11th Nov.

Unilateral Laplace Transform

Given a continuous-time signal $x(t)$, its unilateral laplace transform is defined as

$$X_u(s) = \int_{0^-}^{\infty} x(t) e^{-st}\, dt .$$

If $x(t)$ is a causal signal with $x(t) = 0$ for $t < 0$, $X_u(s) = X(s)$, which is its bilateral Laplace transform.

What can we say about ROC of $X_u(s)$?

Effectively, $X_u(s)$ is bilateral laplace transform of a causal signal.

Hence, its ROC will have the following property:

if $z \in \mathbb{C}$, $z \in ROC$, then $z' \in ROC$ where $Re(z') > Re(z)$.

If $X_u(s)$ is a rational function of s, then

$$ROC = \{ z \in \mathbb{C} \mid Re(z) > \max_{p \in \mathcal{P}} Re(p) \},$$ where

$\mathcal{P}$ is the set of poles of $X_u(s)$.

Therefore, while computing inverse of $X_u(s)$, each term in partial fraction expansion is mapped to its causal time-domain signal.

Even if $x(t)$ is not causal, its values before $t=0$ cannot be inferred from its unilateral Laplace transform.

Examples: (1) $x_1(t) = \delta(t) + e^t u(t)$. Find $X_u(s)$.

$$\underbrace{\int_{0^-}^{\infty} \delta(t) e^{-st}\, dt}_{=1} + \underbrace{\int_{0^-}^{\infty} e^t u(t) e^{-st}\, dt}$$

$$= \int_0^{\infty} e^{-(s-1)t}\, dt = \frac{1}{s-1}, \quad \text{when } \mathrm{Re}(s) > 1$$

$$X_u(s) = 1 + \frac{1}{s-1}, \quad \text{ROC: } \{ z \in \mathbb{C} \mid \mathrm{Re}(z) > 1 \}.$$

Example 2: Suppose $X_u(s) = \frac{1}{(s+1)(s+2)}$. Find $x(t)$.

$$\text{ROC} = \{ z \mid \mathrm{Re}(z) > -1 \}$$

$$X_u(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$x(t) = (e^{-t} - e^{-2t}) u(t).$$

Differentiation property

Let $x(t) \longleftrightarrow X_u(s)$

$\frac{dx(t)}{dt} \longleftrightarrow ?$

$$\int_{0^-}^{\infty} \left(\frac{d}{dt}x(t)\right) e^{-st}\,dt = \underbrace{x(t)e^{-st}\Big|_{0^-}^{\infty}} - \int_{0^-}^{\infty} x(t)(-s)e^{-st}\,dt$$

$$= \underline{sX_u(s) - x(0^-)}$$

Similarly, $\frac{d^2x(t)}{dt^2} = s\left[sX_u(s) - x(0^-)\right] - x'(0^-)$

$$= s^2X_u(s) - sx(0^-) - \underbrace{x'(0^-)}_{\frac{d}{dt}x(t)\Big|_{t=0^-}}$$

Example:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t),$$



Suppose $x(t) = 2u(t)$.

Initial conditions

$y(0^-) = 3,\ y'(0^-) = -5$

Determine $y(t)$ & identify which part of the output is due to external input and which part is driven by initial conditions.

applying unilateral laplace transform, we obtain

$$s^2 Y_u(s) - s y(0^-) - y'(0^-) + 3\left[s Y_u(s) - y(0^-)\right] + 2 Y_u(s) = X_u(s) = \frac{2}{s}$$

$$\Rightarrow Y_u(s)\left[s^2 + 3s + 2\right] - (s+3)\, y(0^-) - y'(0^-) = X_u(s)$$

$$\Rightarrow Y_u(s) = \underbrace{\frac{X_u(s)}{(s+2)(s+1)}}_{\text{response due to external input} \equiv \text{zero state response.}} + \underbrace{\frac{(s+3)\,y(0^-) + y'(0^-)}{(s+2)(s+1)}}_{\text{response due to initial conditions} \equiv \text{zero-input response}} \quad \left(= \frac{3s+4}{(s+2)(s+1)}\right)$$

$y_{in}(t)$ $\qquad$ $y_{st}(t)$

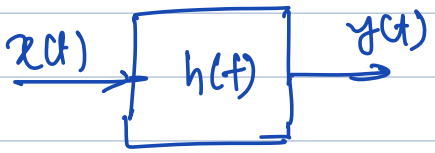
$$\mathcal{UL}\{y_{in}(t)\} = \frac{2}{s(s+2)(s+1)}, \quad \text{ROC: } \underline{\text{Re}(z) > 0}$$

↳ solve via partial fraction expansion..

Note: while defining transfer function using unilateral laplace transform, always make sure that all initial conditions are zero.

$$\mathcal{UL}[h(t)] = \frac{\mathcal{UL}[y(t)]}{\mathcal{UL}[x(t)]} \text{ at zero initial conditions.}$$

$$\mathcal{L}[h(t)] = \frac{(s+2)}{(s+1)(s+2)} = \frac{Y(s)}{X(s)}$$



$$\Rightarrow X(s) = sY(s) + Y(s)$$

$$\Rightarrow x(t) = \frac{d}{dt}y(t) + y(t).$$

$$sX(s) + 2X(s) = s^2Y(s) + 2Y(s) + 3sY(s)$$

$$\Rightarrow \frac{d}{dt}x(t) + 2x(t) = \frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t)$$