LECTURE 17: 2nd September 2026

Partial Fraction

- Suppose $H(j\omega)$ is a **rational function** or **ratio of polynomials**
- Apply **Partial Fraction Expansion** to write $H(j\omega)$ in a form that allows us to determine $h(t)$ from Table 4.2
- **Example:** Stable LTI system described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Then $(j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$

$$\begin{aligned} H(j\omega) &= \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} \\ &= \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \end{aligned}$$

Rewrite $H(j\omega)$ as

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3}$$

$h(t) = A e^{-t} u(t) + B e^{-3t} u(t)$

Partial Fraction Example cont.

- We got

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3} \quad + (j\omega + 3)$$

- Put over a common denominator

$$\begin{aligned} H(j\omega) &= \frac{A(j\omega + 3) + B(j\omega + 1)}{(j\omega + 1)(j\omega + 3)} \\ &= \frac{(A + B)j\omega + (3A + B)}{(j\omega + 1)(j\omega + 3)} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \end{aligned}$$

- Equate real and imaginary parts of the numerators:
  Real part: $3A + B = 2$
  Imaginary part: $A + B = 1$

- Solving for A and B

$$A = \frac{1}{2} \text{ and } B = \frac{1}{2} \Rightarrow$$

$$H(j\omega) = \frac{1}{2}\left(\frac{1}{j\omega + 1}\right) + \frac{1}{2}\left(\frac{1}{j\omega + 3}\right) \Rightarrow h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

if deg. of numerator polynomial $\leq$ deg. of den. poly, the transfer function is called proper,

Partial Fraction General Case

and when $<$, it is called strictly proper.

This course will mainly deal with systems with strictly proper transfer fn.

- For convenience: consider $j\omega$ as a variable v:

$$\begin{aligned} H(v) &= \frac{b_{n-1}v^{n-1} + b_{n-2}v^{n-2} + \cdots + b_1 v + b_0}{v^n + a_{n-1}v^{n-1} + a_{n-2}v^{n-2} + \cdots + a_1 v + a_0} \\ &= \frac{b_{n-1}v^{n-1} + b_{n-2}v^{n-2} + \cdots + b_1 v + b_0}{(v-p_1)^{\sigma_1}(v-p_2)^{\sigma_2}\cdots(v-p_r)^{\sigma_r}} \end{aligned}$$

- We write this as a sum of simpler polynomial ratios as

$$\begin{aligned} H(v) &= \frac{A_{11}}{(v-p_1)} + \frac{A_{12}}{(v-p_1)^2} + \cdots + \frac{A_{1\sigma_1}}{(v-p_1)^{\sigma_1}} \\ &+ \frac{A_{21}}{(v-p_2)} + \frac{A_{22}}{(v-p_2)^2} + \cdots + \frac{A_{2\sigma_2}}{(v-p_2)^{\sigma_2}} \\ &\cdots \\ &+ \frac{A_{r1}}{(v-p_r)} + \frac{A_{r2}}{(v-p_r)^2} + \cdots + \frac{A_{r\sigma_r}}{(v-p_r)^{\sigma_r}} \end{aligned}$$

- We can inverse CTFT each term using $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^n}$ for $a > 0$.

Recall that $\quad t\,x(t) \longleftrightarrow j\frac{d}{d\omega}X(j\omega)$

if $X(j\omega) = \frac{1}{a+j\omega}$, $\quad \frac{d}{d\omega}X(j\omega) = \frac{-1}{(a+j\omega)^2}\cdot j$

$$\Rightarrow j\frac{d}{d\omega}X(j\omega) = \frac{1}{(a+j\omega)^2}$$

$$\text{IFT}\left[\frac{1}{(a+j\omega)^2}\right] = te^{-at}u(t).$$

$$\frac{d}{d\omega}\left[\frac{1}{(a+j\omega)^2}\right] = \frac{-2j}{(a+j\omega)^3} \Rightarrow \frac{j}{2}\frac{d}{d\omega}\left[\frac{1}{(a+j\omega)^2}\right] = \frac{1}{(a+j\omega)^3}$$

$$\text{IFT}\left[\frac{1}{(a+j\omega)^3}\right] = \frac{t^2}{2}e^{-at}u(t).$$

Example

Suppose a causal stable continuous-time LTI system has frequence response

$$H(j\omega) = \frac{4 + j\omega}{6 - \omega^2 + j5\omega}. = \frac{4 + j\omega}{(j\omega + 2)(j\omega + 3)}$$

Determine its impulse response in time-domain.

Recall that $h(t) = \text{IFT}\left[H(j\omega)\right]$

$$= \text{IFT}\left[\frac{A}{j\omega + 2} + \frac{B}{j\omega + 3}\right]$$

$$(A+B)j\omega + (3A+2B) = 1\,(j\omega) + 4$$

$$A + B = 1$$

$$3A + 2B = 4 \Rightarrow 3A + 2 - 2A = 4$$

$$\Rightarrow A = 2, \quad B = -1$$

$$h(t) = \left(2e^{-2t} - e^{-3t}\right)u(t)$$

Q) Determine $x(t)$ applied to the LTI system.

$$x(t) \longrightarrow \boxed{} \longrightarrow y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

$$H(j\omega) = \frac{1}{j\omega + 3}$$

SOL: We know $Y(j\omega) = X(j\omega)\,H(j\omega)$

$$\Rightarrow \frac{1}{j\omega + 3} - \frac{1}{j\omega + 4} = X(j\omega)\,\frac{1}{j\omega + 3}$$

$$\Rightarrow \frac{(j\omega+4-j\omega-3)}{(j\omega+3)(j\omega+4)} = X(j\omega)\frac{1}{j\omega+3}$$

$$\Rightarrow X(j\omega) = \frac{1}{j\omega+4} \Rightarrow x(t) = e^{-4t}u(t).$$

Q) (i) $x(t)$ is real and its FT is $X(j\omega)$.

(ii) $x(t) = 0$ for $t < 0$

(iii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{Re[X(j\omega)]}\, e^{j\omega t}\, d\omega = |t| e^{-|t|}$.

$$= \underline{\frac{1}{2}(x(t) + x(-t))}$$

Determine $\underline{x(t)}$.

$$X(j\omega) + X(j\omega)^* = 2\,Re[X(j\omega)]$$

$$\frac{1}{2\pi}\cdot\frac{1}{2}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega + \underbrace{\frac{1}{2\pi}\cdot\frac{1}{2}\int_{-\infty}^{\infty} X(j\omega)^* e^{j\omega t}\,d\omega}$$

$$= \frac{1}{2}x(t) + \frac{1}{2}x(-t)$$

if $x(t) \longleftrightarrow X(j\omega)$, $\underset{=x(t)}{x(t)^*} \longleftrightarrow \underline{X(-j\omega)^*}$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(-j\bar{\omega})^* e^{-j\bar{\omega}t}\,d\bar{\omega} = x(t)$$

$$\boxed{x(t) = \begin{cases} 0, & t < 0 \\ 2te^{-t}, & t \geq 0 \end{cases}}$$

Q) Consider a system whose input $x(t)$ and output $y(t)$ are related by

$$\frac{d}{dt}y(t) + 10y(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau) z(t-\tau)\, d\tau}_{} - x(t),$$

and $z(t) = e^{-t}u(t) + \underline{3\delta(t)} \Rightarrow Z(j\omega) = \frac{1}{j\omega+1} + 3$

Determine impulse response of this system.

SOL:
$$(j\omega + 10)Y(j\omega) = X(j\omega)Z(j\omega) - X(j\omega)$$
$$= X(j\omega)\left[\frac{1}{j\omega+1} + 2\right]$$
$$\Rightarrow H(j\omega) = \frac{2j\omega+3}{(j\omega+1)(j\omega+10)}$$
$$= \frac{A}{j\omega+1} + \frac{B}{j\omega+10}$$
$$\Rightarrow h(t) = (Ae^{-t} + Be^{-10t})u(t).$$

Q) $x(t) = (e^{-t} + e^{-3t})u(t) \Rightarrow X(j\omega) = \frac{1}{j\omega+1} + \frac{1}{j\omega+3}$

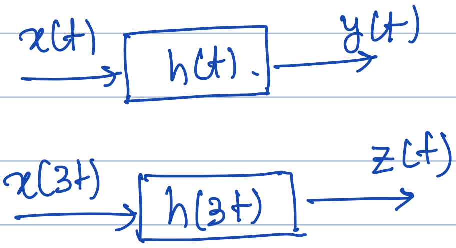
$y(t) = (2e^{-t} - 2e^{-4t})u(t) \Rightarrow Y(j\omega) = \frac{2}{j\omega+1} - \frac{2}{j\omega+4}.$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{2j\omega + 8 - 2j\omega - 2}{(j\omega+1)(j\omega+4)} \cdot \frac{(j\omega+1)(j\omega+3)}{2j\omega+4}$$
$$= \frac{3\cancel{6}(j\omega+3)}{\cancel{2}(j\omega+2)(j\omega+4)} = \frac{3j\omega+9}{(j\omega+2)(j\omega+4)}$$
$$= \frac{3j\omega+9}{(j\omega)^2 + 6j\omega + 8}$$

$$\Rightarrow (j\omega)^2 Y(j\omega) + 6j\omega Y(j\omega) + 8Y(j\omega)$$
$$= 3j\omega X(j\omega) + 9X(j\omega).$$

$$\Rightarrow \quad \frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 3\frac{d}{dt}x(t) + 9x(t).$$

Q.



Express $z(t)$ as

$Ay(Bt)$,

i.e., find scalars A & B

such that $z(t) = \underline{\underline{A}}\, y(Bt)$.

$A = 1/3$, $B = 3$.

SOL: Given: $Y(j\omega) = X(j\omega)\, H(j\omega)$

If $x(t) \longleftrightarrow X(j\omega)$, then $x(3t) \longleftrightarrow \frac{1}{3}X\left(\frac{j\omega}{3}\right)$

$h(t) \longleftrightarrow H(j\omega)$, then $h(3t) \longleftrightarrow \frac{1}{3}H\left(\frac{j\omega}{3}\right)$.

$$Z(j\omega) = \frac{1}{9}\underbrace{X\left(\frac{j\omega}{3}\right)H\left(\frac{j\omega}{3}\right)} = \frac{1}{9}Y\left(j\omega/3\right)$$

$$\text{IFT}\left[\frac{1}{3}\left[\frac{1}{3}Y\left(\frac{j\omega}{3}\right)\right]\right] = \underline{\frac{1}{3}y(3t)}.$$

LECTURE 18: 3rd Sept

$$e^{j\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega_0)\, e^{j\omega_0 t}$$

$$H(j\omega_0) = \mathcal{FT}[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Suppose the input is $x(t) = A\cos(\omega_0 t)$, $= \frac{1}{2}\left[e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}\right] A$

The output

$$y(t) = \frac{1}{2} H(j\omega_0) e^{j\omega_0 t} A + \frac{1}{2} \underline{H(-j\omega_0)}\, e^{-j\omega_0 t} \quad A$$

If $h(t)$ is a real-valued signal, $H(j\omega) = H(-j\omega)^*$

$$\Rightarrow H(-j\omega) = H(j\omega)^*$$

$$\Rightarrow y(t) = \frac{A}{2}\left[H(j\omega_0) e^{j\omega_0 t} + \underline{H(j\omega_0)^*}\ \underline{(e^{j\omega_0 t})^*}\right]$$

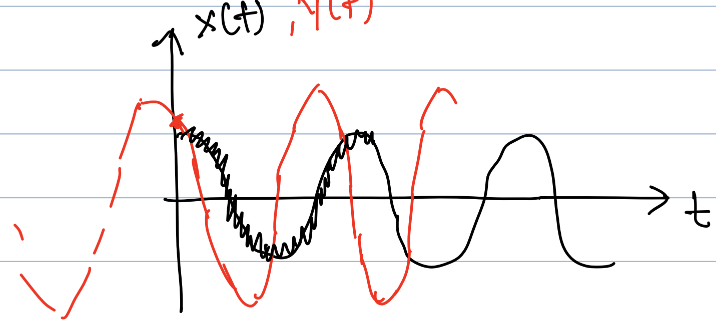
$$= \frac{A}{2}|H(j\omega_0)|\, e^{j(\measuredangle H(j\omega_0) + \omega_0 t)}$$

$$+ \frac{A}{2}|H(j\omega_0)|\, e^{-j(\omega_0 t + \measuredangle H(j\omega_0))}$$

Let $H(j\omega_0) = |H(j\omega_0)|\, e^{j\measuredangle H(j\omega_0)}$

$$H(j\omega_0)^* = |H(j\omega_0)|\, e^{-j\measuredangle H(j\omega_0)}$$

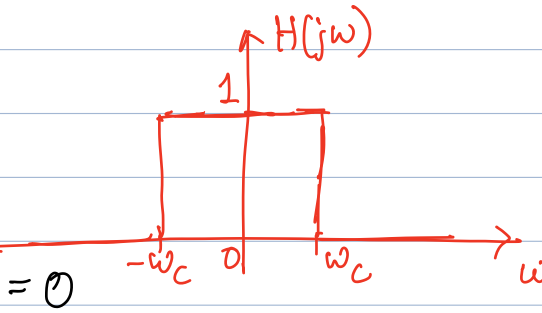
$$= A|H(j\omega_0)| \cos(\omega_0 t + \underline{\measuredangle H(j\omega_0)})$$



Low pass filters

An ideal low-pass filter has frequency response

$$H(j\omega) = \begin{cases} 1, & \text{if } |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$



Let $\omega_c = 1$.

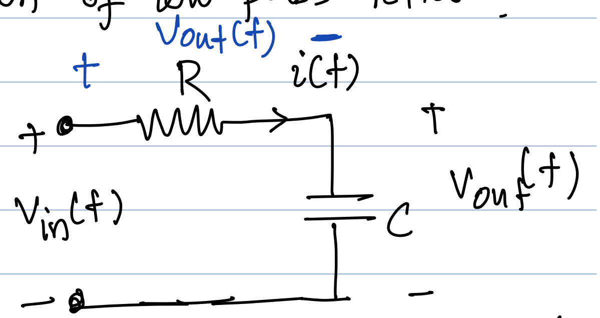
$$x(t) = \underbrace{10\cos(0.5t)} + \underbrace{0.1\cos(10t)}_{|H(j10)| = 0}$$

$$\Rightarrow y(t) = 10\cos(0.5t)$$

Q How to design a filter with a given $H(j\omega)$?

- → analog: ✓
- → digital

RC circuit as an approximation of low-pass filter.



$$V_{in}(t) = RC\frac{dV_{out}(t)}{dt} + V_{out}(t)$$

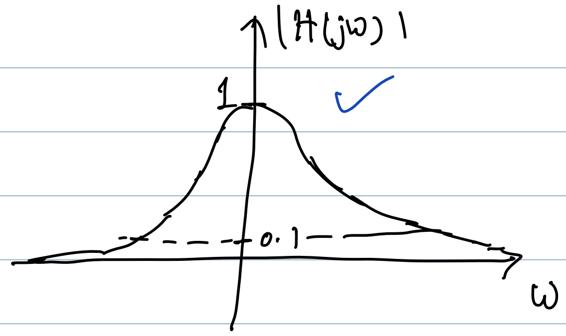
$$i(t) = C\frac{dV_{out}(t)}{dt}$$

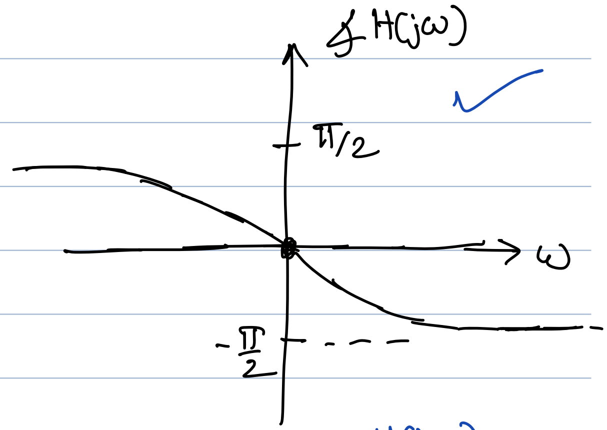
$$\Rightarrow V_{in}(j\omega) = (j\omega RC + 1)V_{out}(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{1}{1 + j\omega RC}$$

→ First order system.

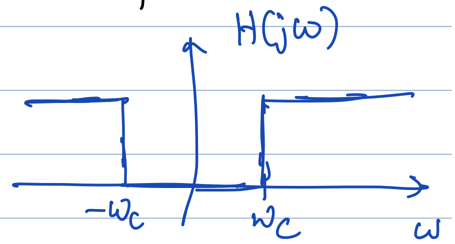
$$|H(j\omega)| = \frac{1}{(1 + \omega^2R^2C^2)^{1/2}}, \quad \measuredangle H(j\omega) = -\tan^{-1}(\omega RC)$$





High pass filter :

$$H(j\omega) = \begin{cases} 0, & |\omega| \le \omega_c \\ 1, & \text{otherwise} \end{cases}$$



Consider the series RC circuit with output being the voltage across the resistor.

$$\cancel{V_{in}(t)} = RC\frac{dV_c(t)}{dt} + V_c(t), \qquad V_c(t) = V_{in}(t) - V_{out}(t)$$

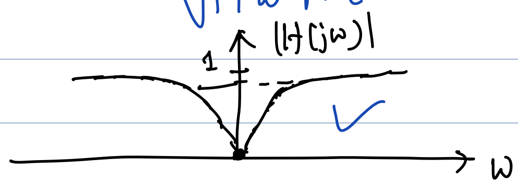
$$= RC\frac{d}{dt}V_{in}(t) - RC\frac{d}{dt}V_{out}(t) + \cancel{V_{in}(t)} - V_{out}(t)$$

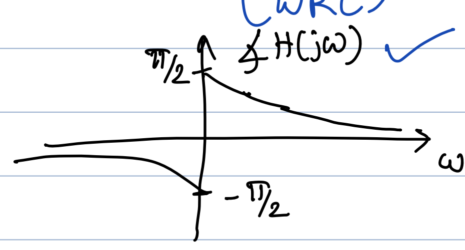
$$\Rightarrow \quad V_{out}(t) + RC\frac{d}{dt}V_{out}(t) = RC\frac{d}{dt}V_{in}(t)$$

$$\Rightarrow \quad (1 + j\omega RC)\,V_{out}(j\omega) = RCj\omega\; V_{in}(j\omega)$$

$$\Rightarrow \quad H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}, \qquad j\omega RC(1 - j\omega RC) = j\omega RC + \omega^2R^2C^2$$

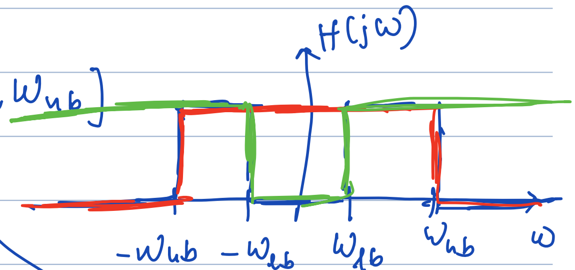
$$|H(j\omega)| = \frac{|\omega RC|}{\sqrt{1 + \omega^2R^2C^2}}, \qquad ∡H(j\omega) = \tan^{-1}\left(\frac{1}{\omega RC}\right).$$

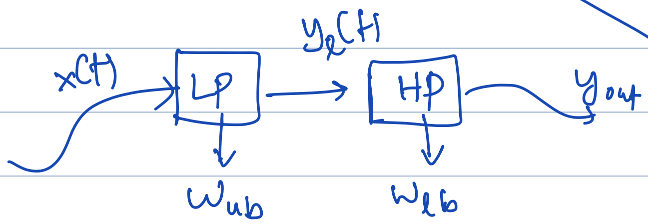




Band-pass filter : Signals with frequency $|\omega| \in [\omega_{lb}, \omega_{ub}]$ reflect on the output without being affected and signals with frequency outside that range is blocked.

$$\underline{\underline{H(j\omega)}} = \begin{cases} 1, & \text{if } |\omega| \in [\omega_{lb}, \omega_{ub}] \\ 0, & \text{otherwise} \end{cases}$$





$$Y_{out}(j\omega) = X(j\omega) \underbrace{H_{lp}(j\omega) H_{hp}(j\omega)}_{H_{eff}(j\omega)}$$

$$|H_{eff}(j\omega)| = |H_{lp}(j\omega)||H_{hp}(j\omega)| \Rightarrow \log |H_{eff}(\omega)|$$

$$\measuredangle H_{eff}(j\omega) = \underline{\measuredangle H_{lp}(j\omega)} + \underline{\measuredangle H_{hp}(j\omega)} \qquad = \log |H_{lp}(j\omega)| + \log |H_{hp}(j\omega)|$$

Bode plot

Given $H(j\omega)$, its Bode plot consists of two plots.

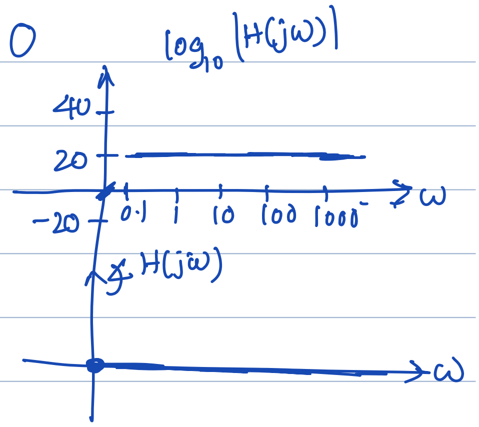
(i) $20\log_{10} |H(j\omega)|$ vs. ω in log scale
↳ (dB)

(ii) $\measuredangle H(j\omega)$ vs. ω in log-scale

Example:

(1) $H(j\omega) = 10$, $\quad 20\log_{10}|H(j\omega)| = 20\,\text{dB}$

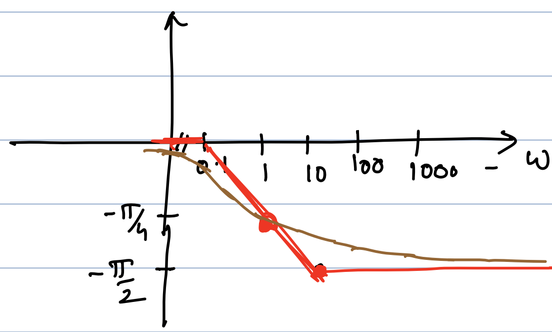
$\measuredangle H(j\omega) = 0$



(2) $H(j\omega) = \dfrac{1}{(j\omega + 1)}$

$$\measuredangle H(j\omega) = -\tan^{-1}(\omega)$$

$$= \begin{cases} 0 & , \ \omega \to 0 \\ -\pi/4 & , \ \omega = 1 \\ -\pi/2 & , \ \omega \to \infty \end{cases}$$



$(j\omega c + 1)$

$(j\omega c + 1)^k$