LECTURE - 6: 29th July

Properties of Systems: Linear

$$\xrightarrow[x[n]]{x(t)} \boxed{\text{System}} \xrightarrow[y[n]]{y(t)}$$

- We say that a system is linear if for all $x_1(t), x_2(t)$ (or $x_1[n]$ and $x_2[n]$) and all scalars a_1, a_2 the response (output) of the system to $a_1x_1(t) + a_2x_2(t)$ is $a_1y_1(t) + a_2y_2(t)$, where y_1 is the response to x_1 and y_2 is the response to x_1

$$x(t) \longrightarrow \boxed{} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

yes.

- Example 1: Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ linear?

$$\int_{-\infty}^{t} \left(a_1x_1(\tau) + a_2x_2(\tau)\right) d\tau = a_1\int_{-\infty}^{t} x_1(\tau)\,d\tau + a_2\int_{-\infty}^{t} x_2(\tau)\,d\tau$$
$$= a_1y_1(t) + a_2y_2(t)$$

- Example 2: What about $y[n] = (x[2n])^2$?

NO.

- Example 3: Is the system $y(t) = (t-1)x(t+2)$ linear? Yes,

Let $x_1(t) \to (t-1)\,x_1(t+2)$
$x_2(t) \to (t-1)\,x_2(t+2)$

Let $\bar{x}(t) = a_1x_1(t) + a_2x_2(t)$.

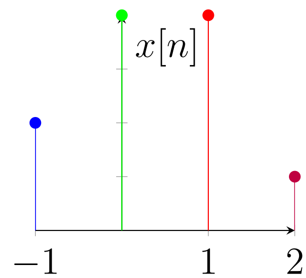
actual output: $(t-1)\,\bar{x}(t+2) = (t-1)\left[a_1x_1(t+2) + a_2x_2(t+2)\right]$

$$= a_1\underline{(t-1)x_1(t+2)} + a_2\,(t-1)x_2(t+2)$$

$$= a_1y_1(t) + a_2y_2(t)$$

Discrete-time LTI Systems

- Definition: We refer to the response (output) of the system to the input $x[n] = \delta[n]$ as **impulse response** of the system.
- Denote the impulse response by $h[n]$.
- Given any signal $x[n]$



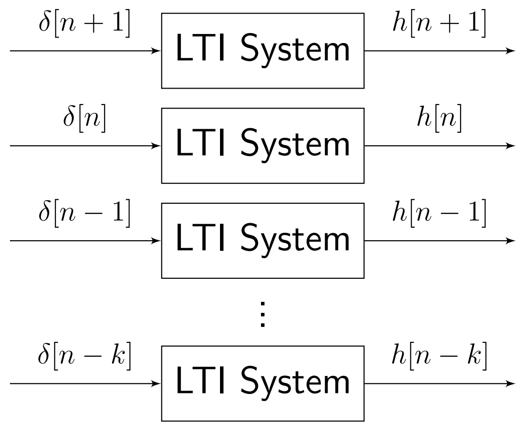
- we can write it as: $x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$
- In short: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ (representation property)

Discrete-time LTI Systems

- By Time-Invariant property:

if input is $a_1 \delta[n+1] + a_0 \delta(n) + a_{-1} \delta[n-1] + \cdots$

output is $a_1 h[n+1] + a_0 h[n) + a_{-1} h[n-1] + \cdots$



- By Linearity:

if we want to know $y[1] = \sum_{k=-\infty}^{\infty} x[k] h[1-k]$

Let $h[n]$ be the impulse response of an LTI system. Then for any input $x[n]$, the output is

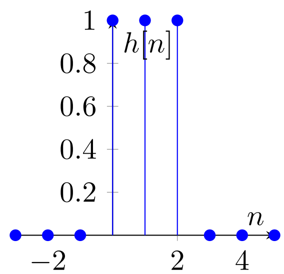
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

- The above sum is called **the convolution (sum)** of $x[n]$ and $h[n]$. Denote it by $y[n] \stackrel{def}{=} x[n] * h[n]$
- Convolution is commutative, i.e.,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k].$$

Discrete-time LTI Systems

- Suppose that we have a DT LTI system and we know that the response (output) to $x[n] = \delta[n]$ is $u[n]$.



What is the response of the system to $x[n] = u[n] - u[n-3]$ (plotted above)?

$$= \delta[n] + \delta[n-1] + \delta[n-2]$$

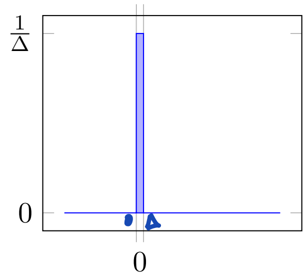
$$u[n] \quad u[n-1] \quad u[n-2]$$

$$y[n] = u[n] + u[n-1] + u[n-2].$$

LECTURE 7 : 5th August

Continuous-time LTI Systems

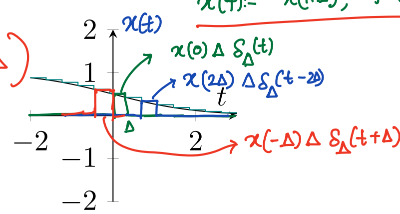
- Reminder: Define $\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } t \in [0, \Delta] \\ 0 & \text{else.} \end{cases}$



- $\Delta\delta_\Delta(t)$ is a pulse of hight 1 and width Δ
- We can approximate a nice-enough signal $x(t) \approx \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t-k\Delta)$

$\hat{x}(t) := x(k\Delta)$, if $t \in [k\Delta, (k+1)\Delta)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t-k\Delta)$$



- Let $h_\Delta(t)$ be the response to the input $\delta_\Delta(t)$
- By Time-Invariant property:

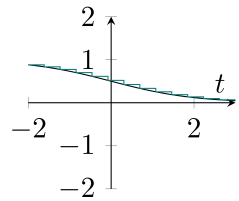
$$\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\text{LTI System}} \longrightarrow h_\Delta(t-k\Delta)$$

- By Linearity:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\text{LTI System}} \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_\Delta(t-k\Delta)$$

Continuous-time LTI Systems

- $x(t) \approx \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t - k\Delta)$



- By LTI property:

$$\xrightarrow{\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t-k\Delta)} \boxed{\text{LTI System}} \xrightarrow{\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_\Delta(t-k\Delta)}$$

- Letting $\Delta \to 0$:
  - $\delta_\Delta(t) \to \delta(t)$
  - $\hat{x}(t) \to x(t)$ (piece-wise approximation becomes exact)
  - $\hat{y}(t) \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ (by the Riemann approximation of the integral),

where $h(t)$ is the response of the system to $\delta(t)$.

Continuous-time LTI Systems

- Define the impulse response of a CT system to be the response of the system to $x(t) = \delta(t)$ and denote it by $h(t)$.

> Let $h(t)$ be the impulse response of an LTI system. Then for any input $x(t)$, the output is
> $$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- The above integral is called **the convolution (integral)** of $x(t)$ and $h(t)$. Denote it by $y(t) \overset{def}{=} x(t) * h(t)$.

MAIN MESSAGE

- For LTI systems, it is sufficient to know the impulse response $h(t)$ (or $h[n]$) to know the response to **any arbitrary** input $x(t)$ ($x[n]$).
- In other words, we know everything about these systems if we know the impulse response and hence, the notation:

$$\xrightarrow{x(t)} \boxed{h(t)} \xrightarrow{y(t)}$$

or

$$\xrightarrow{x[n]} \boxed{h[n]} \xrightarrow{y[n]}$$

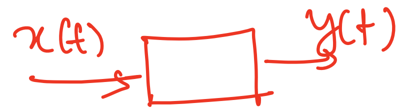
- Only catch: computing convolution is not easy.

Example

- Which of the following systems is linear time-invariant (LTI)?

  a. $y[n] = 3x[n] + 2$

  Not linear.
  time - invariant.



  b. $y(t) = \int_{-\infty}^{\infty} 2\pi x(\tau)\,(u(t-\tau) - u(t-\tau-1))d\tau$

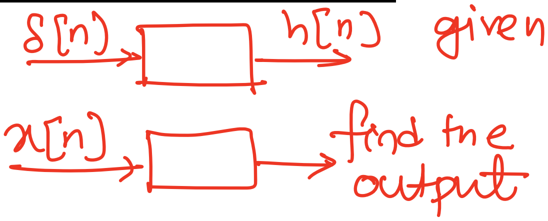
  yes, system is LTI.

  $= x(t) * h(t)$, where
  $\underline{h(t) = 2\pi\,[u(t) - u(t-1)]}$

  c. $y(t) = x(2t)$ – linear, not time-invariant.

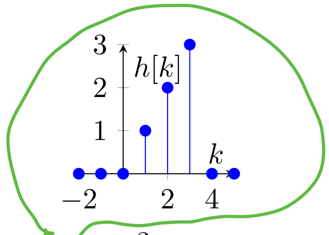
  d. $y[n] = (x[2n])^2$ → neither linear, nor time-invariant.

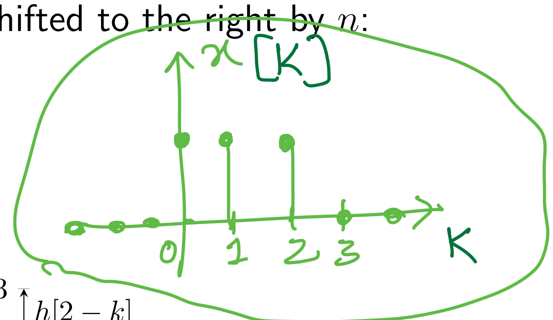
How to compute convolution?

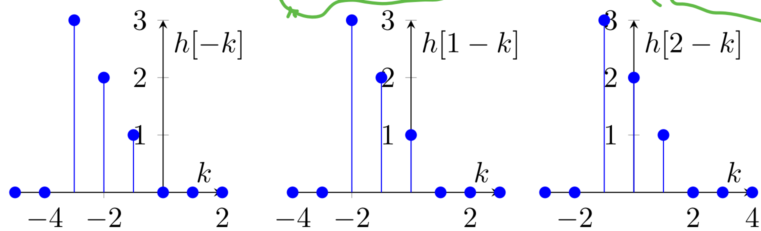


- Fix time n
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- As a function of k: $h[n-k]$ is $h[k]$ flipped and shifted to the right by n:

$h[1-k] = h[-(k-1)]$





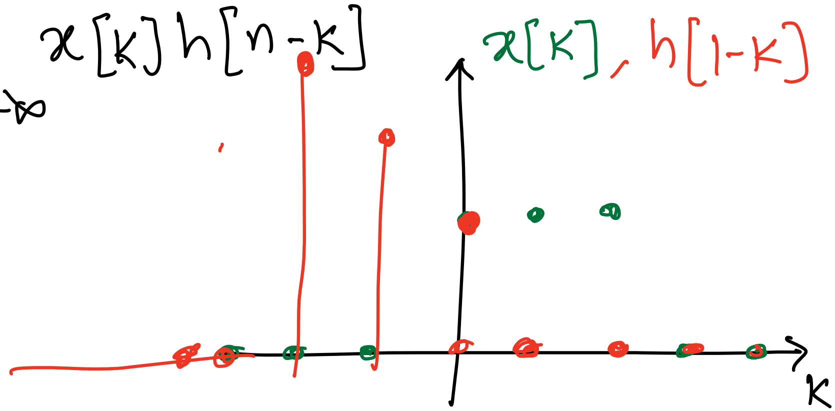


- Output at time n: sample by sample multiply $x[k]$ by $h[n-k]$ and then add
- Example: What is the output of an LTI system with the impulse response $h[n] = n(u[n] - u[n-4])$ to the input $x[n] = u[n] - u[n-3]$?

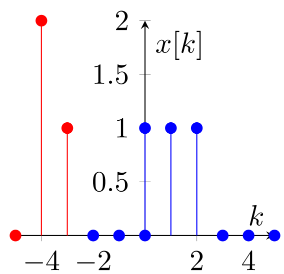
Recall that $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$

$= 0$, $y[1] = 1$

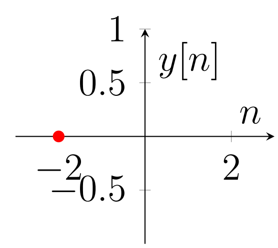


Computing convolution: DT

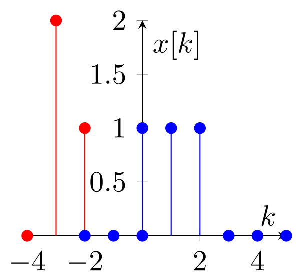


- We want to compute $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ for all n
- Let $n = -2$:

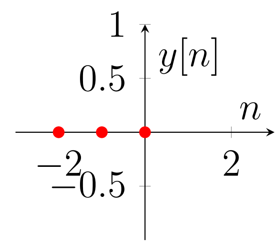
  a. imagine $h[n-k]$
  b. multiply $h[n-k]$ and $x[k]$ point by point
  c. add them up
  d. the result is the output at time n ($y[n]$)



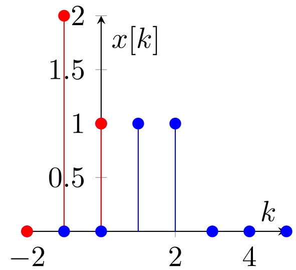
Computing convolution: DT



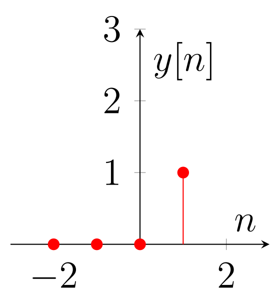
- We want to comput $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ for all n
- For $n < 1$: no intersection of non-zero parts

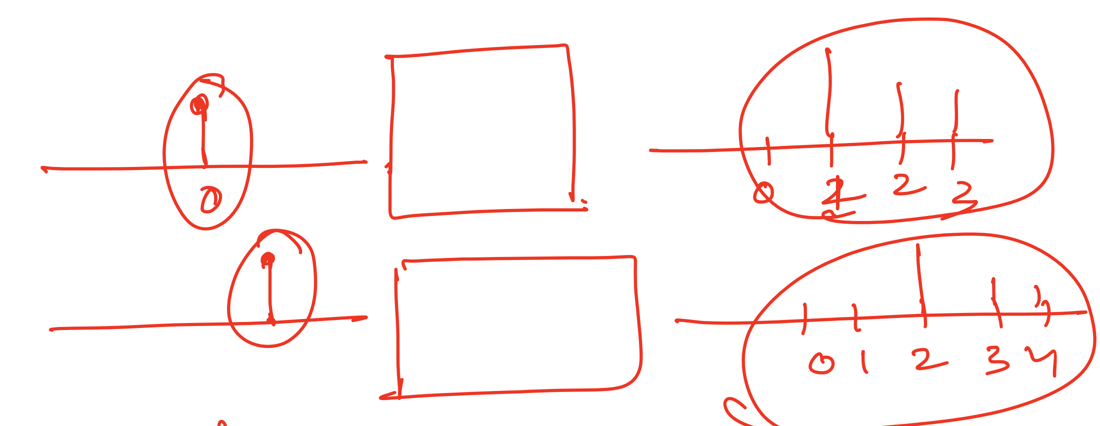


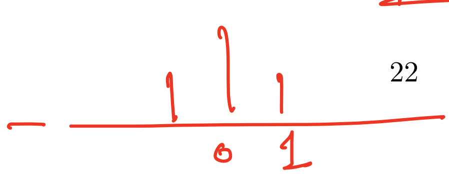
Computing convolution: DT



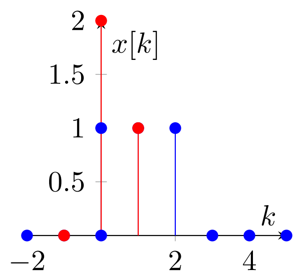
- We want to compute $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ for all n
- Let $n = 1$:
  a. imagine $h[n-k]$
  b. multiply $h[n-k]$ and $x[k]$ point by point
  c. add them up
  d. the result is the output at time n ($y[n]$)



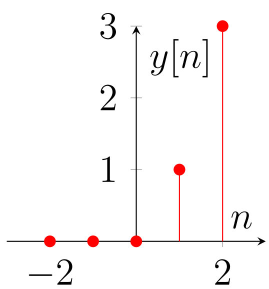




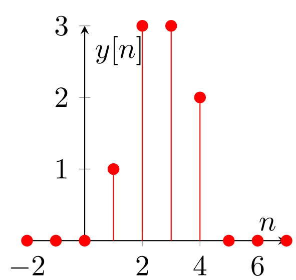
Computing convolution: DT



- We want to compute $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ for all n
- Let $n = 2$:
  a. imagine $h[n-k]$
  b. multiply $h[n-k]$ and $x[k]$ point by point
  c. add them up
  d. the result is the output at time n ($y[n]$)



- Continuing this, we get:



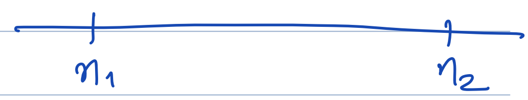
Q) Suppose $x[n] = 0$ for all $n \notin \{n_1, n_1+1, \ldots, n_2\}$

$h[n] = 0$ for all $n \notin \{n_3, n_3+1, \ldots, n_4\}$.

Let $y[n] = x[n] * h[n]$.

Find a range of values of n for which $y[n] = 0$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



$$= \sum_{k=n_1}^{n_2} x[k]h[n-k]$$

if n is such that $n-k < n_3$ or $n-k > n_4$ whenever $n_1 \leq k \leq n_2$, $y[n] = 0$.

$n < k + n_3$

$\Rightarrow$ if $\underline{n < n_1 + n_3}$

if $n > k + n_4$

$\Rightarrow$ $\underline{n > n_2 + n_4}$

Thus $y[n] = 0$ if $n < n_1 + n_3$ or $n > n_2 + n_4$

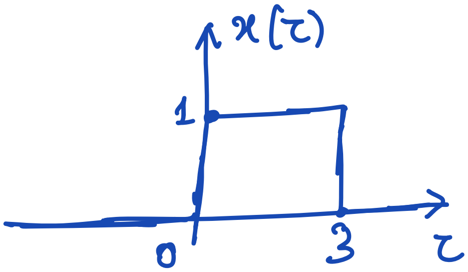
assuming $\underline{n_1 + n_3 < n_2 + n_4}$ ✓

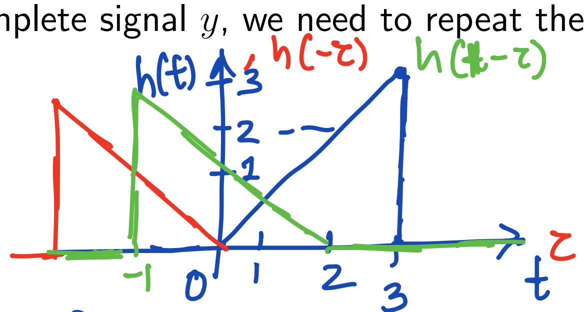
is true by $\underline{\text{defn}}$.

LECTURE 8: 6th August

How to compute convolution in CT?

- Given input $x(t)$ and impulse response $h(t)$
- The main idea: **fix time** t
- $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
- As a function of τ, $x(\tau)$ remains the same
- As a function of τ, $h(t-\tau)$: $h(\tau)$ flipped and shifted to t
- Output at time t: sample by sample multiply $x(\tau)$ by $h(t-\tau)$ and then ~~add~~ integrate
- Example: What is the output of an LTI system with the impulse response $h(t) = t(u(t) - u(t-3))$ to the input $x(t) = u(t) - u(t-3)$?
- To obtain the complete signal y, we need to repeat the above for every t!





$$h(t) = \begin{cases} t, & t \in [0,3] \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^3 h(t-\tau)d\tau = \int (t-\tau)d\tau$$

Evaluate $y(t)$ for $t \in [0,6]$.

$$h(t-\tau) = \begin{cases} t-\tau, & \tau \in [t-3, t] \\ 0, & \text{otherwise} \end{cases}, \quad \tau \in [-2, 1]$$

$$y(1) = \int_0^3 h(1-\tau)d\tau = \int_0^1 (1-\tau)d\tau =$$

case 1: $t \in [0,3]$.

$$y(t) = \int_0^t (t-\tau)\,d\tau = t\tau - \frac{\tau^2}{2}\Big|_0^t = t^2 - \frac{t^2}{2}$$
$$= t^2/2.$$

case 2: $t \in [3,6]$

$$y(t) = \int_{t-3}^{3} (t-\tau)\,d\tau = t\tau - \frac{\tau^2}{2}\Big|_{t-3}^{3}$$

$$= \left(3t - \frac{9}{2}\right) - \left(t(t-3) - \frac{(t-3)^2}{2}\right)$$

= simplify.

Let us determine range of t for which the integral $y(t)$ is zero.
This would be the case when $h(t-\tau) = 0 \quad \forall \tau \in [0,3]$.

we require (a) $t - \tau < 0 \quad \forall \tau \in [0,3]$
$\Rightarrow t < \tau \quad \forall \tau \in [0,3]$
$\Rightarrow \underline{t < 0}$.

(b) $t - \tau > 3 \quad \forall \tau \in [0,3] \Rightarrow t > \tau + 3 \quad \forall \tau \in [0,3]$
$\Rightarrow \underline{t > 6}$.

$\Rightarrow \quad y(t) = 0$ whenever $t < 0$ or $t > 6$.

Properties of Convolution

- *Commutative*: For any signals $x(t)$ (or $x[n]$) and $h(t)$ (or $h[n]$), $x(t)*h(t) = h(t)*x(t)$ (or $x[n]*h[n] = h[n]*x[n]$).
- Proof DT:

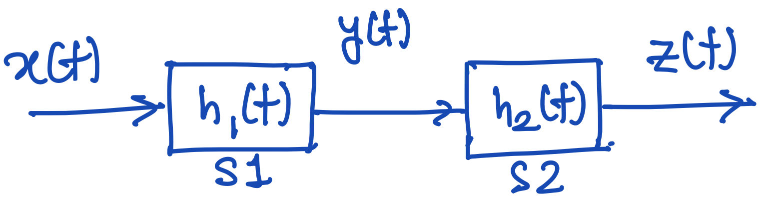
$$x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \overset{n-k\to\ell}{=} \sum_{\ell=-\infty}^{\infty} x[n-\ell]h[\ell] = h[n]*x[n]$$

- Proof of CT is similar.

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Properties of Convolution

- *Associative*: For any three signals $x[n]$, $h_1[n]$, and $h_2[n]$: $(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$ and similarly: $(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$.
- *Distributive*: For any three signals $x[n]$, $h_1[n]$, and $h_2[n]$: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$ and similarly: $(x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t))$.
- What is the system theoretic meaning? (serial and parallel interconnection)

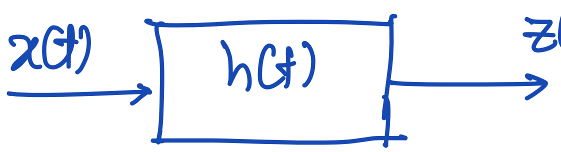


Let S1 & S2 be two LTI systems with impulse response $h_1(t)$ & $h_2(t)$, respectively.

$y(t) = x(t) * h_1(t)$

$z(t) = y(t) * h_2(t) = (x(t) * h_1(t)) * h_2(t)$

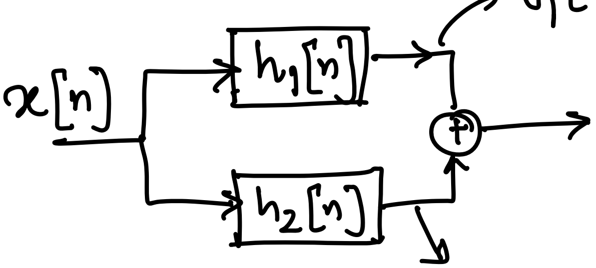
Because of associative property of convolution, we can combine S1 and S2 into a single LTI system with impulse response $h(t) = h_1(t) * h_2(t)$



$z(t) = x(t) * h(t)$

The distributive property enables simplifying parallel interconnection.

$y_1[n] = x[n] * h_1[n]$



$z[n] = y_1[n] + y_2[n]$.

$y_2[n] = x[n] * h_2[n]$

The above system is equivalent to the following.

$$x[n] \longrightarrow \boxed{\begin{array}{l} h_1[n] \\ +h_2[n] \end{array}} \longrightarrow z[n]$$

Impulse Response

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- We know that impulse response tells us everything about a system.
- Can we determine the properties (memorlyess, causality, invertibility, BIBO stability) of an LTI system by investigating h?
- Answer: Yes! For **LTI** systems we can do it through investigating the impulse response.

Lectures 9 & 10 : 7th August

Memoryless

$$\xrightarrow{x(t)} \boxed{h(t)} \xrightarrow{y(t)}$$

A discrete-time LTI system is memoryless if and only if

$$h[n] = a\delta[n], \text{ for some } a \in \mathbb{C}.$$

A continuous-time LTI system is memoryless if and only if

$$h(t) = a\delta(t), \text{ for some } a \in \mathbb{C}.$$

We know that $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

If system is memoryless, $y[n]$ cannot depend on $x[k]$ when $k \neq n$. for any input.

$\Rightarrow \quad x[k]\, h[n-k] = 0$ whenever $k \neq n$.

$\Rightarrow \quad h[n-k] = 0$ whenever $k \neq n$.

Causal

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

A discrete-time LTI system is causal if and only if

$$\boxed{h[n] = 0, \text{ for } n < 0} \tag{1}$$

A continuous-time LTI system is causal if and only if

$$\boxed{h(t) = 0, \text{ for } t < 0.}$$

Let us show this for a continuous-time LTI system. We know

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau + \underbrace{\int_{t}^{\infty} x(\tau)h(t-\tau)d\tau}_{=0}$$

$=0$ for a causal system as it contains input signal beyond t.

Since Causality is a property of the system,

$$\int_{t}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = 0 \quad \forall\, x(t) \text{ signals.}$$

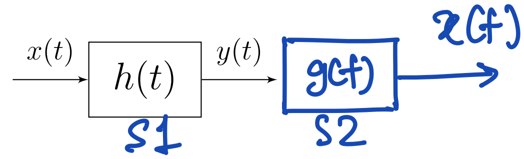
$$\Downarrow$$

$h(t-\tau) = 0$ whenever $\underline{\tau > t}$

or, $h(\bar{t}) = 0$ whenever $\bar{t} := t - \tau < 0$

$\Leftrightarrow$ $\underline{h(\bar{t}) = 0 \text{ whenever } \bar{t} < 0}$

Invertibility



A discrete-time LTI system is invertible if and only if there exists a $g[n]$ such that:

$$g[n] * h[n] = \delta[n].$$

A continuous-time LTI system is invertible if and only if there exists a $g(t)$ such that:

$$\boxed{g(t) * h(t) = \delta(t).}$$

Let us compute the output of S2, which is

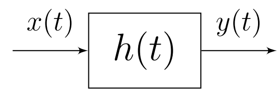
$$y(t) * g(t) = \big(x(t) * h(t)\big) * g(t)$$

$$= x(t) * \big(h(t) * g(t)\big)$$

$$= \underline{x(t)} * \underline{\delta(t)}$$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \underline{\underline{\delta(\tau)}}\, x(t-\tau)\,d\tau = x(t)$$

BIBO Stability



A discrete-time LTI system is BIBO stable if and only if:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{2}$$

A continuous-time LTI system is BIBO stable if and only if:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty.$$

Let us prove the DT case. Suppose the input $x[n]$ is bounded

$\Leftrightarrow \quad |x[n]| \leq B_x \quad \forall\, n \in \mathbb{Z}.$

B_x is a finite constant.

The output

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} \left| h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{|x[n-k]|}_{\leq B_x} \quad \forall\, k$$

$$= B_x \sum_{k=-\infty}^{\infty} |h[k]|$$

(A) if $\sum_{k=-\infty}^{\infty} |h[k]|$ evaluates to a finite quantity, let us say B_h, then

$$|y[n]| \leq B_x B_h \quad \forall n$$

$\Rightarrow$ $x[n]$ being bounded implies $y(n)$ is bounded, so system is BIBO stable.

(B) if $\sum_{k=-\infty}^{\infty} |h[k]|$ diverges, then system is not BIBO stable.

↳ Left as exercise.

$h[n] = 0$ for $n \le 0 \Rightarrow$ causal system.

Practice Problem

→ determine if the system is causal &/or stable.

Consider the DT - LTI system S with impulse response

$$h[n] = n\left(\frac{1}{3}\right)^n u[n-1].$$

$$\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^n < \infty.$$

Which one of the following statements is true:

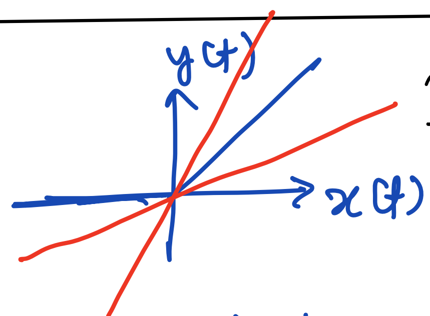
a. S is causal and stable. ✓

b. S is causal but not stable.

c. S is not causal but is stable.

d. S is neither causal nor stable.

(P1)



$x(t) \rightarrow \boxed{} \rightarrow y(t) = \begin{cases} x(t) & \text{if } x(t) > 0 \\ 0 & \text{if } x(t) \le 0 \end{cases}$

→ Is it LTI, causal, stable, memoryless?

time-invariant: ✓

linear :

$x_1(t) = 5 \ \forall t \Rightarrow y_1(t) = 5 \ \forall t$

$x_2(t) = -1 \ \forall t \Rightarrow y_2(t) = 0 \ \forall t$

Suppose I apply $\bar{x}(t) := x_1(t) + x_2(t)$.

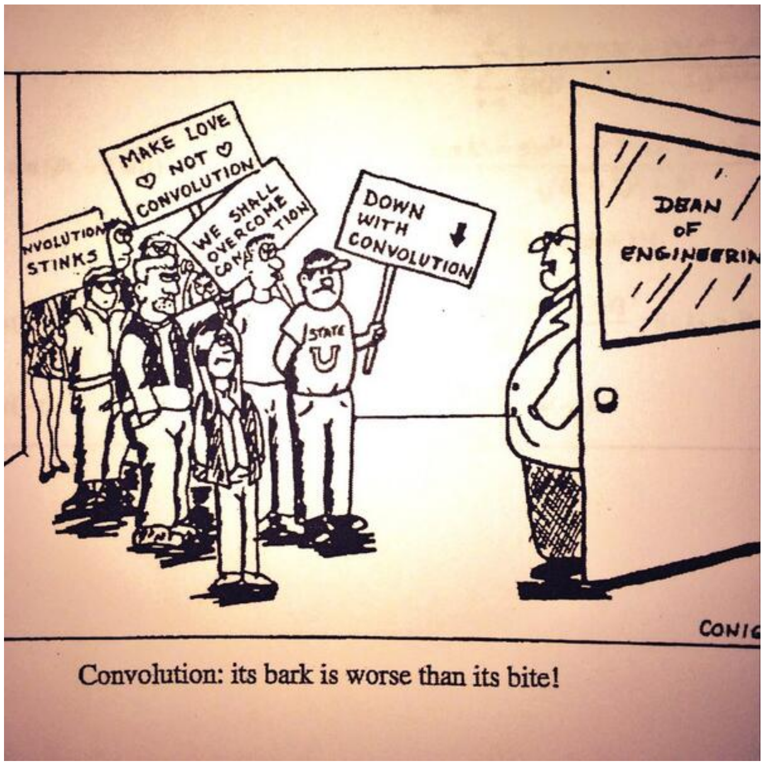
predicted output: $y_1(t) + y_2(t) = 5 \ \forall t$

actual output $4 \ \forall t$

(P2) $\frac{dy(t)}{dt} + 3t\,y(t) = t^2 x(t)$

Is this system linear? - yes - can be readily verified from the definition.

Road map



Convolution: its bark is worse than its bite!

NO CONVOLUTION!!!

Key Idea

- Suppose that we can find a rich family of signals $v_k(t)$ such that:

$$\xrightarrow{v_k(t)} \boxed{h(t)} \xrightarrow{\lambda_k v_k(t)}$$

  for a scalar $\lambda_k \in \mathbb{C}$

- And suppose that you can write a signal $x(t)$ as:

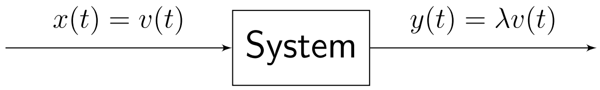
$$x(t) = \sum_{k=-\infty}^{\infty} a_k v_k(t).$$

- Then what would be the response to $x(t)$?

$$\xrightarrow{x(t) = \sum_{k=-\infty}^{\infty} a_k v_k(t)} \boxed{h(t)} \xrightarrow{y(t) = \sum_{k=-\infty}^{\infty} a_k \lambda_k v_k(t)}$$

Eigenfunctions

- A signal (function) $v(t)$ is an eigenfunction of a system if:

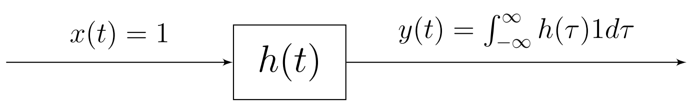


for some $\lambda \in \mathbb{C}$.

- Can we find an eigenfunction for an LTI system?
- Let's try $v(t) = 1$:

$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) d\tau$

H $\forall t$

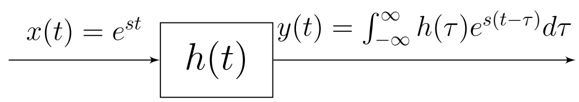


- So, it is an eigenfunction with $\lambda = \int_{-\infty}^{\infty} h(\tau) d\tau$

$y(t) = H x(t)$

Eigenfunctions

- Is there any other eigenfunctions?
- Conjecture: more generally $x(t) = e^{st}$ where $s \in \mathbb{C}$
- Let's investigate:



- We have

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \int_{-\infty}^{\infty} h(\tau)e^{st}e^{-s\tau}d\tau$$
$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \quad = e^{st} H(s)$$

does not depend on t.

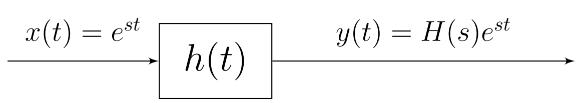
- $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ is called the **transfer function of the system**.
- $H(s)$ is also the **Laplace Transform** of the impulse response $h(t)$.

For any signal $x(t)$, its Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

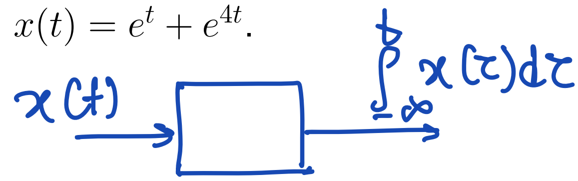
will be discussed later.

> Exponentials e^{st} are eigenfunctions of any CT LTI system for all $s \in \mathbb{C}$ such that $H(s)$ exists. In particular, for $x(t) = e^{st}$ the output will be $y(t) = H(s)x(t) = H(s)e^{st}$.



Example

- Compute the transfer function of the CT integrator system. Using that, compute the output of the system for the input $x(t) = e^t + e^{4t}$.



Step 1: find the impulse response $h(t)$

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases} = u(t), \text{ unit step signal.}$$

Step 2: find the transfer function $H(s)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\,d\tau = \int_{0}^{\infty} 1 \cdot e^{-s\tau}\,d\tau = \frac{-1}{s} e^{-s\tau}\Big|_0^{\infty} = \frac{1}{s}$$

if $Re(s) > 0$

Step 3: express $x(t)$ as linear combination of complex exponential signals

$$x(t) = e^t + e^{4t}$$

Step 4: find output for each component of $x(t)$

output of $e^t \rightarrow e^t H(1)$

" of $e^{4t} \rightarrow e^{4t} H(4)$

Step 5: find the final output.

$$e^t H(1) + e^{4t} H(4)$$

Road map: Discrete Time

- Suppose that we can find a rich family of signals $v_k[n]$ such that:

$$\xrightarrow{v_k[n]} \boxed{h[n]} \xrightarrow{\lambda_k v_k[n]}$$

for a scalar $\lambda_k \in \mathbb{C}$

- And suppose that you can write a signal $x[n]$ as:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k v_k[n].$$

- Then what would be the response to $x[n]$?

$$\xrightarrow{x(t) = \sum_{k=-\infty}^{\infty} a_k v_k[n]} \boxed{h[n]} \xrightarrow{y(t) = \sum_{k=-\infty}^{\infty} a_k \lambda_k v_k[n]}$$

- A signal (function) $v[n]$ is an eigenfunction of a system if:

$$\xrightarrow{x[n] = v[n]} \boxed{\text{System}} \xrightarrow{y[n] = \lambda v[n]}$$

for some $\lambda \in \mathbb{C}$.

- λ is often called an eigenvalue

Eigenfunctions

- Conjecture: $x[n] = z^n$ where $z \in \mathbb{C}$ is fixed, is an eigenfunction
- Let's investigate:

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k}$$

- We have

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = \sum_{k=-\infty}^{\infty} h[k] z^n z^{-k}$$

$$= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

- $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$ is called the **z-transform** of $h[n]$ and also called the transfer function of the system.

> Exponentials z^n are eigenfunctions of any DT LTI system for all $z \in \mathbb{C}$ such that $H(z)$ exists. In particular, for $x[n] = z^n$ the output will be $y[n] = H(z)x[n] = H(z)z^n$.

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z) z^n$$

- Example: Compute the transfer function of the DT derivative (difference) system. Using that, compute the output of the system for the input $x[n] = cos(\omega n)$.

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n] = x[n] - x[n-1].$$

$x[n] = \cos(\omega n)$

Step 1: Find $h[n]$

Step 2: Find $H(z)$

Step 3: Express $x[n] = \sum_k a_k z_k^n$

Step 4 & 5: $y[n] = \sum_k a_k H(z_k) z_k^n$

SOLUTION

Step 1: $h[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$

Step 2: $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} = h[0] z^{-0} + h[1] z^{-1}$

$= 1 - \frac{1}{z},$

well defined $\forall z \neq 0.$

Step 3: $x[n] = \frac{1}{2}\left(e^{j\omega n} + e^{-j\omega n}\right) = \frac{1}{2} z_1^n + \frac{1}{2} z_2^n,$

$z_1 = e^{j\omega}, \quad z_2 = e^{-j\omega}$

Step 4: $\frac{1}{2} e^{j\omega n} \longrightarrow \frac{1}{2} e^{j\omega n} H(e^{j\omega})$

$\frac{1}{2} e^{-j\omega n} \longrightarrow \frac{1}{2} e^{-j\omega n} H(e^{-j\omega})$

Step 5: $y[n] = \frac{1}{2} e^{j\omega n}\left(1 - \frac{1}{e^{j\omega}}\right) + \frac{1}{2} e^{-j\omega n}\left(1 - \frac{1}{e^{-j\omega}}\right)$

Module C: Fourier Series

Exponentials e^{st} are eigenfunctions of any CT LTI system for all $s \in \mathbb{C}$ such that $H(s)$ exists.
In particular, for $x(t) = e^{st}$ the output will be $y(t) = H(s)x(t) = H(s)e^{st}$, where
$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

Exponentials z^n are eigenfunctions of any DT LTI system for all $z \in \mathbb{C}$ such that $H(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ exists.
In particular, for $x[n] = z^n$ the output will be $y[n] = H(z)x[n] = H(z)z^n$, where
$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

Practice Problems

left as exercise

- Compute the transfer function of the CT derivative system.

- Let $x[n] = 3^n$ be the input to a DT LTI system with impulse response $h[n]$ and system function $H(z)$. Suppose the output signal $y[n]$ satisfies $y[2] = 4$. What is $y[n]$?

  a. $y[n] = 4$
  b. $y[n] = 4^{n-1}$
  c. $y[n] = (3^n) - 5$
  d. $y[n] = 4(3^{n-2})$

- How to decompose signals into sum of exponentials (eigenfunctions of LTI systems)?

Decomposing Periodic Signals

- The question: for what signals $x(t)$ we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t},$$

  where $s_k \in \mathbb{C}$?

- Restricted question: for what signals $x(t)$ we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t},$$

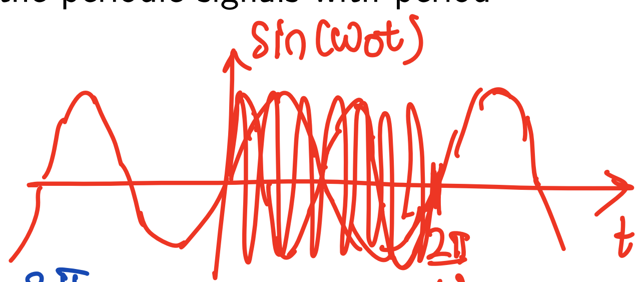
  where $\omega_k \in \mathbb{R}$?

- More restricted question: for what signals $x(t)$ we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \cos(k\omega_0 t) + j\sin(k\omega_0 t)$$

  where $\omega_0 \in \mathbb{R}^+$ doesn't depend on k?

- Answer to the last question: essentially all the periodic signals with period $T = \frac{2\pi}{\omega_0}$!



$$x\left(t+\frac{2\pi}{\omega_0}\right) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0\left(t+\frac{2\pi}{\omega_0}\right)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \underbrace{e^{jk2\pi}}_{=1}$$

$$= x(t).$$

Decomposing Periodic Signals

For any periodic signal $x(t)$ with fundamental period $T > 0$, that has finite energy over a period, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{1}$$

where

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \tag{2}$$

and $\omega_0 = \frac{2\pi}{T}$, and the integral is over any contiguous time interval of length T.

- Eq. (1) is called the Fourier Series representation of $x(t)$.
- Coefficient (2) is called the kth Fourier coefficient of $x(t)$.
- Why a_k should satisfy (2)? Is it unique? Answer: Yes.

Baseline sanity check

$$\frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right) e^{-j2\omega_0 t}\,dt = \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_0^T a_k e^{j(k-2)\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T a_2 e^{j0t}dt + \frac{1}{T}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \int_0^T a_k e^{jm\omega_0 t}\,dt$$

$$a_k\int_0^T \cos m\omega_0 t\,dt + ja_k\int_0^T \sin m\omega_0 t\,dt = 0$$

$= a_2 + 0$

$= a_2.$

Key Equations for CT FS

- For a **periodic** signal $x(t)$ with *fundamental period* T and **fundamental frequency** $\omega_0 = \frac{2\pi}{T}$
- **Synthesis Equation:**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

- **Analysis Equation:**

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

- **Notation**

$$x(t) \longleftrightarrow a_k$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

fundamental period $T_0 = 2$

$$\omega_0 = \frac{2\pi}{2} = \pi$$

CT Fourier Series: Example

- What is the Fourier Series for the signal $x(t) = 1 + \cos(2\pi t) + \sin(3\pi t)$?

(determine without computing any integration)

$$\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$$

$$\frac{1}{2j}e^{j3\pi t} - \frac{1}{2j}e^{-j3\pi t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t}$$

$$= a_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k e^{jk\pi t}$$

$a_0 = 1$

$a_1 = 0 = a_{-1}$

$a_2 = \frac{1}{2}$, $a_{-2} = \frac{1}{2}$

$a_3 = \frac{1}{2j}$, $a_{-3} = \frac{-1}{2j}$

$a_k = 0 \quad \forall\, k \neq 0, 2, -2, 3, -3$