9th September: Lecture 19.

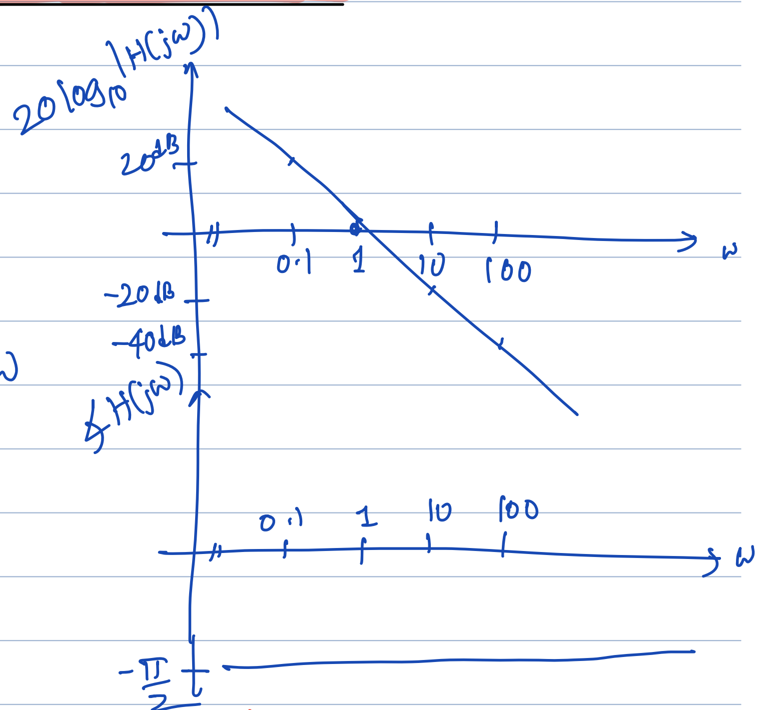
(3) $H(j\omega) = \frac{1}{j\omega}$

$\measuredangle H(j\omega) = -90^\circ$

$|H(j\omega)| = \frac{1}{\omega}$

$20\log_{10}|H(j\omega)| = -20\log_{10}\omega$

$\omega = 1 \Rightarrow 20\log_{10}\omega = 0$

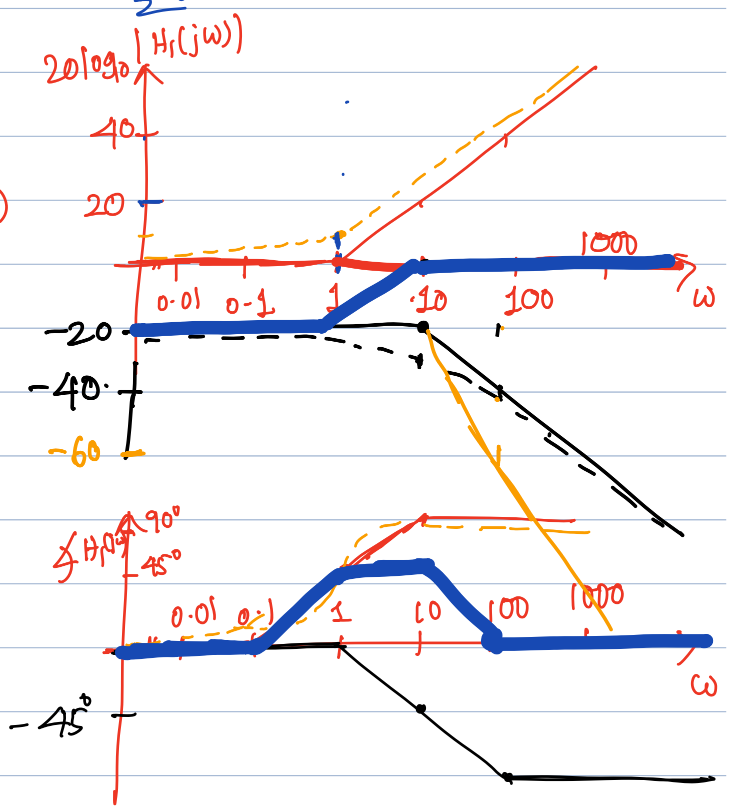


$\underline{20\log_{10}(\sqrt{2}) =}$

(4) $H(j\omega) = \frac{1+j\omega}{10+j\omega}$

Let us sketch $1+j\omega = H_1(j\omega)$

Let us sketch $\frac{1}{10+j\omega}$



(5) $H(j\omega) = \dfrac{1}{(j\omega + c)^k}$ at $\omega \gg c$,

$\angle H(j\omega) = (j)^k$

at $\omega \ll c$: $|H(j\omega)| = \dfrac{1}{c^k}$

$$20\log_{10}|H(j\omega)| = -20k \log_{10} c$$

at $\omega \gg c$: $|H(j\omega)| = \dfrac{1}{\omega^k}$

$$20\log_{10}|H(j\omega)| = -20k \log_{10} \omega$$

$$\measuredangle H(j\omega) = \begin{cases} 0^\circ & \text{at } \omega \ll c \\ -\dfrac{k\pi}{2} & \text{at } \omega \gg c \end{cases}$$

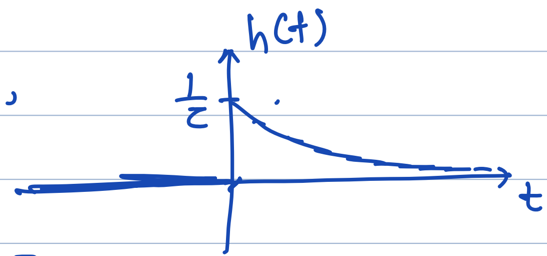
Let us now consider a canonical first order system.

Differential eq^n: $\tau \dfrac{d}{dt} y(t) + y(t) = x(t)$

applying Fourier transform yields: $\tau j\omega Y(j\omega) + Y(j\omega) = X(j\omega)$

$$\Rightarrow H(j\omega) = \frac{1}{1 + j\omega\tau}, \quad \tau : \text{time constant.}$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t).$$



If we apply unit step input $u(t)$, output

$$Y(j\omega) = H(j\omega) X(j\omega)$$

$$= \frac{1}{1 + j\omega\tau} \left[\frac{1}{j\omega} + \pi\delta(\omega) \right]$$

$$= \frac{1}{(j\omega)(1+j\omega\tau)} + \pi\delta(\omega).\frac{1}{1+j\omega\tau}$$

$$= \frac{A}{j\omega} + \frac{B}{1+j\omega\tau} + \pi\delta(\omega).\frac{1}{1+j\omega\tau}$$

$$= \frac{-\tau}{1+j\omega\tau} + \frac{1}{j\omega} + \pi\delta(\omega).\frac{1}{1+j\omega\tau}$$

$A = 1$

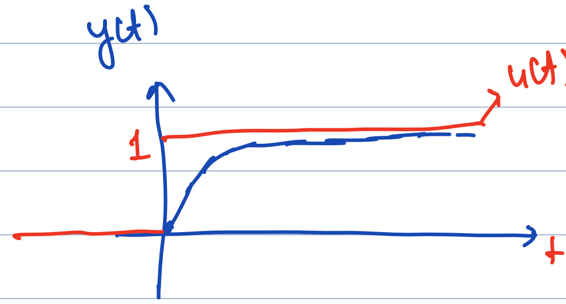
$j\omega\tau A + j\omega B = 0$

$\Rightarrow B = -\tau$

$$y(t) = \mathcal{F}^{-1}\left[\frac{-\tau}{1+j\omega\tau}\right] + \mathcal{F}^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega)\frac{1}{1+j\omega\tau}\right]$$

$$= -e^{-t/\tau}u(t) + u(t)$$

$$= \left[1 - e^{-t/\tau}\right]u(t).$$



Second order system

assume $\zeta, \omega_n > 0$

Differential equation

$$\frac{d^2}{dt^2}y(t) + 2\zeta\omega_n\frac{d}{dt}y(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$$

$$\Rightarrow \left[(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2\right]Y(j\omega) = \omega_n^2 X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Roots of the denominator: $-\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$

Let they be denoted by c_1 & c_2. $= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Case 1 : $\zeta > 1$: $\underbrace{-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}_{c_1}$, $\underbrace{-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}_{c_2}$

Case 2 : $\zeta = 1$: $-\zeta\omega_n$, $-\zeta\omega_n$

Case 3 : $\zeta < 1$: $-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

$-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$

$c_2 - c_1 = -2\omega_n\sqrt{\zeta^2-1}$

$$\begin{cases} c_1 c_2 = \omega_n^2 \\ c_1 + c_2 = 2\zeta\omega_n \end{cases}$$

& sketch.

Find h(t) for all three cases above.

$$H(j\omega) = \frac{\omega_n^2}{(j\omega+c_1)(j\omega+c_2)} = \frac{\omega_n^2}{(j\omega)^2 + (c_1+c_2)j\omega + c_1c_2}$$

when $c_1 \neq c_2$: $= \underbrace{\dfrac{A}{j\omega+c_1}} + \underbrace{\dfrac{B}{j\omega+c_2}}$,

$$= \frac{j\omega(A+B) + Ac_2 + Bc_1}{(j\omega+c_1)(j\omega+c_2)}$$

$\Rightarrow$ $A+B=0$

$Ac_2 + Bc_1 = \omega_n^2$

$\Rightarrow A(c_2-c_1) = \omega_n^2$

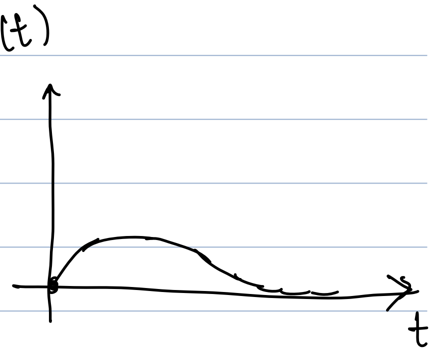
$\Rightarrow A = \dfrac{\omega_n^2}{c_2-c_1}$

$\Rightarrow B = -\dfrac{\omega_n^2}{c_2-c_1}$

$\mathcal{F}^{-1}(\cdot)$

$$h(t) = \left(A e^{+c_1 t} + B e^{+c_2 t}\right)u(t)$$

$$= \underbrace{A\left(e^{+c_1 t} - e^{+c_2 t}\right)u(t)}$$



Case 1 : c_1 and c_2 are both real. and negative

Case 3 : $c_1 = -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n$

$c_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$

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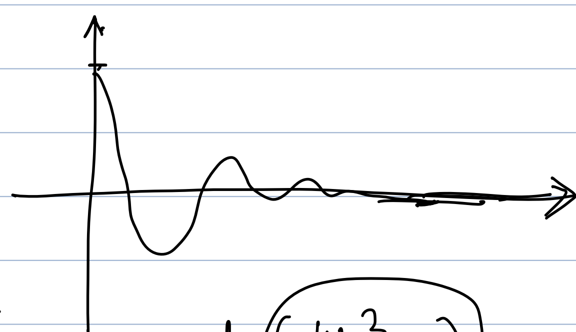
$$C_2 - C_1 = -2j\omega_n\sqrt{1-\zeta^2}$$

$$A = \frac{\omega_n^2}{-2j\omega_n\sqrt{1-\zeta^2}} = \frac{j\omega_n}{2\sqrt{1-\zeta^2}} = \frac{\omega_n}{2\sqrt{1-\zeta^2}} e^{j\pi/2}$$

$$e^{C_1 t} + e^{C_2 t} = e^{-\zeta\omega_n t + j\sqrt{1-\zeta^2}\omega_n} + e^{-\zeta\omega_n t - j\sqrt{1-\zeta^2}\omega_n}$$

$$= e^{-\zeta\omega_n t}\left[2\cos\left(\omega_n\sqrt{1-\zeta^2}\, t\right)\right]$$

$$h(t) = \frac{\omega_n}{2\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left(e^{j\pi/2}\; 2\cos\left(\omega_n\sqrt{1-\zeta^2}\, t\right)\right)$$



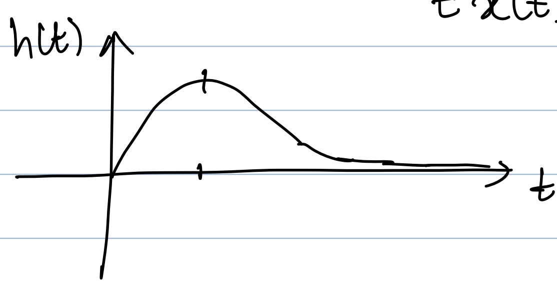
Case 2: $H(j\omega) = \dfrac{\omega_n^2}{(j\omega + C)^2}$

$$\frac{d}{d\omega}\left(\frac{\omega_n^2}{j\omega + C}\right) = -j\frac{\omega_n^2}{(j\omega + C)^2}$$

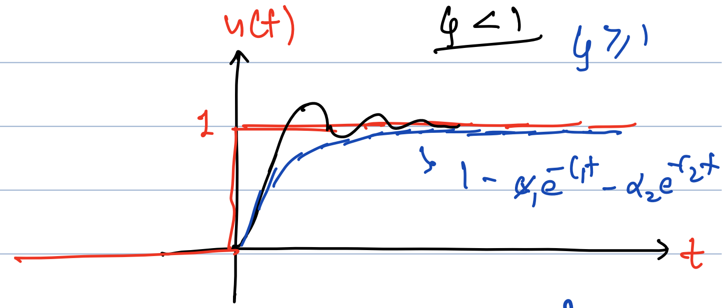
$$h(t) = t\,\omega_n^2\, e^{Ct}\, u(t)$$

$$= \omega_n^2 t\, e^{-\zeta\omega_n t}\, u(t)$$

$$t\,x(t) \longleftrightarrow j\frac{d}{d\omega}X(j\omega)$$



Sketch of Bode plot of $H(j\omega)$



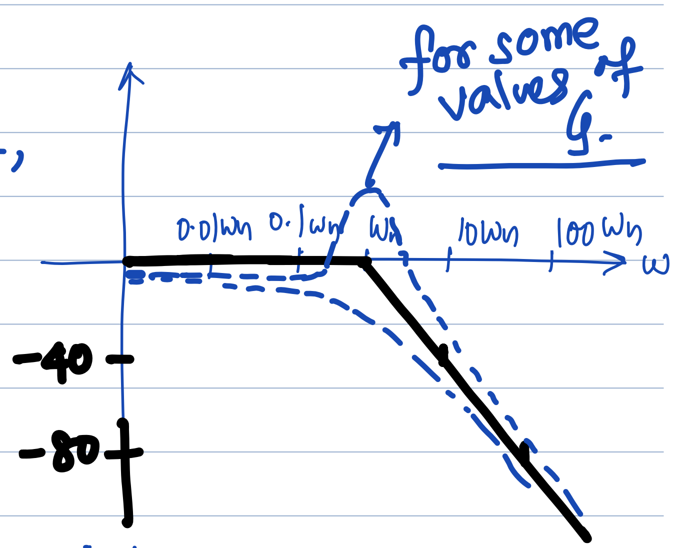
$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

$$\frac{\omega_n^2}{-\omega^2}$$

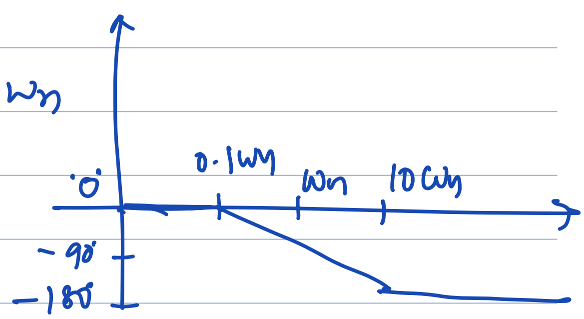
$$|H(j\omega)| = \begin{cases} \dfrac{\omega_n^2}{\omega^2} & \omega \gg \omega_n \\ 1 & \omega \ll \omega_n \end{cases}$$

$$20\log_{10}|H(j\omega)| = \begin{cases} 40\log_{10}\omega_n - 40\log_{10}\omega, & \omega \gg \omega_n \\ 0, & \omega \ll \omega_n \end{cases}$$

HW Find range of ζ for which there is overshoot, i.e., $20\log_{10}(H(j\omega))$ is positive in the vicinity of ω_n.



$$\measuredangle H(j\omega) = \begin{cases} -180^\circ, & \omega \gg \omega_n \\ 0^\circ, & \omega \ll \omega_n \end{cases}$$



Amplitude Modulation

$$\int_{-\infty}^{\infty} X(j\bar{\omega})\, C(j(\omega-\bar{\omega}))\, d\bar{\omega}$$

- By Multiplication Property:

$$x(t)c(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * C(j\omega)$$

- Consider the **sinusoidal carrier signal**

$$c(t) = \cos(\omega_0 t) \longleftrightarrow C(j\omega) = \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

- Let $x(t)$ be a **bandwidth-limited modulating signal** (e.g., music, speech), i.e., $X(j\omega) = 0$ for $|\omega| > \omega_M$
- Consider the signal

$$y(t) = x(t)c(t)$$

- By the **Multiplication Property**

$$x(t)c(t) = x(t)\cos(\omega_0 t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$

$$X(j\omega) * \delta(\omega+\omega_0) = \int_{-\infty}^{\infty} X(j\bar{\omega})\, \delta(\bar{\omega}-\omega-\omega_0)\, d\bar{\omega}$$

$$= X(j(\omega+\omega_0))$$

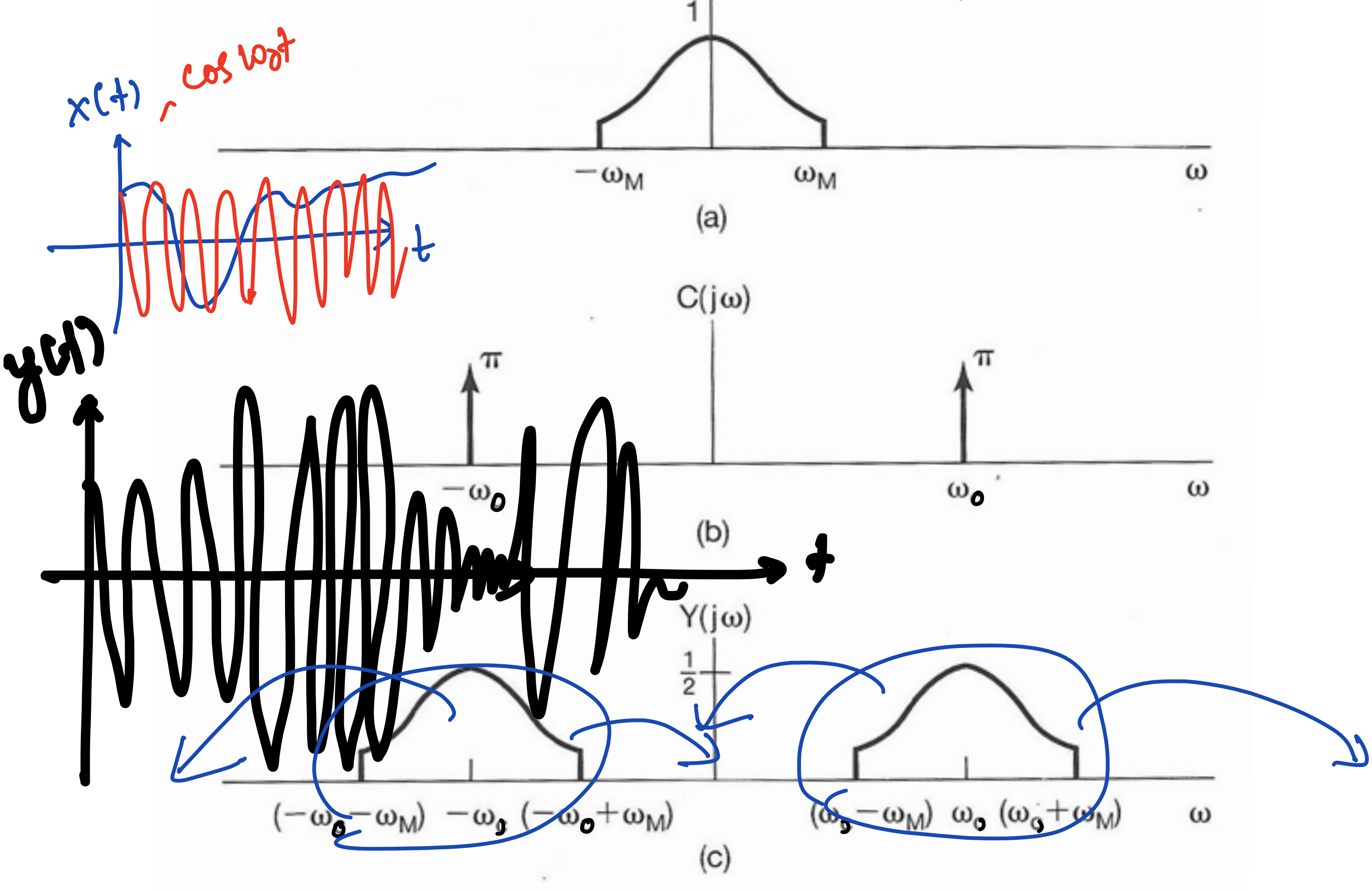
Amplitude Modulation cont.

- Therefore

$$y(t) = x(t)c(t) \longleftrightarrow Y(j\omega) = \frac{1}{2}[X(j(\omega+\omega_0) + X(j(\omega-\omega_0)]$$

We assume $\underline{\omega_0} > \omega_M$

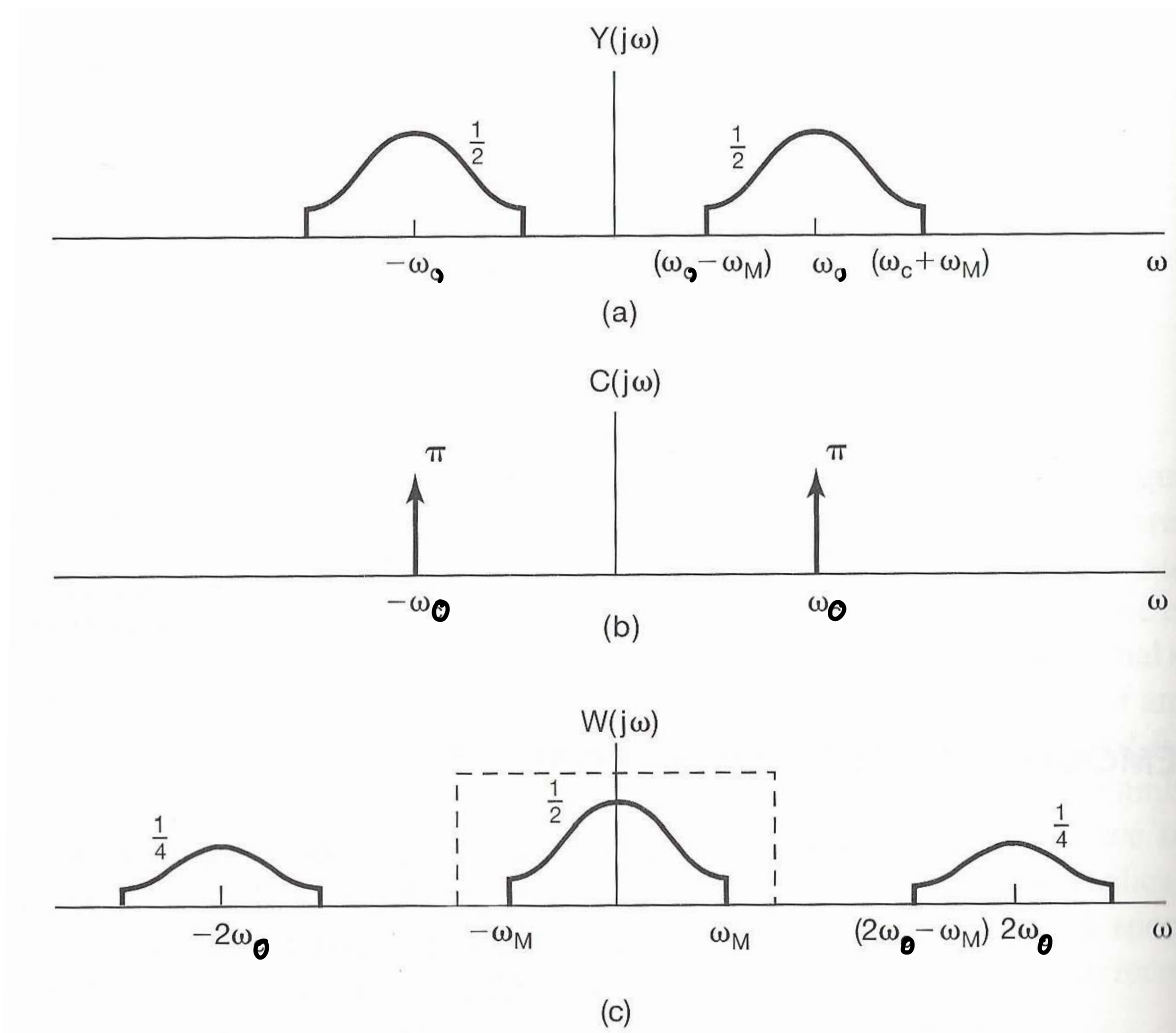
$x(t)$ is called Bandlimited.
ω_M : Bandwidth



Demodulation of AM Signals

- If we have: $w(t) = y(t)c(t)$, then $W(j\omega) = \frac{1}{2}[Y(j(\omega+\omega_0) + Y(j(\omega-\omega_0)]$
- Using: $Y(j\omega) = \frac{1}{2}[X(j(\omega+\omega_0) + X(j(\omega-\omega_0)]$:

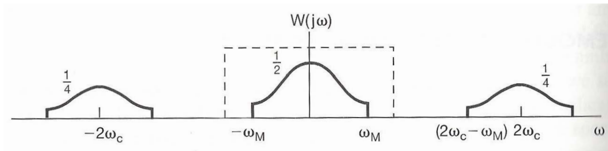
$$W(j\omega) = \frac{1}{4}X(j(\omega+2\omega_0)) + \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega-2\omega_0)).$$



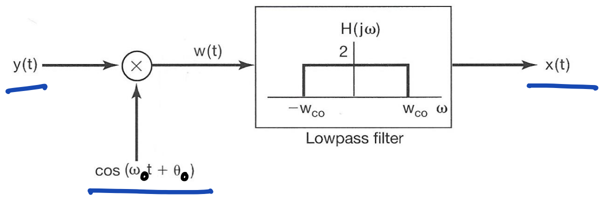
Demodulation of AM Signals

- Therefore, if we low-pass filter $w(t)$ with a gain 2, we get $x(t)$ back:

$$W(j\omega) = \frac{1}{4}X(j(\omega + 2\omega_0)) + \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega - 2\omega_0)).$$



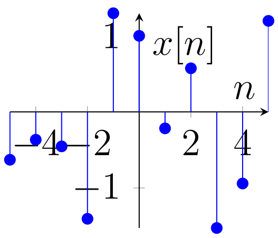
- Therefore, the AM demodulation system is:

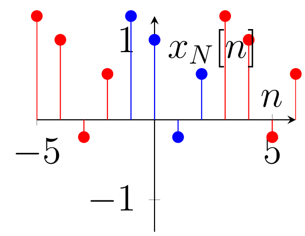


11th September

Discrete-Time Fourier Transform

- Let $x[n]$ be an arbitrary DT signal.





- For an even $N \geq 1$, define $x_N[n]$ to be the signal that:

  a. is periodic with period N ($\omega_0 = \frac{2\pi}{N}$),

  b. $x_N[n] = x[n]$ for $n = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$.

Let $N = 4$

$x_N[n] = x[n]$ for $n = -1, 0, 1, 2$

- The FS representation of $x_N[n]$ would be:

$$x_N[n] = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} a_k e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N} \tag{1}$$

where

$$a_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} x[n] e^{-jk\omega_0 n}.$$

- For large N, we have:

$$a_k = \frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} x[n] e^{-jk\omega_0 n} \approx \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n} =: \underline{\underline{\frac{1}{N} X(e^{jk\omega_0})}}, \tag{2}$$

where $X(e^{j\omega_0})$ is the Fourier Transform, if $x[n]$ is viewed as a continuous-time signal $x(t)$ with impulses at integer values of t.

Continued.

$\omega_0 = \frac{2\pi}{N}$

$\Rightarrow \frac{1}{N} = \frac{\omega_0}{2\pi}$

$\omega_0 \to 0$ as $N \to \infty$

- Replacing (2) in (1), we get:

$$x_N[n] \approx \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \omega_0 X(e^{jk\omega_0}) e^{jk\omega_0 n}.$$

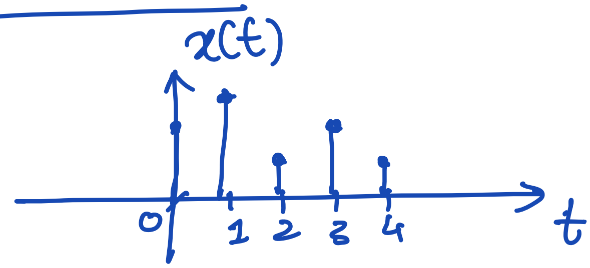
- Since the fundamental period $\omega_0 = \frac{2\pi}{N}$ approaches 0, $k\omega_0$ approaches the continumm

- Note that $\sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \omega_0 X(e^{jk\omega_0}) e^{jk\omega_0 n} \approx \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$.

- Letting N go to infinity, we get:

$$x[n] = \lim_{N\to\infty} x_N[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega.$$

Alternative interpretation



$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, \delta(t-k)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \sum_{k=-\infty}^{\infty} x[k] \int_{-\infty}^{\infty} \delta(t-k) e^{-j\omega t} dt$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x[k] e^{-jk\omega}}$$: discrete-time Fourier transform.

Fourier Transform of DT Signals

For a a discrete-time signal $x[n]$, we have:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{(Synthesis Equation)}$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(Analysis Equation).}$$

$$X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n \underbrace{-j2\pi k n}_{=1.}}$$

- Notation: $x[n] \longleftrightarrow X(e^{j\omega})$

- $X(e^{j\omega})$ is finite for all ω if either (a) $x[n]$ is absolutely summable or (b) $x[n]$ has finite energy.

- DTFT differs from CTFT in the following two ways.
  1. $X(e^{j\omega})$ is periodic in ω with period 2π. Thus, it suffices to specify it over any contiguous interval of ω of length 2π. For a continuous-time signal $x(t)$, its FT $X(j\omega)$ is not necessarily periodic.
  2. The inverse DTFT integral is computed over an interval of length 2π even though $X(e^{j\omega})$ is defined for $\omega \in (-\infty, \infty)$.
  3. Notation: $x[n] \longleftrightarrow X(e^{j\omega})$ while $x(t) \longleftrightarrow X(j\omega)$.

Recall that for a continuous-time periodic signal, $x(t)$, its FS representation was a "discrete-time" function of ω, defined at $k\omega_0$.

Here, we have the complementary setting.

time-domain : discrete-time, not periodic

frequency-domain: continuous-time, periodic.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

DTFT Example

- What is the DTFT of $x[n] = \delta[n]$?

$$= \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$$

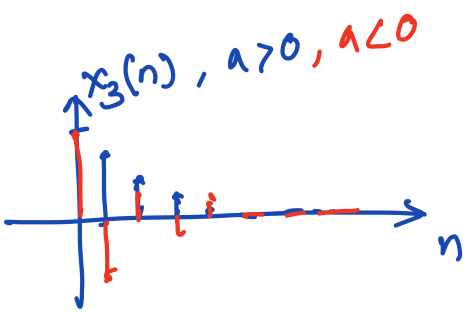
- What is the DTFT of $x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$?

$$= \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \delta[n+1]\right) e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{j\omega}$$

$$= 1 + 2\cos\omega$$

↘ periodic with period 2π.



- What is the DTFT of $x_3[n] = a^n u[n]$ for $|a| < 1$?
- We have:

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

periodic with period 2π.

- $|X(e^{j\omega})|^2 = \dfrac{1}{|1 - a\cos\omega + ja\sin\omega|^2}$

$$= \frac{1}{(1 - a\cos\omega)^2 + (a\sin\omega)^2}$$

DTFT of $a^n u[n]$

Note that: $\frac{1}{1-2a\cos\omega+a^2}$ is maximized (minimized) when $-2a\cos\omega$ is minimized (maximized)

Case A. If $0 < a < 1$, then $\left|X(e^{j\omega})\right|^2$ achieves maximum when $\omega = 0$, and $\left|X(e^{j\omega})\right|^2$ achieves minimum when $\omega = \pi$. Thus,

$$\max\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1-2a+a^2} = \frac{1}{(1-a)^2}$$
$$\min\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1+2a+a^2} = \frac{1}{(1+a)^2}.$$

Case B: If $-1 < a < 0$, then $\left|X(e^{j\omega})\right|^2$ achieves maximum when $\omega = \pi$, and $\left|X(e^{j\omega})\right|^2$ achieves minimum when $\omega = 0$. Thus,

$$\max\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1+2a+a^2} = \frac{1}{(1+a)^2}$$
$$\min\left\{\left|X(e^{j\omega})\right|^2\right\} = \frac{1}{1-2a+a^2} = \frac{1}{(1-a)^2}.$$

DTFT of $a^{|n|}$

- What is the DTFT of $x[n] = a^{|n|}$ for $|a| < 1$?

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=-\infty}^{0} a^{-n} e^{-j\omega n} + \sum_{m=1}^{\infty} a^{m} e^{-j\omega m}$$

$(\bar{n} = -n)$

$$= \sum_{\bar{n}=0}^{\infty} a^{\bar{n}} e^{j\omega \bar{n}} + (ae^{-j\omega}) \sum_{m=0}^{\infty} a^{m} e^{-j\omega m}$$

$$= \frac{1}{1-(ae^{j\omega})} + \frac{ae^{-j\omega}}{1-ae^{-j\omega}}$$

$$= \frac{1-ae^{-j\omega} + ae^{-j\omega} - a^2}{(1-ae^{j\omega})(1-ae^{-j\omega})}$$

$$= \frac{1-a^2}{1-ae^{-j\omega}-ae^{j\omega}+a^2} = \frac{1-a^2}{1+a^2-2a\cos\omega}.$$

periodic wrt ω period 2π.

$$\left(\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-j\omega}}, \quad \left(\tfrac{1}{2}\right)^{-n} u[-n] \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{j\omega}}$$

Exercise

- Note that $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$. Now, given the discrete-time signal $x[n]$ with Fourier transform $X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{j\omega}}$, calculate $x[n]$.

  A. $x[n] = 2^n u[-n]$

  B. $x[n] = 2^{-n} u[-n]$ → diverging .

  C. $x[n] = 2^{n+2} u[n+2]$

if $x[n] \leftrightarrow X(e^{j\omega})$,

$x[-n] \leftrightarrow X(e^{-j\omega})$

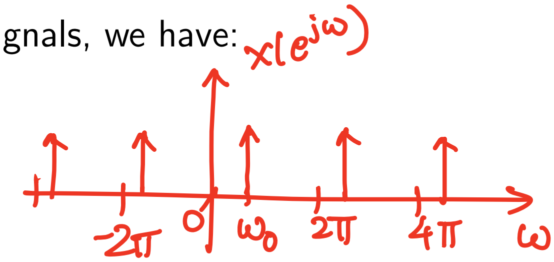
Suppose $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}$

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[-k] e^{-j\omega k} = \underbrace{\sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}}_{X(e^{-j\omega})}$$ if $n=-k$

DTFT of Periodic Signals

- Let $x[n]$ be a periodic signal with period N. What is the FT of this signal?
- By the Fourier series decomposition of periodic signals, we have:

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}.$$



- Show that $e^{j\omega_0 n} \longleftrightarrow \underbrace{2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)}_{X(e^{j\omega})}$
- Using this and linearity of Fourier Transform, we get:

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}).$$

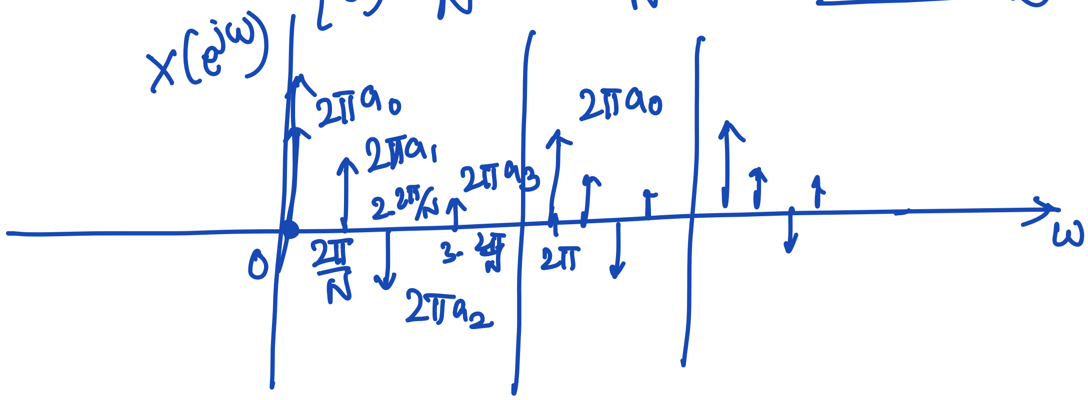
$a_k = a_{k \text{ modulo } N.}$

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{\cancel{2\pi}}\int_0^{2\pi} \cancel{2\pi}\, \delta(\omega - \omega_0) e^{j\omega n} d\omega = e^{j\omega_0 n}$$

Suppose $N = 4$, FS: (a_0, a_1, a_2, a_3), $a_4 = a_0$, $a_5 = a_1$ & so on.

$e^{jk\frac{2\pi}{4}n}$, $k = 0, 1, 2, 3$

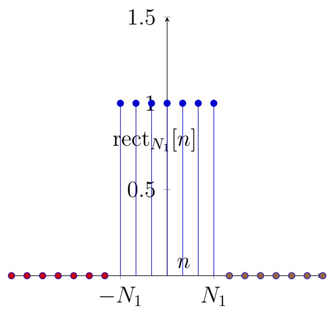
$\left\{0, \frac{2\pi}{N}, 2\cdot\frac{2\pi}{N}, \ldots, (N-1)\frac{2\pi}{N}\right\}$



DTFT of Rectangular Pulse

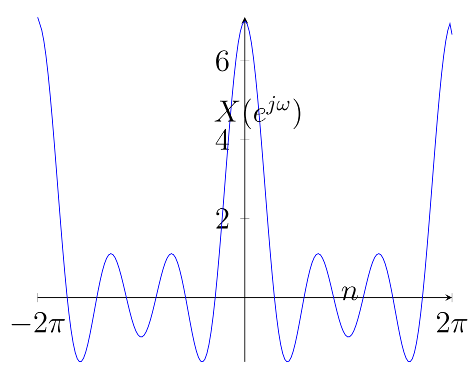
- FT of discrete-time rect

$$x[n] = \text{rect}_{N_1}[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & \text{otherwise} \end{cases}$$



- Recall that $1 + \beta + \beta^2 + \ldots + \beta^{r-1} = \frac{1-\beta^r}{1-\beta}$.
- For $\omega \neq 0$, we have:

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = e^{-j\omega N_1} \frac{e^{2j\omega N_1+1} - 1}{e^{j\omega} - 1}$$



Properties of DTFT

$m = n - n_0$

$$\sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\omega n}$$

- Periodicity of DTFT: $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$
- Linearity: $\alpha x[n] + \beta y[n] \longleftrightarrow \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
- Time-Reversal: $x[-n] \longleftrightarrow X(e^{-j\omega})$
- Time shift: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

$$= \underbrace{\sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m}}_{X(e^{j\omega})} e^{-j\omega n_0}$$

$$= e^{-j\omega n_0} X(e^{j\omega})$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x[n-n_0] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, e^{-j\omega n_0}\, d\omega = \frac{1}{2\pi} \int_{2\pi} \left(X(e^{j\omega}) e^{-j\omega n_0} \right) e^{j\omega n}\, d\omega$$

- Frequency shift: $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

$$\sum_{k=-\infty}^{\infty} e^{j\omega_0 k} x[k] e^{-j\omega k} = \sum_{k=-\infty}^{\infty} x[k] e^{-j(\omega-\omega_0)k} = X(e^{j(\omega-\omega_0)})$$

- Conjugation: $x^*[n] \longleftrightarrow (X(e^{-j\omega}))^*$

$$\sum_{n=-\infty}^{\infty} x[n]^* e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(x[n] e^{j\omega n} \right)^* = \left(\sum_{n=-\infty}^{\infty} x[n] e^{j\omega n} \right)^*$$

$$= \left(X(e^{-j\omega}) \right)^*$$

- Show that if $x[n]$ is real and even, then $X(e^{j\omega})$ is also a real and even function of ω.

if $x[n]$ is real and even,

$x[-n] = x[n] = (x[n])^* \quad \forall n,$

$\Rightarrow X(e^{-j\omega}) = X(e^{j\omega}) = (X(e^{-j\omega}))^*$

$\forall \omega.$

Properties of DTFT

- DT Differentiation: $x[n] - \underline{x[n-1]} \longleftrightarrow \underline{(1 - e^{-j\omega})X(e^{j\omega})}$

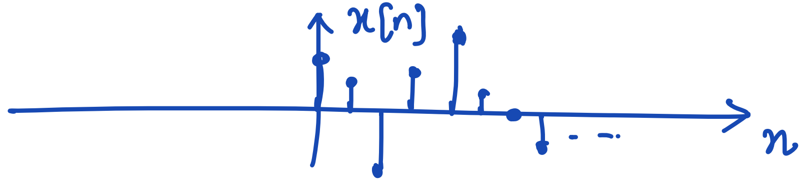
$X(e^{j\omega}) \qquad e^{-j\omega}X(e^{j\omega})$

- DT integration (accumulation):

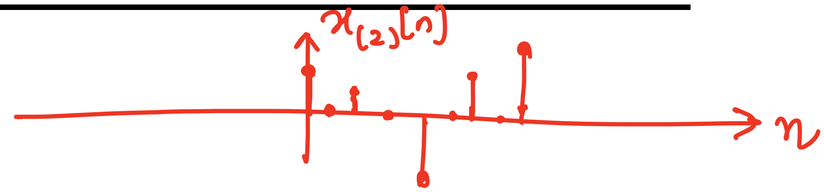
$$\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2k\pi).$$

- Corollary: Fourier transform of the unit step function $u[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2k\pi).$$



Properties of DTFT cont.



- Time-scaling: Define

$$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}] & \text{for } n \text{ a multiple of } k \\ 0 & \text{otherwise} \end{cases}.$$

$n = mk$

Then,

$$\underline{x_{(k)}[n] \longleftrightarrow X(e^{j\omega k}).}$$

$$\sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x_{(k)}[mk]\, e^{-j\omega mk}$$

$$X(e^{j\omega k})$$

$$\|$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega mk}$$

- Differentiation in frequency domain:

$$nx[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega}).$$

$$j\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{-j\omega n}\right)' = \sum_{n=-\infty}^{\infty} n x[n]\, (-j)\, \underset{(j)}{e^{-j\omega n}}$$

$$= \sum_{n=-\infty}^{\infty} \underline{\underline{n x[n]}}\, e^{-j\omega n}$$

- Parseval's Relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

DTFT Properties

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix}X(e^{j\omega})\\Y(e^{j\omega})\end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}d\{x[n]\}$ [$x[n]$ real]	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$		

Exercise

Determine the DTFT of the signal

$$x[n] = \begin{cases} 1, & n \in \{0, 2, 4\} \\ 2, & n \in \{1, 3, 5\} \\ 0, & \text{otherwise.} \end{cases}$$

$$X(e^{j\omega}) = 1 \cdot \left(e^{-j\omega \cdot 0} + e^{-j\omega 2} + e^{-j\omega 4}\right)$$
$$+ 2\left(e^{-j\omega} + e^{-j\omega 3} + e^{-j\omega 5}\right)$$

Duality of DTFT and Continuous-time Fourier Series

- We noticed that $X(e^{j\omega})$ is always periodic with period 2π.
- Let $z(t) = X(e^{jt})$ which is periodic with period 2π.
- Then, $z(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt}$ where:

$$a_k = \frac{1}{2\pi}\int_{2\pi} z(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{-jk\omega}d\omega = x[-k].$$

- $x[n] \longleftrightarrow X(e^{j\omega}) \longleftrightarrow x[-n]$.

k-th FS coefficient of $X(e^{jt})$ is the value of $x[n]$ at $n = -k$.