16th September

Properties of DTFT: Convolution Property

- As in the case of CT, synthesis equation provides *decomposition of a discrete-time signal into exponential components.*
- From convolution, we have

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

$$y[n] = e^{j\omega n} * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}_{\text{DTFT of } h[n]} = e^{j\omega n} H(e^{j\omega})$$

- Therefore:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \frac{1}{2\pi}\int_{2\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega$$

Inverse DTFT of $H(e^{j\omega})X(e^{j\omega})$

- Therefore: For any signals $x[n]$ and $h[n]$, we have:

$$y[n] = x[n] * h[n] \longleftrightarrow X(e^{j\omega})H(e^{j\omega}) = Y(e^{j\omega}).$$

both are periodic with period 2π.

- Multiplication:

$$y[n] = x[n] \times z[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\underbrace{\int_{2\pi} X(e^{j\theta})Z(e^{j\omega - j\theta})d\theta}_{X(e^{j\omega}) * Z(e^{j\omega})}.$$

periodic convolution

integral over any contiguous interval of width 2π.

use: $x[n] \longleftrightarrow X(e^{j\omega})$, then $nx[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$

$x[n-1] \longleftrightarrow e^{-j\omega}X(e^{j\omega})$

Exercise

Consider an LTI System with inpulse response $h[n] = \alpha^n u[n]$ with $|\alpha| < 1$. Let the input to this system be $x[n] = \beta^n u[n]$ with $|\beta| < 1$. Determine $y[n]$ when $\alpha = \beta$ and $\alpha \neq \beta$.

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = ?$$

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

case-1: $\alpha \neq \beta$

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})} = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$y[n] = A\alpha^n u[n] + B\beta^n u[n].$$

$$= \frac{A+B-(\beta A+\alpha B)e^{-j\omega}}{(\quad)(\quad)}$$

comparing the coefficients

$$A+B=1$$

$$\beta A+\alpha B=0$$

$$\Rightarrow A = \frac{-\alpha}{\beta}B,$$

$$B\left(1-\frac{\alpha}{\beta}\right) = 1$$

$$\Rightarrow B = \frac{\beta}{\beta-\alpha},$$

$$A = \frac{-\alpha}{\beta-\alpha}$$

Case-2: $\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

$$j\frac{d}{d\omega}H(e^{j\omega}) = j\frac{d}{d\omega}\frac{1}{(1-\alpha e^{-j\omega})}$$

$$= j\,\frac{-1\,\frac{d}{d\omega}(-\alpha e^{-j\omega})}{(1-\alpha e^{-j\omega})^2}$$

$$= (-j)\,\frac{(-\alpha)(-j)\,e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$= \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} \longleftrightarrow n\alpha^n u[n]$$

$$\frac{1}{(1-\alpha e^{-j\omega})^2} \longleftrightarrow (n+1)\alpha^{n+1}u[n+1]\cdot\frac{1}{\alpha}$$

$$= (n+1)u[n+1]\alpha^n$$

Frequency Response of Systems Characterized by Constant-Coefficient Difference Equation

- Similar to CT, in DT, many practical systems are defined by **constant coefficient difference equations**

$$
\begin{aligned}
& a_N y[n-N] + \cdots + a_1 y[n-1] + a_0 y[n] \\
& \qquad = b_M x[n-M] + \cdots + b_1 x[n-1] + b_0 x[n].
\end{aligned}
$$

- Taking FT of both sides we get:

$$
\begin{aligned}
& (a_N e^{-j\omega N} + \cdots + a_1 e^{-j\omega} + a_0) Y(e^{j\omega}) \\
& \qquad = (b_M e^{-j\omega M} + \cdots + b_1 e^{-j\omega} + b_0) X(e^{j\omega})
\end{aligned}
$$

- Therefore,

$$
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_M e^{-j\omega M} + \cdots + b_1 e^{-j\omega} + b_0}{a_N e^{-j\omega N} + \cdots + a_1 e^{-j\omega} + a_0}.
$$

Problems

- Problem 1: What is the frequency response of a system whose input-output behavior is given by:

$$y[n] - y[n-1] = 3x[n] + 2x[n-1] + x[n-2].$$

$$Y(e^{j\omega})(1 - e^{-j\omega}) = (3 + 2e^{-j\omega} + e^{-j2\omega})X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{3 + 2e^{-j\omega} + e^{-j2\omega}}{1 - e^{-j\omega}}$$

- Problem 2: Suppose $x[n] \longleftrightarrow X(e^{j\omega})$. Determine DTFT of the following signals in terms of $X(e^{j\omega})$.
  - $x_1[n] = x[-1-n] + x[1-n]$.
  - $x_2[n] = (n-1)^2 x[n]$. $= n^2 x[n] - 2n\,x[n] + x[n]$

$$\rightarrow X(e^{j\omega}) - 2j\frac{d}{d\omega}X(e^{j\omega}) - \frac{d^2}{d\omega^2}X(e^{j\omega})$$

$y[n] = x[-n]$, $\quad x_1[n] = y[n-1] + y[n+1]$

$Y(e^{j\omega}) = X(e^{-j\omega})$ $\quad X_1(e^{j\omega}) = Y(e^{j\omega})(e^{j\omega} + e^{-j\omega})$

$$= 2\cos\omega\, X(e^{-j\omega})$$

- Problem 3: When $x[n] = (\frac{4}{5})^n u[n]$ is applied to a DT LTI system, the output $y[n] = n(\frac{4}{5})^n u[n]$. Determine the difference equation relating $x[n]$ and $y[n]$.

$X(e^{j\omega}) = \dfrac{1}{1 - \frac{4}{5}e^{-j\omega}}$, $\quad Y(e^{j\omega}) = j\dfrac{d}{d\omega}X(e^{j\omega})$

$$= j\frac{-1}{(1 - \frac{4}{5}e^{-j\omega})^2}\cdot\left(-\frac{4}{5}\right)(-j)$$

$$= \frac{4}{5}\,\frac{e^{-j\omega}}{(1 - \frac{4}{5}e^{-j\omega})^2}$$

$$H(e^{j\omega}) = \frac{4}{5}\cdot\frac{e^{-j\omega}}{(1 - \frac{4}{5}e^{-j\omega})} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\Rightarrow 4e^{-j\omega}X(e^{j\omega}) = 5Y(e^{j\omega}) - 4e^{-j\omega}Y(e^{j\omega})$$

$$4x[n-1] = 5y[n] - 4y[n-1]$$

Problem 4: DT LTI system $\quad x[n] \rightarrow \boxed{} \rightarrow y[n]$

$$y[n] - 0.5\,y[n-1] = x[n]$$

when $x[n] = \delta[n] - 2\delta[1-n]$, determine DTFT of output.

$$X(e^{j\omega}) = 1 - 2e^{-j\omega}, \qquad Y(e^{j\omega})\left(1 - 0.5e^{-j\omega}\right) = X(e^{j\omega})$$

$$\Rightarrow Y(e^{j\omega}) = \frac{1-2e^{-j\omega}}{1-0.5e^{-j\omega}}.$$

Practice problems from textbook

Chapter 4: 24, 26, 28(a), 30, 31, 36, 12, 17, 2, 6, 13, 14

Chapter 5: 5, 6, 9, 13, 14, 15, 22(c), 28, 33, 41, 43, 49, 50

Chapter 6: 9, 10, 11, 12

Midsem syllabus:
- Chapters 1-5
- Chapter 6.5, 6.6

17th September

First Order Discrete-Time System

- A discrete-time first order LTI system is given by:

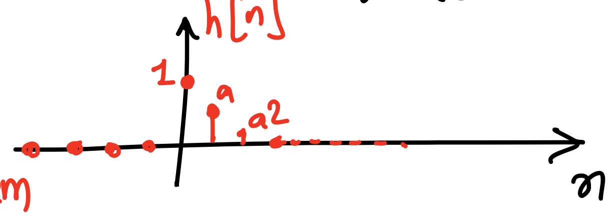
$$y[n] - ay[n-1] = x[n] \quad \text{with} \quad |a| < 1. \qquad (1)$$

- Determine its frequency response $H(e^{j\omega})$ and impulse response $h[n]$.
- Determine its step response $s[n]$.
- The parameter a plays the role of time constant. If $|a|$ is close to 1, the response is slow, and if $|a|$ is close to 0, the response is fast.
- If $a < 0$, we see oscillations and overshoot.

applying DTFT on both sides of (1), we obtain

$$Y(e^{j\omega})\left(1 - ae^{-j\omega}\right) = X(e^{j\omega}) \Rightarrow H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$\Rightarrow h[n] = a^n u[n]$$

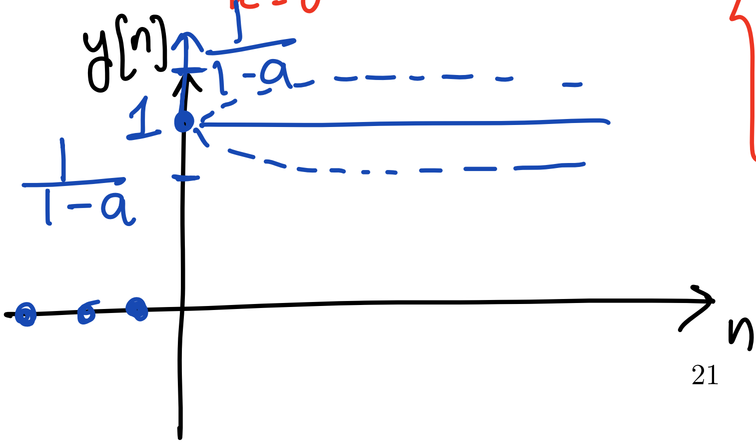


Step-response: output of the system when input $x[n] = u[n]$

$$y[n] = u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k]\, u[n-k]$$

$$= \sum_{k=0}^{\infty} a^k \underbrace{u[n-k]} = \begin{cases} 0, & \text{if } n < 0 \\ \sum_{k=0}^{n} a^k, & \text{if } n \geq 0 \end{cases}$$

$$= \frac{1 - a^{n+1}}{1 - a}, \text{ if } n \geq 0$$



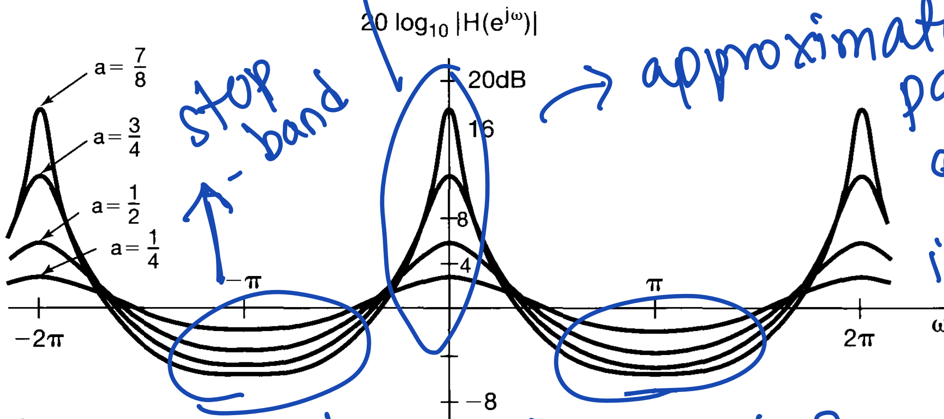
$$\lim_{n \to \infty} y[n] = \frac{1}{1 - a}$$

Log-Magnitude and Phase of Frequency Response

When $a > 0$:

passband

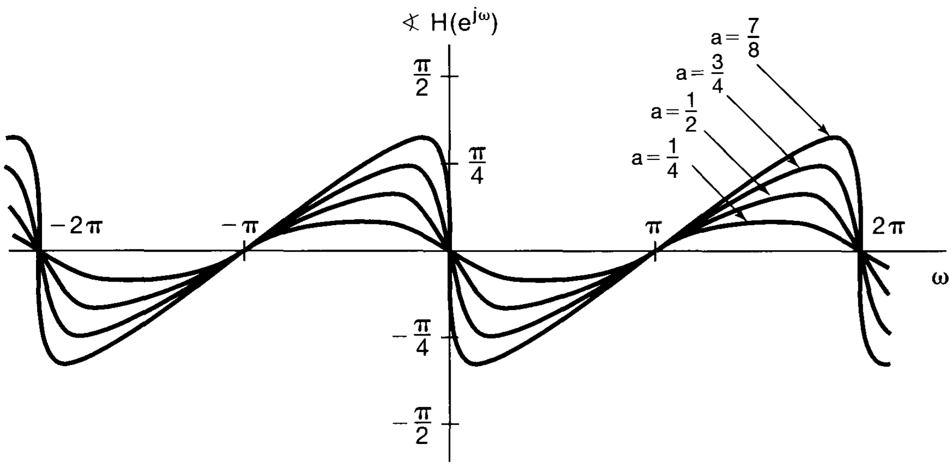
$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



stop band

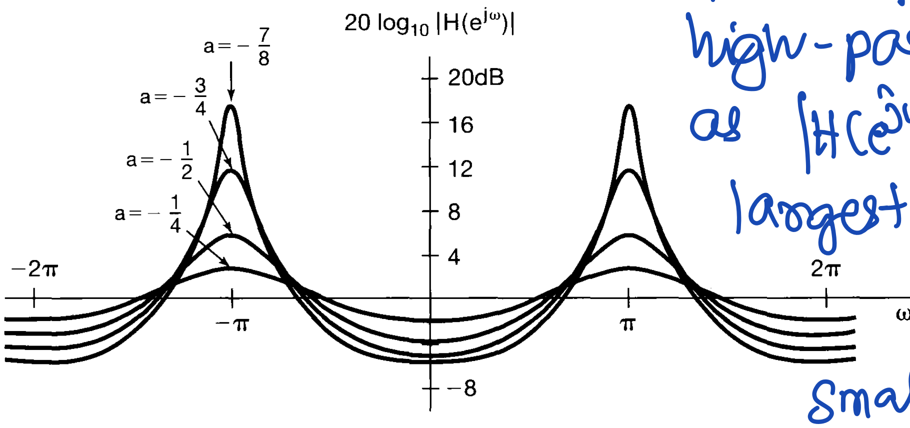
approximates a low pass filter as $|H(e^{j\omega})|$ is highest at $\omega = 0$,

Since $H(e^{j\omega})$ is periodic with period 2π, frequency near π or $-\pi$ are "high frequency" region.

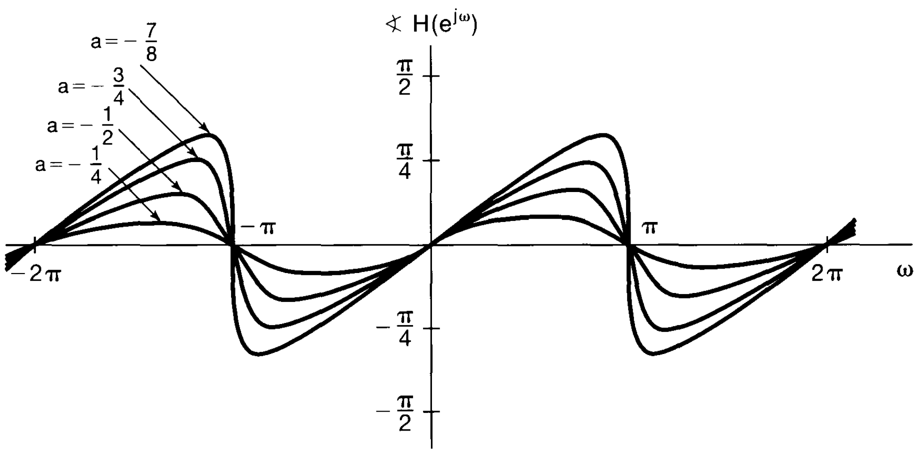


Log-Magnitude and Phase of Frequency Response

When $a < 0$:



approximates the behavior of a high-pass filter as $|H(e^{j\omega})|$ is largest for $\underline{\omega \cong \pi, -\pi}$ and smallest for $\underline{\omega \cong 0}$.



Second Order Discrete-Time System

- A discrete-time ~~first~~ second order LTI system is given by:

$$y[n] - 2r\cos(\theta)y[n-1] + r^2 y[n-2] = x[n],$$

where $r \in (0,1)$ and $0 \le \theta \le 2\pi$.

- Determine its frequence response $\underline{H(e^{j\omega})}$ and impulse response $\underline{h[n]}$.

applying DTFT on both sides, we obtain

$$Y(e^{j\omega})\left[1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}\right] = X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{(1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega})}$$

$$= \frac{1}{(1 - r(e^{j\theta} + e^{-j\theta})e^{-j\omega} + r^2 e^{-j2\omega})}$$

$$= \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}$$

$$= \frac{A}{1 - re^{j\theta}e^{-j\omega}} + \frac{B}{1 - re^{-j\theta}e^{-j\omega}}$$

$$h[n] = A(re^{j\theta})^n u[n] + B(re^{-j\theta})^n u[n]$$

<u>Numerator</u>: $A - rAe^{-j\theta}e^{-j\omega} + B - Bre^{j\theta}e^{-j\omega} = 1$

$$\Rightarrow A + B - re^{-j\omega}(Ae^{-j\theta} + Be^{j\theta}) = 1$$

– needs some calculations to determine A and B.

Q. DT system: $y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$

(i) Determine its frequency response

(ii) " the frequency ω where gain is 0 dB, i.e., $\underline{20\log_{10}|H(e^{j\omega})| = 0}$.

Solution:

$$H(e^{j\omega}) = \frac{1}{3}\left(1 + 2\cos\omega\right)$$

$$20\log_{10}|H(e^{j\omega})| = 0$$

$$\Rightarrow |H(e^{j\omega})| = 1$$

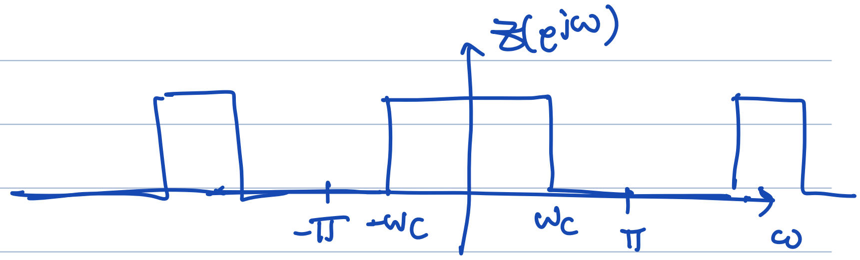
$$\Rightarrow |1 + 2\cos\omega| = 3$$

or $\omega = 0$ or $2n\pi$ for $n \in \mathbb{Z}$.

Q. Find inverse DTFT of $Y(e^{j\omega})$ is $\left(\frac{\sin\omega_c n}{\pi n}\right)^2$, where $0 < \omega_c < \pi$. Find $Y(e^{j\omega})$.

Find DTFT of $\frac{\sin\omega_c n}{\pi n} = x[n]$

Suppose $Z(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise.} \end{cases}$



$$z[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\cdot\frac{1}{jn}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$= \frac{1}{2j}\cdot\frac{1}{\pi n}\cdot 2j \sin \omega_c n$$

$$= \left(\frac{\sin \omega_c n}{\pi n}\right)$$

So $$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(e^{j\bar{\omega}})\, Z(e^{j(\omega-\bar{\omega})})\, d\bar{\omega}$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \underline{Z(e^{j(\omega-\bar{\omega})})}\, d\bar{\omega}$$

further calculations required.

$$-\omega_c \le \omega - \bar{\omega} \le \omega_c$$

$$\underline{\bar{\omega} - \omega_c \le \omega \le \bar{\omega} + \omega_c}$$

5.28 $\quad x[n] \leftrightarrow X(e^{j\omega}), \quad g[n] \leftrightarrow G(e^{j\omega})$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\, G(e^{j(\omega-\theta)})\, d\theta = 1 + e^{-j\omega}$$

$$x[n]\, g[n] \longleftrightarrow \delta[n] + \delta[n-1]$$

(i) if $x[n] = (-1)^n$, determine $g[n]$.

Is $g[n]$ unique? $g[n] = \begin{cases} 1 & \text{if } n=0 \\ -1 & \text{if } n=1 \\ 0 & \text{o.w.} \end{cases}$

(unique)

(ii) Repeat the above when $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

Solution:

$$g[n] = \begin{cases} 1, & n=0 \\ 2 & n=1 \\ 0 & , n \geq 2 \\ * & , n < 0 \end{cases}$$

not unique.