LECTURE - 11, 13th August

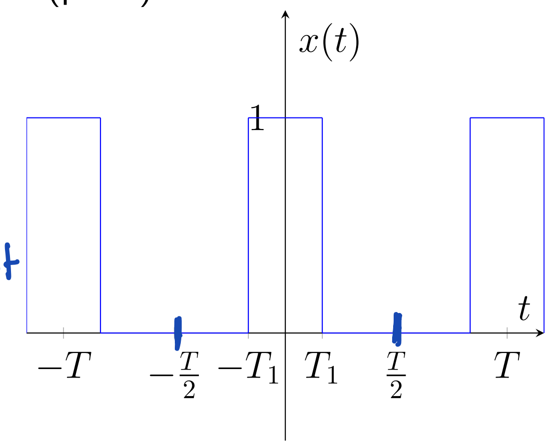
CT Fourier Series: Periodic Pulse

- Example: What is the Fourier series of the periodic rectangular wave (pulse)?

Fundamental period $= T$

$\omega_0 = \dfrac{2\pi}{T}$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\underbrace{e^{-j(0)\omega_0 t}}_{=1}\,dt$$



$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

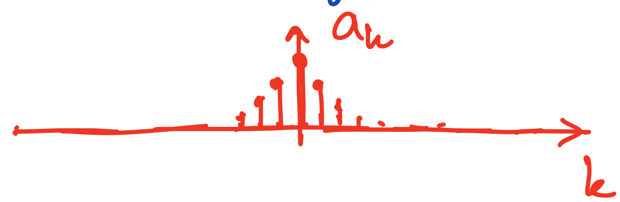
$$x(t) = \begin{cases} 1 & \text{if } |t| \le T_1 \\ 0 & \text{if } T_1 < |t_1| \le \frac{T}{2} \end{cases}.$$ for $t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$ and is periodic with fn. period T.

$$= \frac{1}{T}\int_{-T_1}^{T_1} 1\cdot dt = \frac{2T_1}{T}.$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} 1\cdot e^{-jk\omega_0 t}\,dt = \frac{1}{T}\cdot\frac{1}{(-jk\omega_0)}\, e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{1}{T}\cdot\frac{1}{-jk\omega_0}\left[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right]$$

$$= \frac{1}{T}\,\frac{-2j\sin(k\omega_0 T_1)}{-jk\omega_0} = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}.$$



CT Fourier Series: Impulse Train

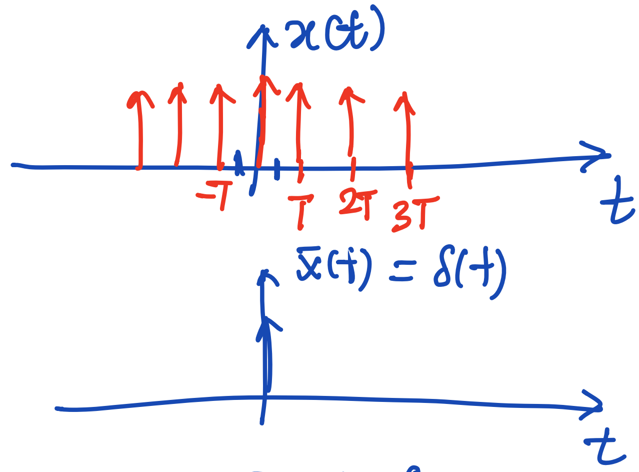
- Example: What is the Fourier series of the impulse train $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$?

Fundamental period: T

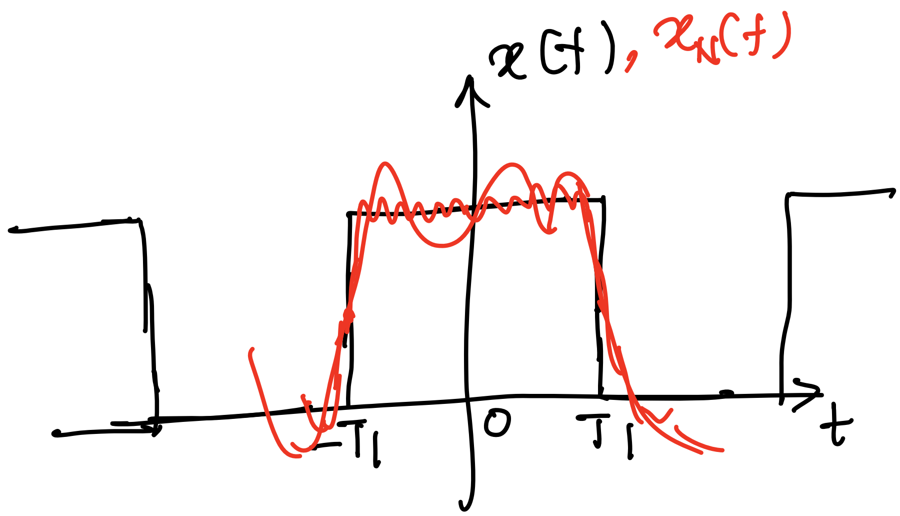
$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T} \cdot \quad \forall k \in \mathbb{Z}$$



$x(t) = \bar{x}(t)$ for $t \in [-T/2, T/2]$



$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Fourier Series: Convergence Meaning

- What does $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ mean for a periodic $x(t)$?
- Define $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ and $e_N(t) = x(t) - x_N(t)$.
- Let's look at $x_N(t)$ for various values of N

> Let $x(t)$ be a signal and suppose that $x(t)$ has finite energy over $[0, T]$. Then,
> $$\lim_{N\to\infty} \int_0^T |e_N(t)|^2 dt = 0.$$
> Moreover, for all $t \in [0, T]$, $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_n t} = \frac{1}{2}(x^+(t) + x^-(t))$, where $x^+(t)$ is the right-limit of x and $x^-(t)$ is the left-limit of x at t.

- Least Square Approximation Interpretation.

Suppose I define $\bar{e}_N(t) = x(t) - \sum_{k=-N}^{N} b_k e^{jk\omega_0 t}$

To find best possible approximation of $x(t)$, one approach is to find $(b_{-N}, \ldots b_0, \ldots b_N)$ that minimize the square of the error signal.

$$\min_{b_{-N}, b_{-N+1}, \ldots b_0, b_1 \ldots b_N} \varphi(b_{-N}, b_{-N+1}, \ldots b_0, \ldots b_N) := \int_0^T \Big(x(t) - \sum_{k=-N}^{N} b_k e^{jk\omega_0 t}\Big)^2 dt$$

The optimal set of coefficients are nothing but the Fourier series coefficients, i.e., $b_k^* = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$

Existence of Fourier Series

- When does a CT periodic signal have a Fourier Series representation?

A CT periodic signal $x(t)$ has a Fourier Series representation if all three of the following conditions are satisified.

1. The signal is absolutely integrable over one period:
$$\int_{t_0}^{t_0+T} |x(t)|dt < \infty.$$
2. In any finite interval of time, $x(t)$ has bounded variation, i.e., it only have a finiite number of maxima and minina during any single period of the signal.
3. In any finite interval of time, there are only a finite number of discontinuities and each of these discontinuities are finite.

- The above conditions are called **Dirichlet conditions**.
- The first conditions guarantees that the Fourier series coefficients are finite.
- **Gibb's Phenomenon**:
  - If $x(t)$ is continuous at some τ, then its value is equal to the value of its Fourier series representation (FSR).
  - If $x(t)$ is discontinuous at some τ, then the value of its Fourier series representation is equal to $0.5(x(\tau^+) + x(\tau^-))$.
  - At a point of discontinuity, the FSR typically shows an overshoot compared to the original signal. As the number of terms in the FSR increases, the width of the overshoot decreases, but the height remains unchanged.

will be illustrated for the next class.

Practice problems from texbook

Chapter 1: 1.11, 1.13, 1.20, 1.26, 1.28, 1.31
1.32, 1.34, 1.40, 1.41, 1.42, 1.43

Chapter 2: 2.14, 2.15, 2.16, 2.19, 2.28, 2.31
2.38, 2.44, 2.46, 2.48, 2.50

Chapter 3: 3.4, 3.13, 3.18, 3.19, 3.21
3.26, 3.40, 3.44, 3.48,
3.63, 3.64, 3.65, 3.69

Properties of Fourier Series

Let $x(t)$ and $y(t)$ have fundamental period T.

- Linearity: If $x(t) \longleftrightarrow \{a_k\}$ and $y(t) \longleftrightarrow \{b_k\}$, then $\alpha x(t) + \beta y(t) \longleftrightarrow \{\alpha a_k + \beta b_k\}$
- Time-shift: If $x(t) \longleftrightarrow \{a_k\}$ then $x(t - t_0) \longleftrightarrow \{a_k e^{-jk\omega_n t_0}\}$
- Time-reversal: If $x(t) \longleftrightarrow \{a_k\}$ then $x(-t) \longleftrightarrow \{a_{-k}\}$
- Time-scaling: If $x(t) \longleftrightarrow \{a_k\}$ then $x(\alpha t) \longleftrightarrow \{a_k\}$!
- Multiplication: If $x(t) \longleftrightarrow \{a_k\}$ and $y(t) \longleftrightarrow \{b_k\}$, then $x(t)y(t) \longleftrightarrow \{a_k * b_k\}$ [assuming that $x(t)y(t)$ have same fundamental period as $x(t)$ & $y(t)$].
- Conjugation: If $x(t) \longleftrightarrow \{a_k\}$ then $x^*(t) \longleftrightarrow \{a^*_{-k}\}$
- PLEASE consult **Table 3.1** of the book for more properties of Fourier series of CT-periodic signals.

For any signal

$$z(t) = \sum_{k=-\infty}^{\infty} \underbrace{(\equiv)}_{k\text{-th Fourier series coefficient}} e^{+jk\omega_0 t}$$

Linearity: we know

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{+jk\omega_0 t}$$

$$\alpha x(t) + \beta y(t) = \alpha \left(\sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t}\right) + \beta \left(\sum_{k=-\infty}^{\infty} b_k e^{+jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{(\alpha a_k + \beta b_k)}_{k\text{th Fourier Series coefficient of } \alpha x(t) + \beta y(t).} e^{+jk\omega_0 t}$$

Properties of Fourier Series Cont.

<u>Time-Shifting:</u> $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t}$

$$\Rightarrow x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_0}\right) e^{jk\omega_0 t}$$

kth Fourier Series representation of $x(t-t_0)$.

<u>Time-Scaling:</u> $y(t) := x(at)$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (at)}$$

$y(t)$ is periodic with Fundamental period T/a.

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk(a\omega_0)t}$$

$\hat{\omega}_0 = \frac{2\pi a}{T} = a\omega_0$

Thus, k-th Fourier series coefficient of $y(t)$ is also a_k, but it corresponds to a different frequency component.

LECTURE 12 & 13 - 14th August

If $x(t)$ is even, then $x(t) = x(-t) \Rightarrow \boxed{a_k = a_{-k}}$

Properties of Fourier Series Cont.

Time-Reversal: When $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, then

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t} = \sum_{\bar{k}=-\infty}^{\infty} (a_{-\bar{k}}) e^{j\bar{k}\omega_0 t} \quad \text{if } \bar{k} = -k$$

$\Rightarrow$ Kth Fourier series coefficient of $x(-t)$ is the $-k$-th F.S. coefficient of $x(t)$.

Multiplication: We know $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, $y(t) = \sum_{\bar{k}=-\infty}^{\infty} b_{\bar{k}} e^{j\bar{k}\omega_0 t}$

$$x(t)y(t) = \sum_{k=-\infty}^{\infty} \sum_{\bar{k}=-\infty}^{\infty} a_k b_{\bar{k}} e^{jk\omega_0 t} e^{+j\bar{k}\omega_0 t}$$

Let us now determine l-th FS coefficient of $x(t)y(t)$.

$$c_l = \frac{1}{T} \int_0^T x(t)y(t) e^{-jl\omega_0 t} dt$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{\bar{k}=-\infty}^{\infty} a_k b_{\bar{k}} \underbrace{\int_0^T e^{jk\omega_0 t} e^{j\bar{k}\omega_0 t} e^{-jl\omega_0 t} dt}_{= \int_0^T e^{j(k+\bar{k}-l)\omega_0 t} dt \;=\; \begin{cases} T, & \text{if } k+\bar{k}=l \\ 0, & \text{otherwise.} \end{cases}}$$

$$= \sum_{k=-\infty}^{\infty} a_k b_{l-k}$$

which is nothing but convolution sum of two discrete-time signals $\{a_k\}_{k=-\infty}^{\infty}$ and $\{b_k\}_{k=-\infty}^{\infty}$.

Conjugation: If $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, then

$$(x(t))^* = \sum_{k=-\infty}^{\infty} (a_k)^* e^{-jk\omega_0 t}$$

$$= \sum_{\bar{k}=-\infty}^{\infty} b_{\bar{k}} e^{j\bar{k}\omega_0 t},$$

then $b_{\bar{k}} = (a_{-k})^*$.

If the signal is real, $x(t) = x(t)^*$

$$\Rightarrow a_k = (a_{-k})^*$$

Then, we can write $x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{-1} (a_k)^* e^{+jk\omega_0 t}$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[a_k e^{jk\omega_0 t} \right]$$

Suppose $a_k = r_k e^{j\theta_k}$,

Then $x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[r_k e^{j(k\omega_0 t + \theta_k)} \right]$

$$= a_0 + \sum_{k=1}^{\infty} 2 r_k \cos(k\omega_0 t + \theta_k).$$

Differentiation: Let $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$,

$$\frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} a_k (jk\omega_0)\, e^{jk\omega_0 t}$$

K-th Fourier series coefficient of $\frac{d}{dt}x(t)$.

$x(t) = \cos\left(\frac{\pi}{2}(t-1)\right)$, $T_0 = 4$

$\omega_0 = \frac{\pi}{2}$.

Practice Problem

Determine the Fourier series coefficients of the signal $x(t) = \cos\left(\frac{\pi}{2}(t-1)\right)$.

a. The signal is not periodic!

b. $a_1 = a_{-1} = -j/2$, all other coefficients 0

c. $a_1 = -j/2$, $a_{-1} = j/2$, all other coefficients 0

d. $a_1 = j/2$, $a_{-1} = -j/2$, all other coefficients 0

We have

$$\begin{aligned} x(t) = \cos\left(\frac{\pi}{2}(t-1)\right) &= \frac{1}{2}e^{j\frac{\pi}{2}(t-1)} + \frac{1}{2}e^{-j\frac{\pi}{2}(t-1)} \\ &= \frac{1}{2}e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{2}t} + \frac{1}{2}e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{2}t} \\ &= \frac{-j}{2}e^{j\frac{\pi}{2}t} + \frac{j}{2}e^{-j\frac{\pi}{2}t} \end{aligned}$$

Clearly $\omega_0 = \frac{\pi}{2}$. So $a_1 = -j/2, a_{-1} = j/2$ and $a_k = 0$ for all other k.

Alternatively:
$y(t) = \cos\left(\frac{\pi}{2}t\right) \xleftrightarrow{\text{F.S.}} a_1 = a_{-1} = 1/2, a_k = 0, \text{all other } k.$
Time-shifting property says:
$x(t) = y(t-t_0) \xleftrightarrow{\text{F.S.}} b_k = e^{-jk\omega_0 t_0}a_k$. Here, $t_0 = 1$, so $e^{-jk\omega_0 t_0} = e^{-jk\frac{\pi}{2}} = (-j)^k$.

Q) Let $x(t)$ has FS coefficients $a_k = \begin{cases} 2 & \text{when } k=0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{else.} \end{cases}$

Determine if i) $x(t)$ is real → NO

ii) $x(t)$ is even → Yes.

iii) $\frac{d}{dt}x(t)$ is real

iv) $\frac{d}{dt}x(t)$ is even.

$\frac{d}{dt}x(t) \longleftrightarrow \left\{ j\left(\frac{1}{2}\right)^{|k|} jw_0 k \right\}$

$a_{-k} = j\left(\frac{1}{2}\right)^{|k|}$

$(a_{-k})^* = -j\left(\frac{1}{2}\right)^{|-k|}$

$= \left(\frac{1}{2}\right)^{|k|}(-j)$

$\neq a_k$.

If a signal is real $a_k = (a_{-k})^*$

$\Rightarrow$ Signal is not real.

If signal is even, $a_k = a_{-k}$, which is true for $x(t)$

k-th FS coefficient of $\frac{d}{dt}x(t)$ is $\underline{-\left(\frac{1}{2}\right)^{|k|} k w_0} = b_k$

here $b_k \neq b_{-k}$ $\Rightarrow$ $\frac{d}{dt}x(t)$ is not a even signal.

$b_k = (b_{-k})^*$ $\Rightarrow$ $\frac{d}{dt}x(t)$ is not a real signal.

Parseval's Relation

- For a periodic signal $x(t)$ we have $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_n t}$.

For a periodic signal $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_n t}$, we have:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Parseval Relation: Average power of a signal is equal to sum of the energy of its harmonic components!

$$\frac{1}{T}\int_0^T (x(t))^* x(t)\, dt = \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{\infty} (a_k)^* e^{-jk\omega_0 t}\right)\left(\sum_{\bar{k}=-\infty}^{\infty} a_{\bar{k}} e^{j\bar{k}\omega_0 t}\right) dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{\bar{k}=-\infty}^{\infty} (a_k)^* a_{\bar{k}} \underbrace{\int_0^T e^{-jk\omega_0 t} e^{j\bar{k}\omega_0 t}\, dt}_{= \begin{cases} 0 & \text{when } k \neq \bar{k} \\ T & \text{when } k = \bar{k} \end{cases}}$$

$$= \sum_{k=-\infty}^{\infty} (a_k)^* a_k$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$

if $x(t)$ is real,
$a_1 = (a_{-1})^*$

Problem: Let $x(t)$ be a real periodic signal with fundamental period $T=4$; $\omega_0 = \frac{2\pi}{4} = \pi/2$

$x(t) \leftrightarrow \{a_k\}$. $a_k = 0$ for all $|k| > 1$

Let $y(t)$ be a signal with FS coefficients $b_k = e^{-j\pi/2 k} a_{-k}$

$y(t)$ is odd.

Finally, $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2} = |a_0|^2 + |a_1|^2 + |a_{-1}|^2 = 2a_1^2 \Rightarrow |a_1| = \frac{1}{2}$

Determine $x(t) = a_0 + a_1 e^{j\frac{\pi}{2}t} + a_{-1} e^{-j\frac{\pi}{2}t}$

For $y(t)$, $b_0 = a_0$, $b_1 = a_{-1} e^{-j\pi/2}$, $b_{-1} = a_1 e^{j\pi/2}$

Since $y(t)$ is odd, $b_1 = -b_{-1} \Rightarrow a_{-1} e^{-j\pi/2} = -a_1 e^{j\pi/2}$

$\Rightarrow a_{-1} = -a_1 e^{j\pi} = a_1$

$y(t)$ is odd

$$b_0 = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\,dt = \frac{1}{T}\int_{-T/2}^{0} y(t)dt + \frac{1}{T}\int_{0}^{T/2} y(t)dt$$

$$= 0 = a_0.$$

Thus, $a_1 = a_{-1} = (a_{-1})^*$ & $|a_1| = \frac{1}{2}$

$\Rightarrow a_1 = \frac{1}{2} = a_{-1}$. , or $a_1 = a_{-1} = -\frac{1}{2}$.

either $x(t) = \frac{1}{2} e^{j\frac{\pi}{2}t} + \frac{1}{2} e^{-j\frac{\pi}{2}t} = \cos\left(\frac{\pi}{2}t\right)$

or $x(t) = -\cos\left(\frac{\pi}{2}t\right)$.

Frequency Response

- Remember:

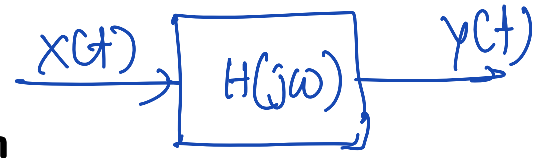
$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

- When $s = j\omega$ (we are only interested in harmonic exponentials), $H(s) = H(j\omega)$ is called the **frequency response** of the system.

- Recap: For a periodic signal $x(t)$:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

K-th Fourier series coefficient of $y(t)$ is $a_K H(jK\omega_0)$



Practice Problem

Consider an ideal CT low-pass filter S whose frequency response is

$$H(j\omega) = \begin{cases} 1, & |\omega| \le 10 \\ 0, & |\omega| > 10. \end{cases}$$

$\omega_0 = \frac{2\pi}{T} = \underline{3}$

Suppose the input to this filter is a periodic signal $x(t)$.
$x(t)$ has fundamental period $T = 2\pi/3$ and Fourier series a_k.
We find that the output $y(t)$ is identical to $x(t)$, i.e., $y(t) = x(t)$.

For what values of k must a_k necessarily be equal to 0.

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk3t}$

a. All values of k.

b. $|k| \le 3$

c. $|k| \ge 3$

d. $|k| \ge 4$

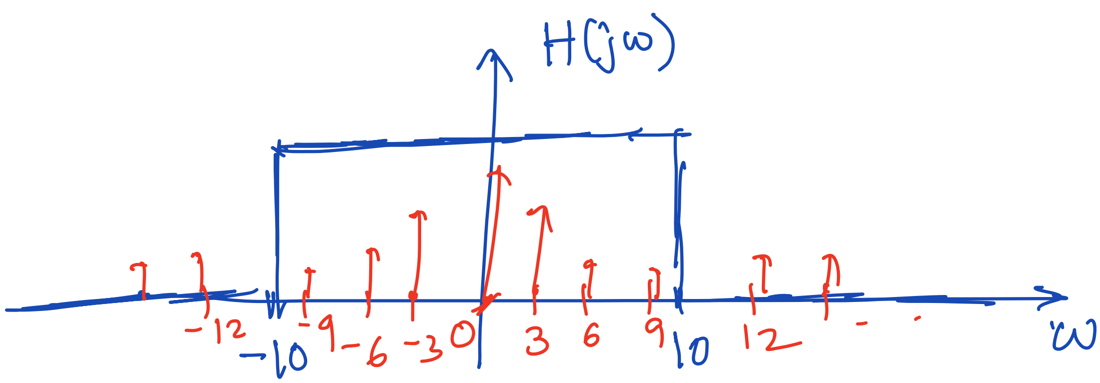
$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk3) e^{jk3t}$

If $x(t) = y(t)$, then whenever $\underline{H(jk3)} = 0$, we must have $a_k = 0$

$k = 1 \Rightarrow H(jk3) = H(j3) \neq 0$

similarly for $|k| \le 3$, $H(jk3) \neq 0$

otherwise $H(jk\omega_0) = 0$.



Road map: Discrete Time

- We have:

$$\xrightarrow{z^n} \boxed{h[n]} \xrightarrow{H(z)z^n}$$

  for $H(z) = \sum_{n=-\infty}^{\infty} h[k] z^{-k}$

- Suppose that you can write a signal $x[n]$ as:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k z_k^n.$$

- Then what would be the response to $x[n]$?

$$\xrightarrow{x(t) = \sum_{k=-\infty}^{\infty} a_k z_k^n} \boxed{h[n]} \xrightarrow{y(t) = \sum_{k=-\infty}^{\infty} H(z_k) a_k z_k^n}$$

- **For what DT signals $x[n]$, we can write $x[n] = \sum_k a_k e^{jk\omega_0 n}$?**

Fourier Series for Discrete-time Periodic Signals

For a periodic DT signal $x[n]$ with fundamental period $N \geq 1$ and fundamental frequency $\omega_n = \frac{2\pi}{N}$, we have:

$$x[n] = \sum_{k=n_0}^{n_0+N-1} a_k e^{jk\omega_n n},$$

$a_k = a_{k+N}$ for all k.

where

$$a_k = \frac{1}{N} \sum_{k=n_0}^{n_0+N-1} x[n] e^{-jn\omega_n k}.$$

$$a_{k+N} = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-jn\omega_n k} \underbrace{e^{-jn \overset{=2\pi}{N\omega_n}}}_{1}$$

$$e^{-jn2\pi} = 1$$

- The sequence $\{a_k\}$ is called *the Fourier series* of $x[n]$.
- Each a_k is called a Fourier coefficient of $x[n]$.
- Important feature: $\{a_k\}$ is periodic with period N since there are only N different complex exponentials in the discrete-time harmonic familiy associated with fundamental period N.
- It is easy to see that $a_{k+N} = a_k$ for all k.
- Therefore, we only need to know $a_0, a_1, \ldots, a_{N-1}$.

Key Equations for DT FS

- For a **periodic** signal $x[n]$ with *fundamental period* N and **fundamental frequency** $\omega_0 = \frac{2\pi}{N}$
- **Synthesis Equation:**

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n},$$

- **Analysis Equation:**

$$a_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jk\omega_0 n}$$

- **Notation**

$$x[n] \longleftrightarrow a_k$$

- Important feature: $a_{k+N} = a_k$
- A very useful identity for DT FS: $(1-\beta)(1+\beta+\ldots+\beta^r) = 1-\beta^{r+1}$ for all $\beta \in \mathbb{C}$.
- Equivalently: $1+\beta+\ldots+\beta^r = \frac{1-\beta^{r+1}}{1-\beta}$

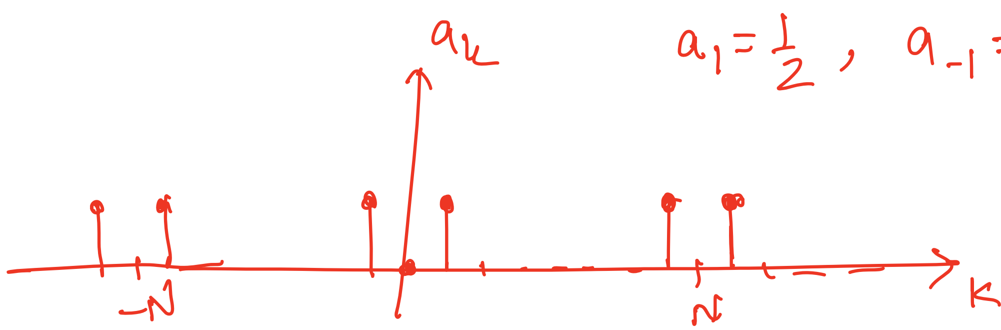
Formula: $x[n] = \sum_{K=0}^{N-1} a_K e^{jK\omega_n n}$ $\qquad e^{j\omega_n n(N-1)} = e^{-j\omega_n n}$

Example of DT FS

- What is the Fourier series of $x[n] = \cos(\omega_n n)$ for $\omega_n = \frac{2\pi}{N}$ for some $N \geq 1$?

$= \frac{1}{2} e^{j\omega_n n} + \frac{1}{2} e^{-j\omega_n n}$

$a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2} = a_{N-1}$



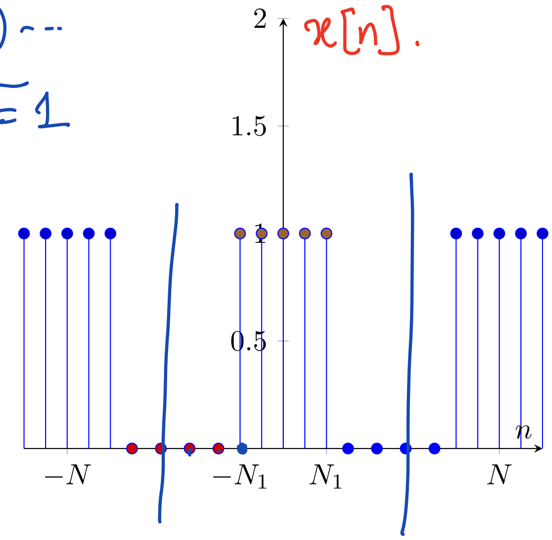
$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N-N_1-1} x[n]e^{-jk\omega_n n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\frac{2\pi}{N}n} \qquad \omega_n = \frac{2\pi}{N}.$$

Fourier Series for Discrete-time Periodic Signals: Example

- What is the Fourier-series of the following discrete-time pulse?

$$a_0 = \frac{1}{N}\sum_{-N_1}^{-N_1+N-1} x[n]\underbrace{e^{-j(0)\cdots}}_{=1}$$

$$= \frac{2N_1+1}{N}.$$



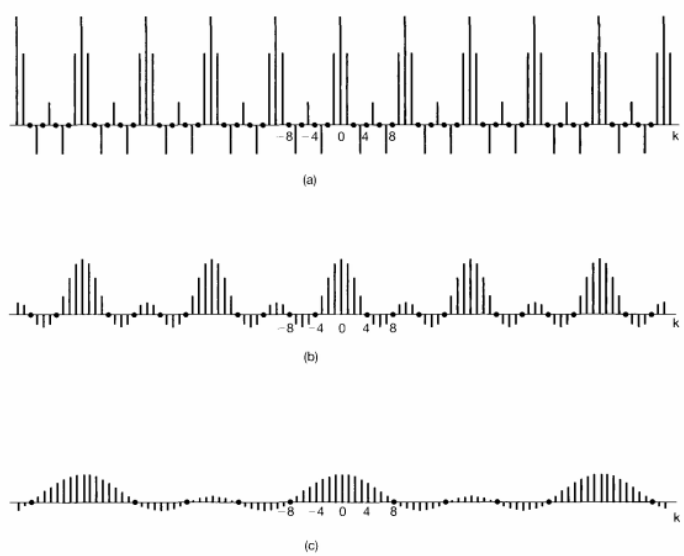


Figure 3.17 Fourier series coefficients for the periodic square wave of Example 3.12; plots of Na_k for $2N_1 + 1 = 5$ and (a) $N = 10$; (b) $N = 20$; and (c) $N = 40$.

Properties of DT FS

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l}x[n]\\y[n]\end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l}a_k\\b_k\end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic}\\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only}\\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$

Parseval's Relation for DT FS

- For a periodic signal $x[n]$ we have $x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_n n}$.

> For a periodic signal $x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_n n}$, we have:
>
> $$\frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |a_k|^2$$

- Note that $|a_k|^2$ is the average energy of the k'th harmonic component
- Parseval Relation: Average power of a signal is equal to sum of the energy of its harmonic components!