\omega/2)} \text{rect}_{\omega_s/2}(\omega).$$

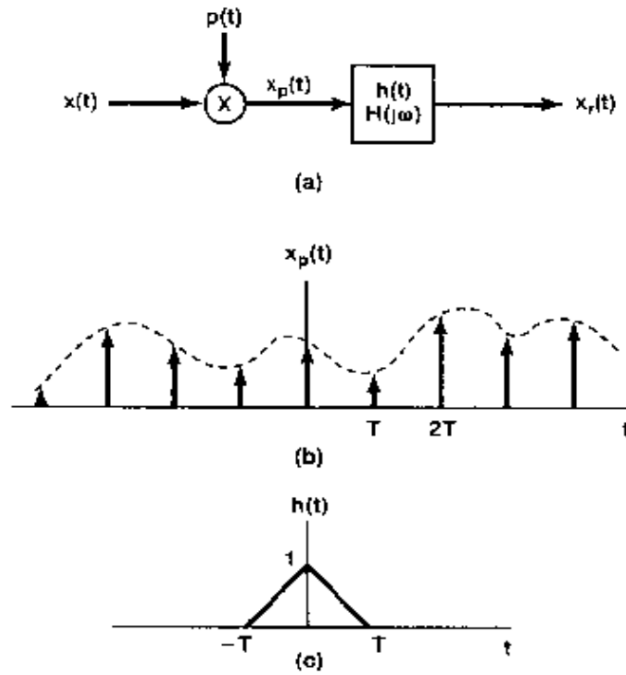
$$\frac{1}{H_1(j\omega)}.$$

- Note that the first zero of $\frac{\sin(T\omega/2)}{\omega}$ occurs at $\omega_1 = \frac{2\pi}{T} = \omega_s$. Since, $\text{rect}_{\omega_s/2}(\omega_s) = 0$, the zeros of $\frac{\sin(T\omega/2)}{\omega}$ are not going to be problematic in the above fraction.

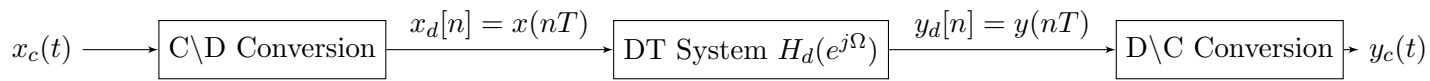


First Order Hold Sampling

- Corresponds to linear interpolation of the impulse train samples.

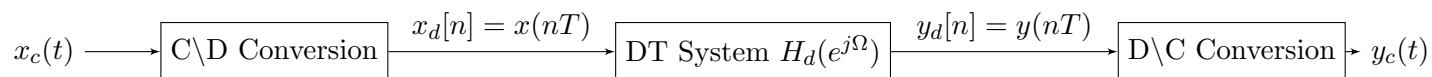


9th October: The following portion was taught on blackboard. Annotated DT Processing of CT Signals Slides not available.

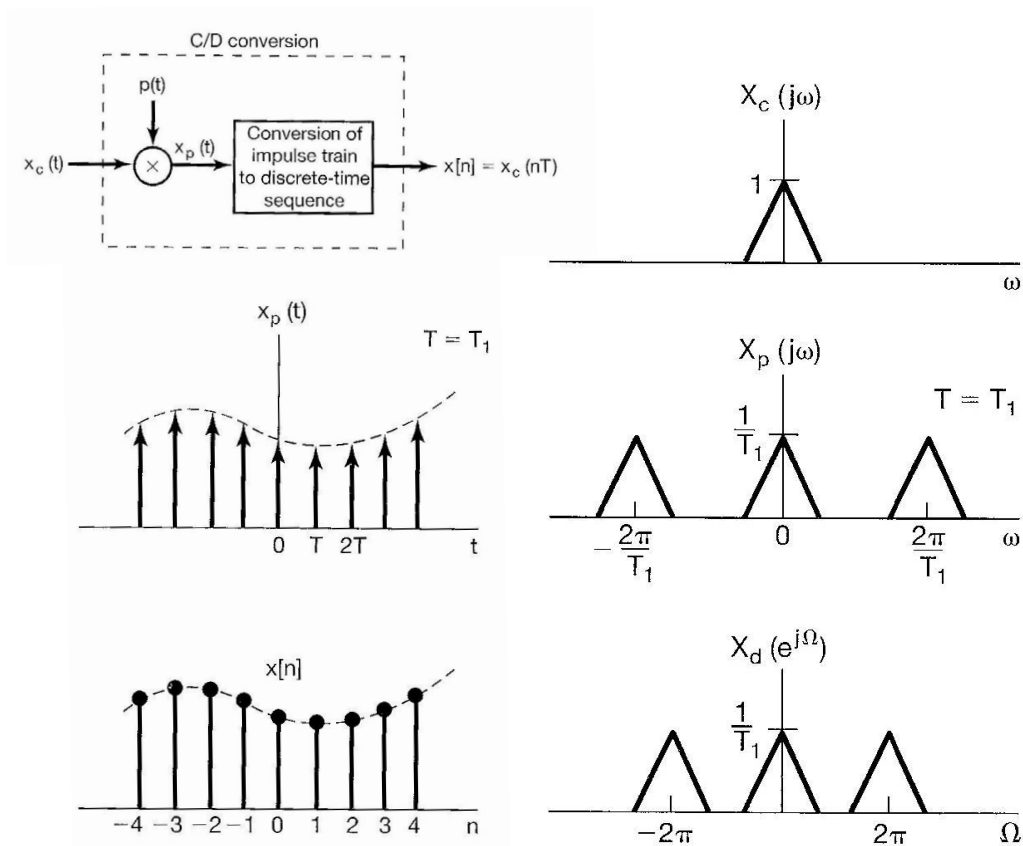


- How to process a CT signal using discrete-time processors?
 1. Convert $x(t)$ to $x_d[n]$ by C/D conversion (with sampling period T)
 2. (design and implement) Filter $x_d[n]$, $X_d(e^{j\Omega})$ in discrete-time with $h_d[n]$, $H_d(e^{j\Omega})$ to get desired $y_d[n]$, $Y_d(e^{j\Omega})$.
 3. Convert $y_d[n]$ to $y(t)$ by D/C conversion (with sampling period T).

DT Processing of CT Signals



- The discrete-time sequence $x_d[n] = x_c(nT)$.
- What is the relationship between $X_c(j\omega)$ and $X_d(e^{j\Omega})$?
- We use ω to denote frequency in continuous-time FT and Ω to denote frequency in discrete-time FT.



DT Processing of CT Signals

- CT Sampling - Perspective 1: $x_p(t) = x_c(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$ and

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$$

- CT Sampling - Perspective 2: On the other hand $\delta(t - nT) \longleftrightarrow e^{-j\omega nT}$ and hence:

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

- DT Sampling: $x_d[n] = x(nT)$ for n and hence

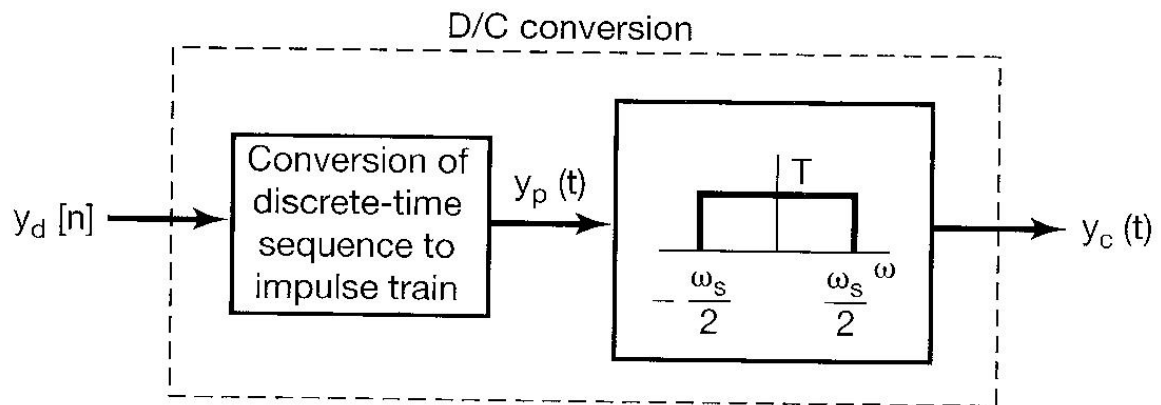
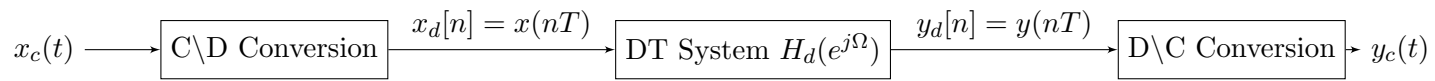
$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega n}$$

- Conclusion:

$$X_d(e^{j\Omega}) = X_p(j\Omega/T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega/T - k\omega_s))$$

- Converting $x_p(t)$ to $x_d[n]$ scales the time-axis by $1/T$. Consequently, in the frequency domain, $X_d(e^{j\Omega})$ is obtained by scaling $X_c(j\omega)$ by factor T , i.e., $\Omega = T\omega$.

DT Processing of CT Signals



DT Processing of CT Signals

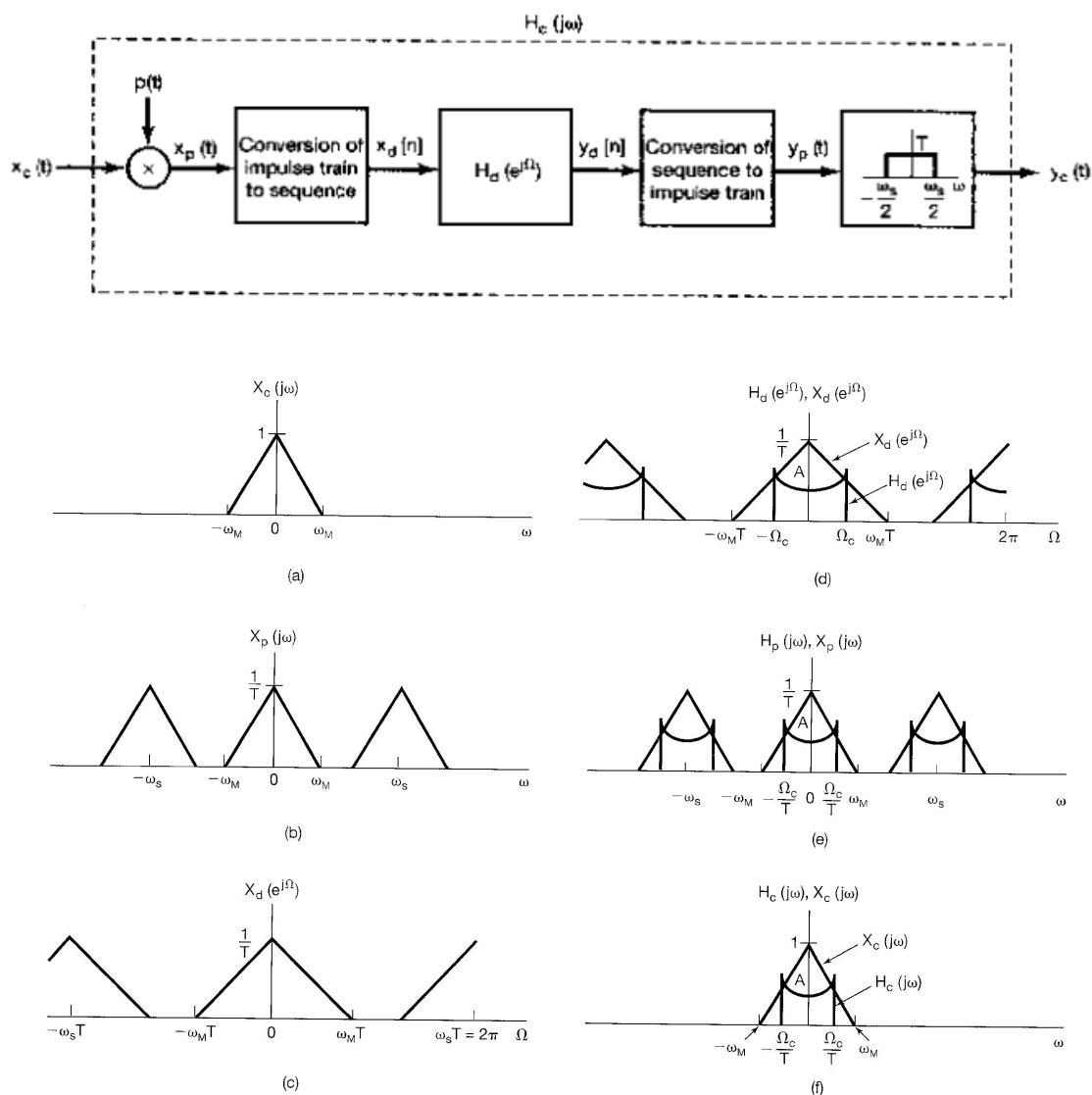


Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.