LECTURE 14: 19th August

$$x[n] = \sum_{k=0}^{5} a_k e^{jk\omega_0 n}$$

Practice Problem

$$a_k = \frac{1}{6} \sum_{n=0}^{5} x[n] e^{-jk\omega_0 n}$$

Consider a discrete-time signal $x[n]$ for which the following properties are true.

a. $x[n]$ is periodic with fundamental period $N = 6$. $\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$.

b. $\sum_{n=0}^{5} x[n] = 2$ $\rightarrow$ $a_0 = \frac{1}{6} \sum_{n=0}^{5} x[n] \Rightarrow a_0 = \frac{2}{6} = \frac{1}{3}$.

c. $\sum_{n=2}^{7} (-1)^n x[n] = 1$

d. $x[n]$ has minimum average power among all signals satisfying the above three conditions.

Determine $x[n]$.

$$e^{-j\pi n} = (-1)^n$$

$$a_3 = \frac{1}{6} \sum_{n=2}^{7} x[n] e^{-j3\frac{\pi}{3}n}$$

$$= \frac{1}{6} \sum_{n=2}^{7} x[n] (-1)^n = \frac{1}{6}.$$

From Parseval's relation, we can conclude that $a_1 = a_2 = a_4 = a_5 = 0$.

$$x[n] = a_0 + a_3 e^{j3\pi/3 n} = \frac{1}{3} + \frac{1}{6} e^{j\pi n}$$

$$\Rightarrow x[n] = \frac{1}{3} + \frac{1}{6}(-1)^n, \quad 0 \le n \le 5$$

which repeats after every T steps.

the restriction of energy being minimum resulted in a signal with period = 2.

Q1) Consider a discrete-time LTI system with $h[n] = \left(\frac{1}{2}\right)^{|n|}$. Determine the FS representation of the output for the following two input signals.

i) $x_1[n] = \sin\left(\frac{3\pi n}{4}\right)$

ii) $x_2[n] = j^n + (-1)^n$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

Step 1: find $H(z)$

Step 2: express $x_i[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$

Step 3: $y_i[n] = \sum_{k=0}^{N-1} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n}$$

(assume that z is such that the series converges)

$$= 1 + \frac{1}{2z}\left(1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \cdots\right) + \sum_{m=1}^{\infty} \left(\frac{z}{2}\right)^m$$

$$= 1 + \frac{\frac{1}{2z}}{1 - \frac{1}{2z}} + \frac{z/2}{1 - \frac{z}{2}}.$$

$$H(z) = 1 + \frac{1}{2z-1} + \frac{z}{2-z}.$$

$$H(e^{j\pi/2}) = \left(1 + \frac{1}{2j-1} + \frac{j}{2-j}\right) \qquad H(e^{j\pi}) = 1 + \frac{1}{-3} + \frac{-1}{3}$$

$$= +1/3$$

Let us consider $x_2[n] = j^n + (-1)^n$

$$x_2[n] = e^{j\frac{\pi n}{2}} + e^{j\pi n}$$

$x_2[0] = 1 + 1 = 2$
$x_2[1] = j - 1$
$x_2[2] = -1 + 1 = 0$
$x_2[3] = -j - 1$
$x_2[4] = j^4 + (-1)^4 = 2$

fundamental period $= 4$

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$x_2[n] = \sum_{k=0}^{3} a_k e^{jk\frac{\pi}{2}n},$$

$a_0 = 0$
$a_1 = 1$
$a_2 = 1$
$a_3 = 0$

$$y_2[n] = \sum_{k=0}^{3} a_k H\left(\underline{e^{jk\pi/2}}\right) e^{jk\frac{\pi}{2}n}$$

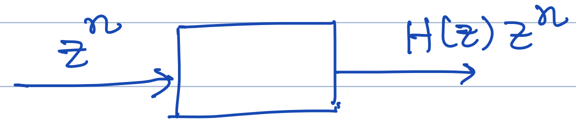
$$= H\left(e^{j\pi/2}\right)\underline{(j)}^n + H\left(e^{j\pi}\right)\underline{(-1)}^n$$

Q2) Consider discrete-time impulse train $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$

If $x[n]$ is applied as input to a LTI system, then output $y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right)$

Determine $H(e^{jk\pi/2})$ for $k = 0, 1, 2, 3$.

Recall:



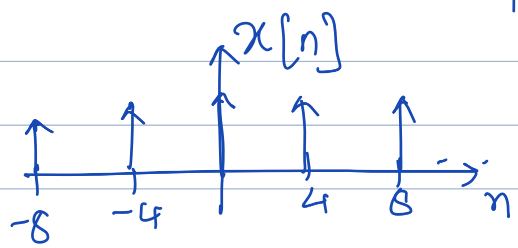
Let $h[n]$ be the impulse response.

$H(z) = \sum_{k=-\infty}^{\infty} h[n] z^n$

if input $x[n] = \sum_{K=0}^{N-1} a_k e^{jk\omega_0 n}$,

then output $y[n] = \sum_{k=0}^{N-1} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$

$x[n]$



, $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$.

Step1: Find FS coefficients of $x[n]$, (a_0, a_1, a_2, a_3)

Step2: Predict $y[n]$ from $x[n]$, & relate it with the given $y[n]$

Step3: determine $H(e^{jk\pi/2})$ for $k = 0, 1, 2, 3$

Step1: $a_k = \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\omega_0 n} = \frac{1}{4} \quad \forall k = 0, 1, 2, 3$

Step2: $y[n] = \sum_{k=0}^{3} \frac{1}{4} H(jk\frac{\pi}{2}) e^{jk\frac{\pi}{2}n}$

$= \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right)$

$= \frac{1}{2} e^{j\frac{5\pi}{2}n + j\frac{\pi}{4}} + \frac{1}{2} e^{-j\frac{5\pi}{2}n - j\frac{\pi}{4}}$

$$= \frac{1}{2} e^{j\left(2\pi + \frac{\pi}{2}\right)n} e^{j\pi/4} + \frac{1}{2} e^{-j\left(2\pi + \frac{\pi}{2}\right)n} e^{-j\pi/4}$$

$$= \underbrace{\frac{1}{2} e^{j\pi/4} e^{j\frac{\pi}{2}n}}_{\text{corresponds to } k=1} + \underbrace{\frac{1}{2} e^{-j\pi/4} e^{-j\frac{\pi}{2}n}}_{\text{corresponds to } k=-1 \text{ or } k=3}$$

$H\left(e^{j\frac{\pi}{2}k}\right) = 0$ for $k = 0, 2$

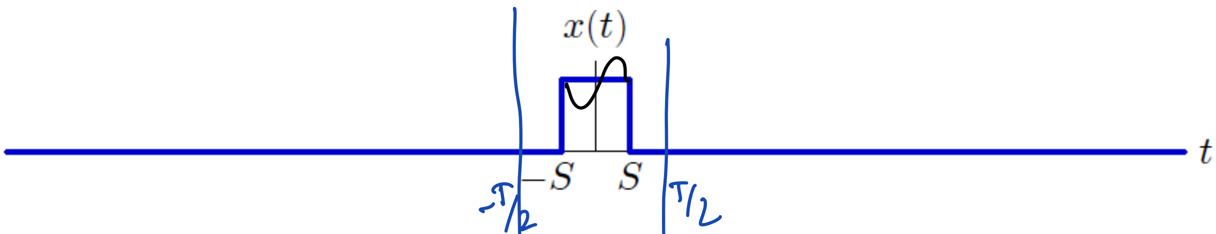
$H\left(e^{j\pi/2}\right) \cdot \frac{1}{4} = \frac{1}{2} e^{j\pi/4} \Rightarrow \underline{H\left(e^{j\pi/2}\right) = 2e^{j\pi/4}}$

$H\left(e^{j\frac{3\pi}{2}}\right) = H\left(e^{-j\pi/2}\right) = \frac{1}{2} e^{-j\pi/4}.$

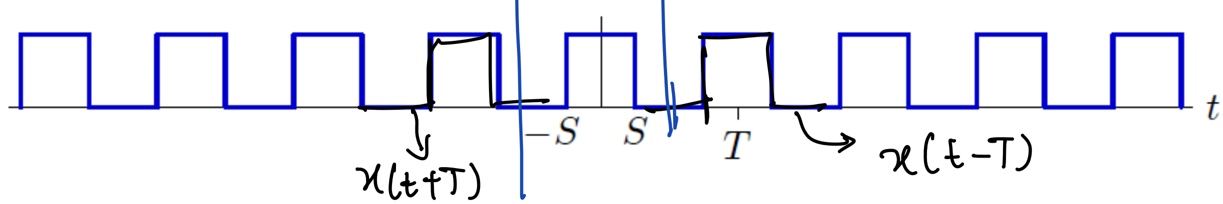
LECTURE 14: 20th August 26

Module D: Fourier Transform of Continuous-Time Signals

- We will now consider continuous-time signals that are not necessarily periodic.
- We start with a motivating example.
- Consider an aperiodic signal $x(t)$ that has finite duration, i.e., $x(t) = 0$ for $|t| > S$ and $x(t) = 1$ for $|t| \leq S$.
- Clearly, $x(t)$ is not a periodic signal.



- We construct a periodic signal $\tilde{x}(t)$ with period T:

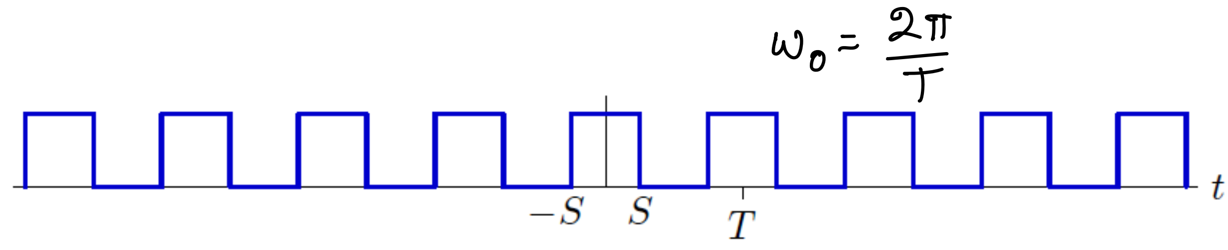


- It is easy to see that $\tilde{x}(t) := \sum_{k=-\infty}^{\infty} x(t + kT)$.

T is chosen to be > S

so that there is
no overlap of the non-zero parts of the signals.

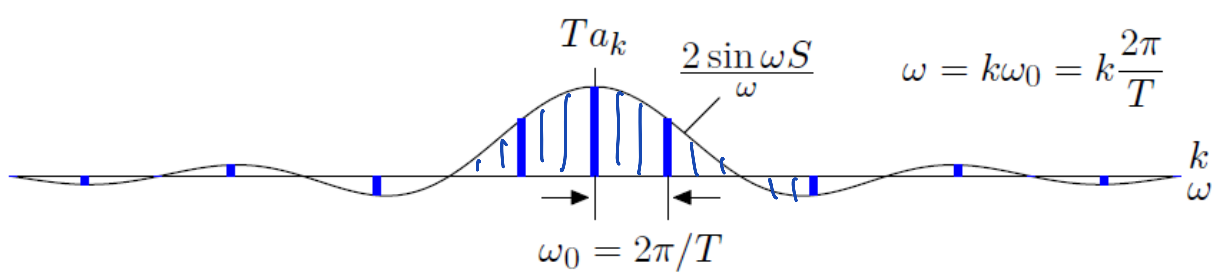
Fourier Transform: Motivation



- Since $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t+kT)$ is periodic, we may express $\tilde{x}(t)$ using Fourier Series:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t},$$

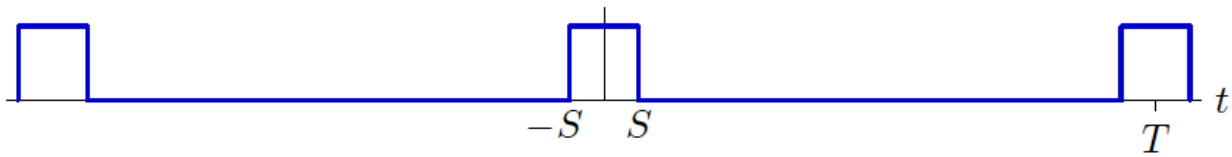
where
$$\begin{aligned} a_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi}{T}kt}dt \\ &= \frac{\sin(\frac{2\pi kS}{T})}{\pi k} = \frac{2}{T}\frac{\sin(\omega_0 Sk)}{\omega_0 k}, \\ a_0 &= \frac{2S}{T}, \qquad\qquad \omega_0 = \frac{2\pi}{T}. \end{aligned}$$



- Note that over the interval $[-\frac{T}{2}, \frac{T}{2}]$, $\tilde{x}(t)$ coincides with $x(t)$. Therefore,

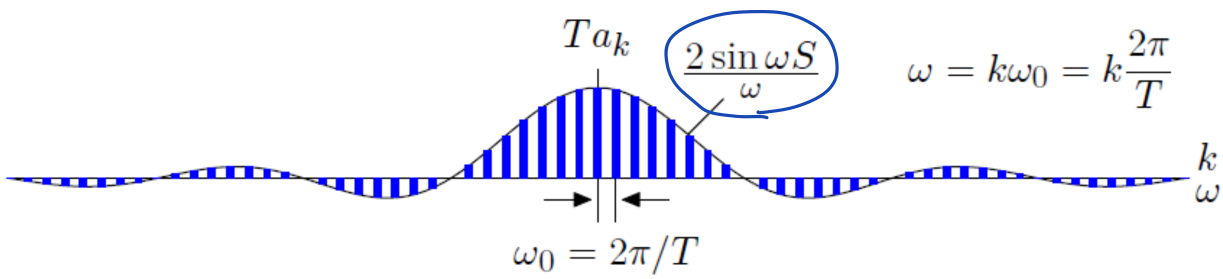
$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-j\frac{2\pi}{T}kt}dt.$$

Fourier Transform



What happens wehrn T increases?

$$a_k = \frac{\sin(\frac{2\pi k S}{T})}{\pi k} = \frac{2}{T}\frac{\sin(\omega S)}{\omega}$$



Fourier Transform

- Define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,$$

then

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-j\frac{2\pi}{T}kt}dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt \\ &= \frac{X(jk\omega_0)}{T}, \qquad \omega_0 = \frac{2\pi}{T}. \end{aligned}$$

- Substituting this in the synthesis equation, we get

$$\tilde{x}(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t} = \frac{\omega_0}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\,\omega_0$$

as $\omega_0 \to 0$,

$$\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$

$\Rightarrow$ phase shifted version of $e^{j\omega t}$

Fourier Transform

When the period $T \to \infty$ $(\omega_0 \to 0)$, the periodic signal $\tilde{x}(t)$ approaches $x(t)$. That is,

$\omega \in \mathbb{R}$

$$x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) e^{j\omega t} d\omega.$$

Hence, this is called the Synthesis Equation because we are gathering the Fourier domain information to reconstruct the time signal.

The Analysis Equation, because we are analyzing the time signal in the Fourier domain, is given by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Fourier Transform of an Aperiodic CT Signal

The Fourier Transform $X(j\omega)$ is given by the

- Analysis Equation:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

and the inverse Fourier Transform is given by the

- Synthesis Equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$X(j\omega)$ is called the spectrum of the signal and it represents the contribution of frequency ω to the signal $x(t)$.

$$= \frac{-1}{+(a+j\omega)} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

Let $x(t) = e^{-at} u(t)$, $\underline{\underline{a > 0}}$.

$$= \frac{-1}{a+j\omega} \left[0 - 1 \right]$$

$$\lim_{t \to \infty} \underline{\underline{e^{-at}}} e^{-j\omega t} = 0$$

$$= \frac{1}{a+j\omega}.$$

Example 1

Consider the signal $x(t) = e^{-at}u(t)$, for $a > 0$. Find its Fourier transform.

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} e^{-at-j\omega t}dt \\
&= \frac{-1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_{0}^{\infty} \\
&= \frac{1}{a+j\omega}. \quad = r(\omega)\, e^{j\theta(\omega)}
\end{aligned}
$$

To visualize $X(j\omega)$, we need to plot its magnitude and phase with respect to ω on separate plots. We will revisit this later.

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1 \qquad \forall\, \omega \in \mathbb{R}$$

Example 2

Consider the unit impulse signal $x(t) = \delta(t)$. Find its Fourier transform.

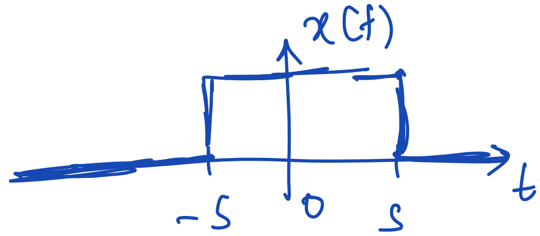
The Fourier Transform is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt \\ &= 1. \end{aligned}$$

In other words, the spectrum of the impulse signal has equal contribution from all frequencies.

Example 3

Find the Fourier transform of $x(t)$ which takes value 0 for $|t| > S$ and $x(t) = 1$ for $|t| \leq S$.



$$X(j\omega) = \int_{-S}^{S} e^{-j\omega t} \, d\omega$$

$$= \frac{1}{-j\omega}\left(e^{-j\omega S} - e^{j\omega S} \right)$$

$$= \frac{2\sin(\omega S)}{\omega}.$$

Example 4

Find the signal whose Fourier transform is given by:

$$X(j\omega) = \begin{cases} 1, & |\omega| \leq W, \\ 0, & |\omega| > W. \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} \, d\omega = \frac{1}{2\pi} \cdot \frac{1}{jt} \left(e^{jWt} - e^{-jWt} \right)$$

$$= \frac{1}{\cancel{2}\pi} \cdot \frac{1}{\cancel{j}t} \, \cancel{2j} \sin(Wt)$$

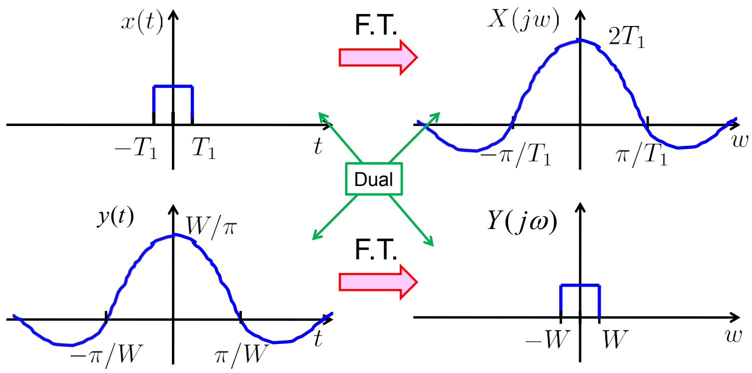
$$= \frac{\sin(Wt)}{\pi t}$$

Let $X(j\omega) = \delta(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi}, \quad \forall t$$

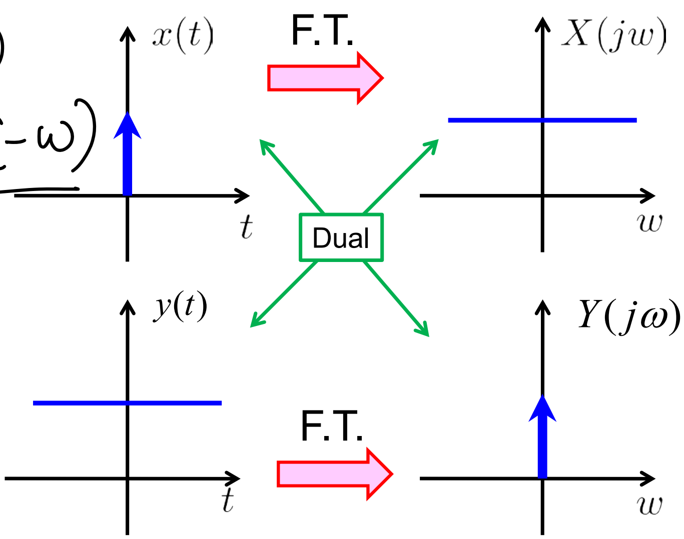
Duality



$$x(t) \longrightarrow \mathcal{F}(x(t)) = X(j\omega)$$

$$\tilde{x}(t) = X(jt)$$

$$\mathcal{F}(\tilde{x}(t)) = 2\pi x(-\omega)$$



Properties of Fourier Transform: Duality

- **Duality**: FT and IFT are *very similar*. Mathematically, for a signal $x(t)$

$$\boxed{FT(FT(x(t))) = 2\pi x(-\omega).}$$ → can we prove it ?

- Example: We know that $\delta(t) \longleftrightarrow 1$. What is the $IFT\{\delta(\omega)\}$?
  - By duality: it is $\frac{1}{2\pi}$.
- Example 2: We know that $e^{-t}u(t) \longleftrightarrow \frac{1}{1+j\omega}$. What is $FT\{\frac{1}{1+jt}\}$?
  - Using duality: $FT\{\frac{1}{1+jt}\} = 2\pi e^{\omega}u(-\omega)$.

$$FT(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = X(j\omega)$$

$$\underline{\underline{X(j\tau)}} = \int_{-\infty}^{\infty} x(t)\, e^{-jt\tau}\, dt$$

$$IFT\left(2\pi x(-\omega)\right) = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2\pi}\, x(-\omega)\, e^{+j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} x(-\omega)\, e^{j\omega t}\, d\omega.$$

let $\bar{\omega} = -\omega$

$$= \underbrace{\int_{-\infty}^{\infty} x(\bar{\omega})\, e^{-j\bar{\omega} t}\, d\bar{\omega}} = \underline{\underline{X(jt)}}$$

LECTURES 15 & 16: 21st August

Existence of Fourier Transform

A CT signal $x(t)$ has a Fourier Transform if all three of the following conditions are satisfied.

1. The signal is absolutely integrable:
$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty.$$
2. In any finite interval of time, $x(t)$ has bounded variation, i.e., it only have a finite number of maxima and minima during any finite interval of time.
3. In any finite interval of time, there are only a finite number of discontinuities and each of these discontinuities are finite.

- The above conditions are called **Dirichlet conditions**.
- The above conditions are only sufficient, not necessary.
- An alternative sufficient condition is that the signal has finite energy, i.e., it is square integrable:
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty.$$

Fourier Transform of a Periodic CT Signal

- Do all periodic signals have Fourier Transforms?
- We start backwards, start with a frequency domain signal and find its inverse Fourier Transform.
- Let the Fourier Transform of a signal $x(t)$ be given by

$$X(j\omega) = 2\pi\delta(\omega - \omega_0).$$

- Then,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega-\omega_0)e^{j\omega t}\,d\omega$$

$$= \underline{e^{j\omega_0 t}}.$$

- Thus, we have

$$e^{j\omega_0 t} \stackrel{F.T.}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$

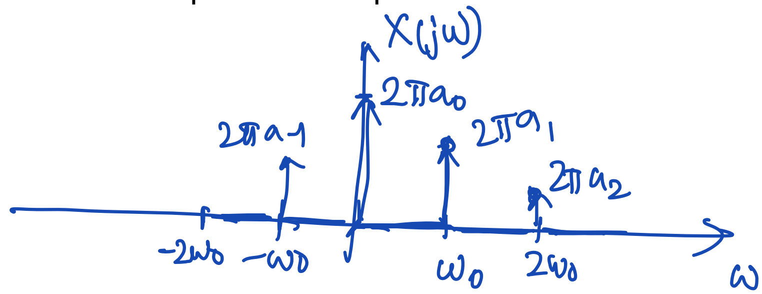
- For a general periodic signal:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

we have

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

- Thus, for a periodic signal, the FT consists of a sequence of impulse functions at multiples of ω_0 with height $2\pi a_k$.



Example 5

Find the Fourier transform of $x(t) = \cos(\omega_0 t)$.

Since this a sinusoidal signal, we can use the synthesis equation to obtain the Fourier series coefficients:

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2},$$

which implies $a_1 = a_{-1} = \frac{1}{2}$ and $a_k = 0$ otherwise.

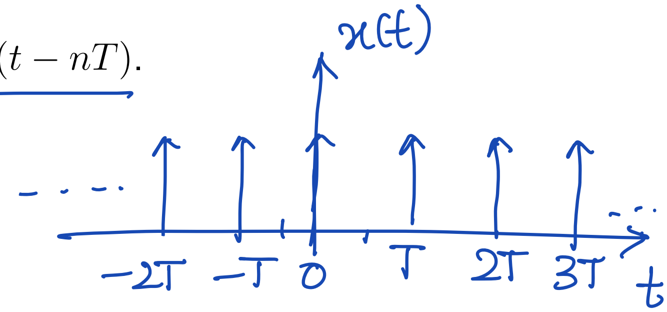
The Fourier transform is:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$

Example 6

Find the Fourier transform of $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

periodic with $\omega_0 = \frac{2\pi}{T}$.



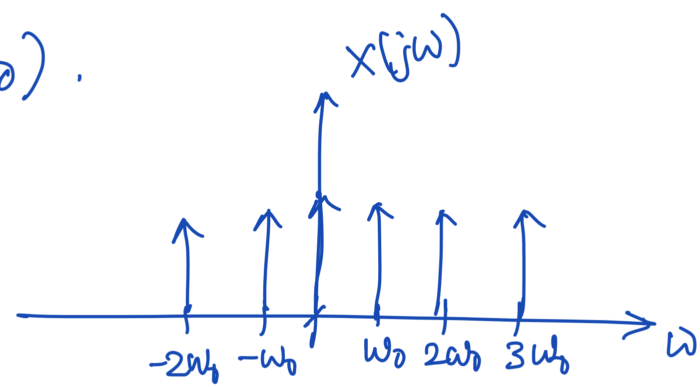
FS coefficients:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{jk\omega_0 t}\, dt = \frac{1}{T}, \quad \forall k.$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\frac{2\pi}{T}t}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi\, \delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$= \sum_{k=-\infty}^{\infty} \omega_0\, \delta(\omega - k\omega_0).$$



Let $x(t) \longleftrightarrow X(j\omega)$
$y(t) \longleftrightarrow Y(j\omega)$

Properties of Continuous-time Fourier Transform

$z(t) = \alpha x(t) + \beta y(t)$

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt$$

$$= \alpha \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + \beta \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

- Notation: $x(t) \xleftrightarrow{F.T.} X(j\omega)$
- We also write

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

and

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$$

as alternative notations.

- **Linearity:** FT and IFT are both linear:

$$\alpha x(t) + \beta y(t) \xleftrightarrow{F.T.} \alpha X(j\omega) + \beta Y(j\omega).$$

, $\forall \alpha, \beta \in \mathbb{C}$

IFT: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \Rightarrow x(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{\boxed{X(j\omega) e^{-j\omega t_0}}} \, e^{j\omega t} d\omega$$

- **Time-Shifting:**

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega).$$

This holds as:

$$x(t-t_0) = \frac{1}{2\pi} \int_{\infty}^{\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega = \frac{1}{2\pi} \int_{\infty}^{\infty} \underbrace{\left(e^{-j\omega t_0} X(j\omega)\right)}_{\text{FT of } x(t-t_0)} e^{j\omega t} d\omega.$$

- **Frequency-shift:**

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(j(\omega - \omega_0)).$$

$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$, let $t = \tau - t_0$, $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau - t_0) e^{-j\omega(\tau - t_0)} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau - t_0) e^{-j\omega\tau} \left(e^{j\omega t_0}\right) d\tau$$

$$\Rightarrow e^{-j\omega t_0} X(j\omega) = \underbrace{\int_{-\infty}^{\infty} x(\tau - t_0) e^{-j\omega\tau} d\tau}_{FT(x(\tau - t_0))}$$

Let $y(t) = e^{j\omega_0 t} x(t)$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(j(\omega-\omega_0))$$

Properties of Fourier Transform and Inverse Fourier Transform cont.

- Conjugate and Conjugate Symmetry:

$$x(t) \longleftrightarrow X(j\omega) \implies x^*(t) \longleftrightarrow X^*(-j\omega).$$

In particular for **real-valued** signals:

$$x(t) \longleftrightarrow X(j\omega) \implies x(t) \longleftrightarrow X^*(-j\omega).$$

This implies the **Conjugate Symmetry Property:** $\boxed{X^*(-j\omega) = X(j\omega).}$

- In other words, for any ω, $X(j\omega)$ has the same magnitude as $X(-j\omega)$, and the phase of $X(j\omega)$ is negative of the phase of $X(-j\omega)$.

- If $x(t)$ is even, show that $X(-j\omega) = X(j\omega)$.

- If $x(t)$ is real-valued and even, then $X(j\omega)$ is real-valued and even.

- If $x(t)$ is real-valued and odd, then $X(j\omega)$ is purely imaginary and odd.

$$x(t) = x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{-j\omega t} d\omega, \qquad \bar{\omega} := -\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(-j\bar{\omega}) e^{j\bar{\omega} t} d\bar{\omega}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow (x(t))^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} (X(j\omega))^* e^{-j\omega t}\, d\omega \quad , \quad \text{let } \bar{\omega} = -\omega$$

$$d\omega = -d\bar{\omega}$$

$$= \frac{1}{2\pi} \int_{\infty}^{-\infty} (X(-j\bar{\omega}))^* e^{j\bar{\omega} t} (-) d\bar{\omega}$$

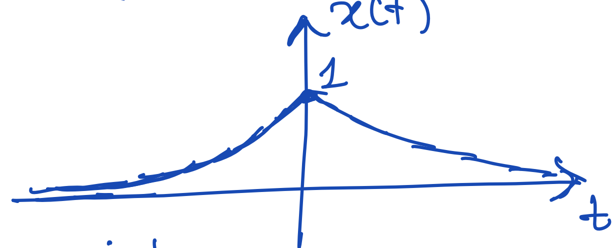
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (X(-j\bar{\omega}))^* e^{j\bar{\omega} t}\, d\bar{\omega}$$

$$\Rightarrow (x(t))^* \longleftrightarrow (X(-j\omega))^*$$

Example 7

real and even

Verify the above statements for $x(t) = e^{-a|t|}$ where a is a positive real number.



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{-a|t|} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-a|t|} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a-j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^{0} + \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a+j\omega+a-j\omega}{(a+j\omega)(a-j\omega)} = \frac{2a}{a^2+\omega^2}.$$

real and even function of ω.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[j\omega X(j\omega) \right] e^{j\omega t} d\omega$$

Properties of Fourier Transform and Inverse Fourier Transform cont.

- **Differentiation**: Suppose that $x(t)$ is a differentiable signal. Then:

$$\underline{\frac{dx(t)}{dt} \longleftrightarrow j\omega X(j\omega).}$$

$$x(at) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega a t}\, d\omega \,, \quad \bar{\omega} = \omega a \Rightarrow \omega = \frac{\bar{\omega}}{a}, \quad d\omega = \frac{d\bar{\omega}}{a}$$

let $a > 0$:

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j\frac{\bar{\omega}}{a}\right) e^{j\bar{\omega}t}\, \frac{d\bar{\omega}}{a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a} X\left(j\frac{\bar{\omega}}{a}\right) \cdot e^{j\bar{\omega}t}\, d\bar{\omega}$$

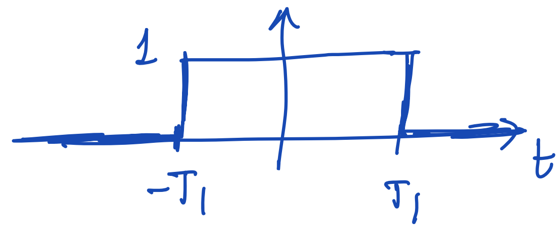
- **Time and Frequency Scaling:**

$$x(at) \longleftrightarrow \frac{1}{|a|}X(j\frac{\omega}{a}).$$

Implication: $x(-t) \longleftrightarrow X(-j\omega)$. Shrinking a time-domain signal expands it in the frequency domain.

If $a < 0$, $\bar{\omega}$ limit will be from ∞ to $-\infty$, which will result in $\frac{1}{|a|}$.

Key CT FT Pairs

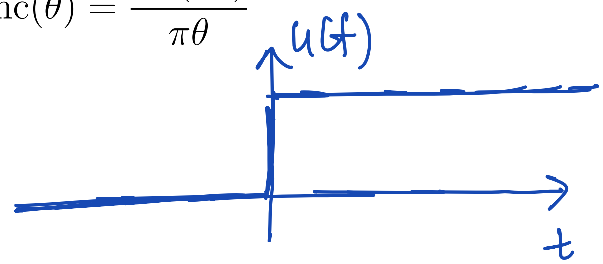


- $e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$

- $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$ for a with $Re(a) > 0$

- $\text{rect}_{T_1}(t) \longleftrightarrow \frac{2\sin(\omega T_1)}{\omega} = 2T_1 \,\text{sinc}(\frac{T_1}{\pi}\omega)$ where

$$\text{rect}_{T_1}(t) = \begin{cases} 1 & |t| \leq T_1 \\ 0 & \text{otherwise} \end{cases}, \qquad \text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$



- $\delta(t) \longleftrightarrow 1$

- $u(t) \longleftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$
  - We cannot use analysis equation.
  - We see $u(t)$ as $\lim_{a\to 0} u(t)e^{-at}$
  - $u(t)e^{-at} \longleftrightarrow \frac{1}{a+j\omega}$ and:

$$\frac{1}{a+j\omega} = \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

  - $\lim_{a\to 0} \frac{-j\omega}{a^2+\omega^2} = \frac{1}{j\omega}$
  - For the first term: $\lim_{a\to 0} \frac{a}{a^2+\omega^2} = \infty$ for $\omega = 0$ and otherwise, it is zero
  - On the other hand, $\int_{-\infty}^{\infty} \frac{a}{a^2+\omega^2} d\omega = \tan^{-1} \frac{\omega}{a}\Big|_{-\infty}^{\infty} = \pi$
  - Therefore, $\lim_{a\to 0} \frac{a}{a^2+\omega^2} = \pi\delta(\omega)$

Properties of Fourier Transform and Inverse Fourier Transform: Example

- Example: We know that $\mathrm{rect}_{T_1}(t) \longleftrightarrow 2T_1 \,\mathrm{sinc}(\frac{T_1}{\pi}\omega)$. Using this, calculate the FT of the following signal.

$$x(t) = \begin{cases} -1 & t \in [-1, 0] \\ 1 & t \in (0, 1] \\ 0 & \text{otherwise} \end{cases}.$$

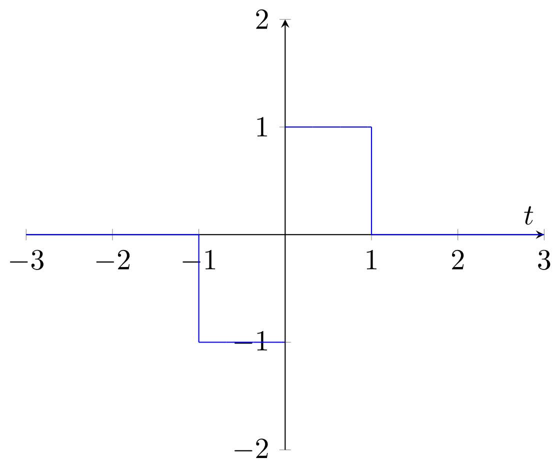
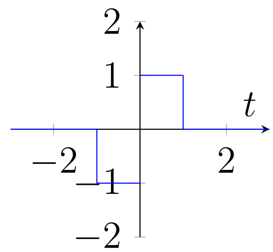


Figure 1: Plot of $x(t)$.

Properties of Fourier Transform and Inverse Fourier Transform: Example



Solution: Note that $x(t) = \mathrm{rect}_1(t) - 2\,\mathrm{rect}_{0.5}(t+0.5)$ (why?).

$$\begin{aligned} X(j\omega) &= FT\{\mathrm{rect}_1(t))\} - 2FT\{\mathrm{rect}_{0.5}(t+0.5)\} \\ &= 2\,\mathrm{sinc}(\frac{\omega}{\pi}) - 2e^{j\frac{\omega}{2}}(2\frac{1}{2}\,\mathrm{sinc}(\frac{\omega}{2\pi})) \\ &= 2\,\mathrm{sinc}(\frac{\omega}{\pi}) - 2e^{j\frac{\omega}{2}}\,\mathrm{sinc}(\frac{\omega}{2\pi}). \end{aligned}$$

Properties of Fourier Transform cont.

- **Parseval's Theorem**: applicable for signals with finite energy.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Let $z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

$$Z(j\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau\right] e^{-j\omega t}\, e^{j\omega\tau}\, e^{-j\omega\tau}\, dt$$

- **Convolution**:

$$x(t) * y(t) \longleftrightarrow X(j\omega)Y(j\omega)$$

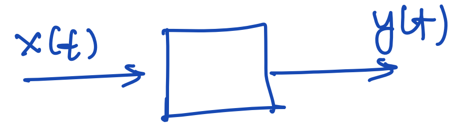
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right) e^{-j\omega(t-\tau)}\, e^{-j\omega\tau}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega(t-\tau)} d(t-\tau)$$

$$= \underline{X(j\omega)Y(j\omega)}.$$

For an integrator, impulse response of $\delta(t)$ is $u(t)$.

Properties of Fourier Transform cont.



- **Integration**: Let $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Then:

$$y(t) \longleftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).$$

$u(t-\tau) = 1$ when $t-\tau > 0$

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\,d\tau.$$

$$Y(j\omega) = X(j\omega)\,U(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

- **Multiplication**:

$$x(t)y(t) \longleftrightarrow \frac{1}{2\pi}X(j\omega) * Y(j\omega)$$ $\rightarrow$ left as exercise.

If $x(t) \longleftrightarrow X(j\omega)$, then $tx(t) \longleftrightarrow j\frac{d}{d\omega}X(j\omega)$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} x(t)\,(-jt)\,e^{-j\omega t}\,dt$$

$$j\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} x(t)t\,\underbrace{(-j \times j)}_{=1.}\,e^{-j\omega t}\,dt$$

Properties of FT

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Fourier Transform and LTI Systems

- We know:

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

where $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$ is the frequency response of the system.

- Interestingly: $h(t) \longleftrightarrow H(j\omega)$
- Does $H(j\omega)$ exist for all LTI systems?
- Therefore:

$$e^{j\omega t}d\omega \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}d\omega$$

- Hence:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \longrightarrow \boxed{h(t)} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

- As a result: $y(t) = x(t) * h(t) \longleftrightarrow X(j\omega)H(j\omega) = Y(j\omega)$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \underbrace{H(j\omega)}_{\text{frequency response}}$$

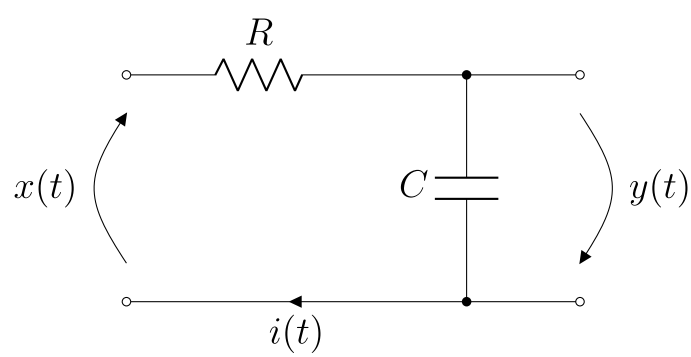
frequency response → only a property of the system, and independent of the input being applied.

If system is not LTI, $y(t) \neq x(t) * h(t)$,

hence $\frac{Y(j\omega)}{X(j\omega)}$ will depend on the input-output pair.

CT LTI Systems Described by Differential Equations

- Differential equations provide a bridge between math (engineering) and physical world
- Almost any engineering (and even many of the economical) systems behavior is modeled by ODE



- Example 1: RLC networks:
- In this case: $x(t) = RC\frac{dy}{dt} + y(t)$

assume that initial conditions are zero.

apply Fourier transform to both sides.

$$X(j\omega) = Y(j\omega) + j\omega\, Y(j\omega) RC = Y(j\omega)\left[1 + j\omega RC\right]$$

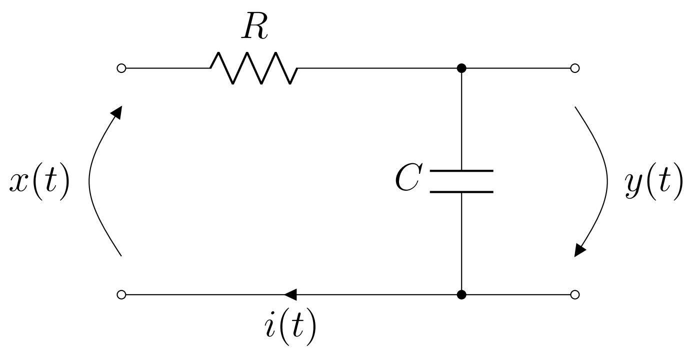
$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + j\omega RC}$$

→ independent of the input-output pair that we encounter.

↑ intrinsic to the system.

There are of course LTI systems whose impulse response $h(t)$ is such that Fourier transform of $h(t)$ is not defined. Such systems don't have a well-defined frequency response.

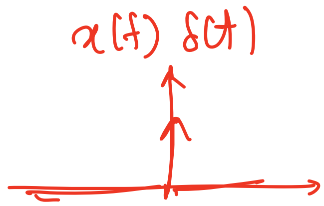
LTI Systems Described by Differential Equations

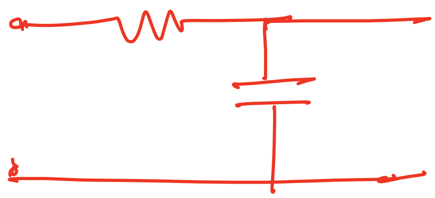


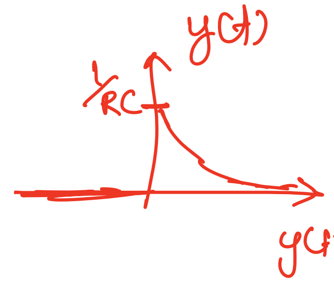
- Example 1: RLC networks:
- In this case: $x(t) = RC\frac{dy}{dt} + y(t)$
- Therefore: $X(j\omega) = RC(j\omega)Y(j\omega) + Y(j\omega) \Rightarrow H(j\omega) = \frac{1}{1+RC(j\omega)} = \frac{1}{RC}\frac{1}{\frac{1}{RC}+(j\omega)}$

$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$

- Hence: $h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$







LTI Systems Described by Differential Equations

- Let S be a **stable** LTI system described by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Apply the CTFT to both sides

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}.$$

- From the **Linearity Property** and **Differentiation Property**

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

- By the **Convolution Property**, the **frequency response** is

$$H(j\omega) = Y(j\omega)/X(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

- Question: Can we get the system **impulse response** from Table? Yes, through partial fraction!

Therefore, it is important to develop techniques to find inverse fourier transform of H(jω) that is a ratio of polynomials.

Partial Fraction

- Suppose $H(j\omega)$ is a **rational function** or **ratio of polynomials**
- Apply **Partial Fraction Expansion** to write $H(j\omega)$ in a form that allows us to determine $h(t)$ from Table 4.2
- **Example:** Stable LTI system described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Then

$$(j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$$

$$\begin{aligned} H(j\omega) &= \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} \\ &= \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \end{aligned}$$

Rewrite $H(j\omega)$ as

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3}$$

$$h(t) = A e^{-t} u(t) + B e^{-3t} u(t)$$

Partial Fraction Example cont.

- We got

$$H(j\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3}$$

- Put over a common denominator

$$\begin{aligned} H(j\omega) &= \frac{A(j\omega + 3) + B(j\omega + 1)}{(j\omega + 1)(j\omega + 3)} \\ &= \frac{(A + B)j\omega + (3A + B)}{(j\omega + 1)(j\omega + 3)} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \end{aligned}$$

- Equate real and imaginary parts of the numerators:
  Real part: $3A + B = 2$
  Imaginary part: $A + B = 1$

- Solving for A and B

$$A = \frac{1}{2} \text{ and } B = \frac{1}{2} \Rightarrow$$

$$H(j\omega) = \frac{1}{2}\left(\frac{1}{j\omega + 1}\right) + \frac{1}{2}\left(\frac{1}{j\omega + 3}\right) \Rightarrow h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$