28th October

$H(z) = \frac{}{(z-a_1)(z-a_2)\cdots(z-a_m)}$, with $|a_1| \le |a_2| \le \cdots \le |a_m|$

System Properties through ROC

If $h[n]$ is a causal signal, then ROC $= \{z \in \mathbb{C} \mid |z| > |a_m|\}$

- **Causality:** an LTI system is causal when its impulse response $h[n] = 0$ for $n < 0$. In other words, $h[n]$ is a causal signal.
- Thus, the z-transform of $h[n]$, denoted $H(z)$ has ROC as the exterior of a circle including ∞.
- The radius of this circle is larger than the maximum value of the magnitude of its poles.
- Since the ROC includes ∞, the order of the numerator is less than or equal to the order of the denominator.

If $H(z)$ is not infinite when $z = \infty$, we must have $m \le n$.

$\frac{z^{m-n}(\text{const})\, z^m\left(\cdots - \frac{1}{z} - \frac{1}{z^2}\right)}{z^n(\cdots)}$

- **Stability:** an LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
- In other words, the ROC of the z-transform of $h[n]$ includes the unit circle.

> A causal system with rational $H(z)$ is stable if and only if all the poles are inside the unit circle.

$|H(z)| = \left|\sum_{n=-\infty}^{\infty} h[n] z^{-n}\right| \le \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$ unit circle belongs to ROC.

- Example: Is a causal system with the transfer function $H(z) = \frac{1}{z-2j}$ stable?

pole is $2j$

causality $\Rightarrow$ ROC $= \{z \in \mathbb{C} \mid |z| > |2j| = 2\}$

$\Rightarrow$ unit circle is not part of ROC

$\Rightarrow$ system is not stable.

Examples

Consider the following z-transforms of impulse responses of several LTI systems. Determine if these systems are stable, causal, both or neither for different possible ROCs.

$$H_1(z) = \frac{z^3 - 2z^2 + z}{z^2 + 0.25z + 1/8}$$

$$H_2(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

$$H_3(z) = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}}$$

→ not causal as deg. of num > deg. of den.

For $H_1(z)$: poles: $\frac{-1+\sqrt{7}j}{8}$, $\frac{1-j\sqrt{7}}{8}$, $\alpha = \sqrt{\frac{1}{8^2} + \frac{7}{8^2}} = \frac{1}{2\sqrt{2}} < 1$

Stable, not causal ← $ROC_1 = \{z \mid |z| > \alpha\}$

not stable, not causal ← $ROC_2 = \{z \mid |z| < \alpha\}$

For $H_2(z)$: poles: 2 and 0.5

$ROC_1 = \{|z| > 2\}$ → causal, not stable

$ROC_2 = \{0.5 < |z| < 2\}$, not causal, stable

$ROC_3 = \{|z| < 0.5\}$ → neither causal, nor stable

$H_3(z)$: poles: $re^{j\theta}$, $re^{-j\theta}$

causal ← $\{z \mid |z| > r\}$ → if $r > 1$ ⇒ not stable; $r < 1$ ⇒ stable

$\{z \mid |z| < r\}$ → not causal; $r > 1$ ⇒ stable; $r < 1$ ⇒ not stable.

LTI Systems - Linear Constant Coefficient Difference Equations

- Consider the following general form of linear difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where $x[n]$ is the input and $y[n]$ is the output.

applying z-transform to both sides, we obtain

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

- Then, we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

The ROC is determined from whether the system is stable, causal, etic.

Examples

- Consider a LTI system whose input and output satisfy

$$y[n] - 0.5y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

Determine its impulse response for different possible ROCs.

$Y(z) - 0.5z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$

$\Rightarrow \frac{Y(z)}{X(z)} = H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-0.5z^{-1}}$

$ROC_1 = \{z \mid |z| > 0.5\}$

$ROC_2 = \{z \mid |z| < 0.5\}$

$= \frac{1-0.5z^{-1}+0.5z^{-1}+\frac{1}{3}z^{-1}}{1-0.5z^{-1}}$

$= 1 + \frac{\frac{5}{6}z^{-1}}{1-\frac{1}{2}z^{-1}}$

For ROC_1, $h[n] = \delta[n] + \frac{5}{6}\left(\frac{1}{2}\right)^{n-1}u[n-1]$

For ROC_2: $h[n] = \delta[n] + \frac{5}{6}\left[-\left(\frac{1}{2}\right)^{-(n-1)}u[-(n-1)-1]\right]$

$= \delta[n] + \frac{5}{6}\left(-\left(\frac{1}{2}\right)^{-n+1}u[-n]\right)$

Example 10.26

Q If $x_1[n] = \left(\frac{1}{6}\right)^n u[n]$, $y_1[n] = \underbrace{\left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n\right]}_{h[n]}$. $x[n] \rightarrow$ [ ] $\rightarrow y[n]$

If $x_2[n] = (-1)^n$, $y_2[n] = \frac{7}{4}(-1)^n$

Determine $H(z)$, value of a and whether system is causal

Solution: Since $x_2[n] = (-1)^n$, $y_2[n] = \frac{7}{4}(-1)^n$,

we know $H(-1) = \frac{7}{4}$.

$$Y_1(z) = a\frac{1}{1-\frac{1}{2}z^{-1}} + 10\frac{1}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{6}$$

$$H(z) = \left(1-\frac{1}{6}z^{-1}\right)\left[\frac{a}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}}\right]$$

causal $\Rightarrow |z| > \frac{1}{2}$

$$H(-1) = \left(1+\frac{1}{6}\right)\left[\frac{a}{\left(1+\frac{1}{2}\right)} + \frac{10}{\left(1+\frac{1}{3}\right)}\right] = \frac{7}{4}$$

$$\Rightarrow \frac{7}{6}\left[\frac{a}{3/2} + \frac{10}{4/3}\right] = \frac{7}{4}$$

$$\Rightarrow \frac{2a}{3} + \frac{30}{4} = \frac{3}{2} \quad \text{---} \quad \Rightarrow a = ?$$

29th October 2026

Example 10.27

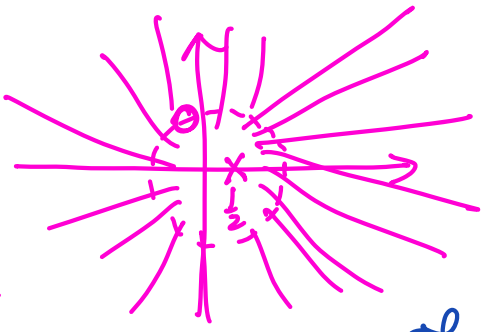
A discrete-time system is causal & stable. $H(z)$ has a pole at $\frac{1}{2}$ and a zero on the unit circle.
Determine if the following are true, false or inconclusive.

(a) DTFT of $2^{-n}h[n]$ exists → true.

(b) $H(e^{j\omega}) = 0$ for some ω → true

(c) $h[n]$ has finite duration → false

(d) $h[n]$ is real signal → inconclusive.



(e) $g[n] = n\left(h[n] * h[n]\right)$ is impulse response of a stable system → True.

$$\sum_{n=-\infty}^{\infty} 2^{-n} h[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} h[n] \underbrace{(2e^{-j\omega})^{n}}_{z}$$

↓ z-transform when $|z| = 2$

From the information given, $2 \in$ ROC.

(e) $h[n] \longleftrightarrow H(z)$

$h[n] * h[n] \longleftrightarrow (H(z))^2$

$n(h[n] * h[n]) \longleftrightarrow -z\frac{d}{dz}(H(z))^2 = -z\,2H(z)\frac{d}{dz}H(z)$

↓ poles of $G(z) \subseteq$ poles of $H(z)$

⇒ poles of $G(z)$ lie inside unit circle.
& $g[n]$ is causal ⇒ unit circle is part of ROC of $G(z)/g[n]$.

29th October

Q1) $X_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$

$$x_1[n+3] \longleftrightarrow \frac{z^3}{1-\frac{1}{2}z^{-1}}, \quad \left\{ z \in \mathbb{C} \;\middle|\; |z| > \tfrac{1}{2}, z \neq \infty \right\}$$

$$X_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$x_2[-n] = \frac{1}{1-\frac{z}{3}}, \quad |z| < 3$$

$$x_2[-n+1] = x_2[-(n-1)] = \frac{z^{-1}}{1-\frac{z}{3}}, \quad \{|z| < 3, \; z \neq 0\}$$

$$Y(z) = \frac{z^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{z}{3}\right)}, \quad \text{ROC: } \frac{1}{2} < |z| < 3$$

Q2) $$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{m=-\infty}^{\infty} x_1[2m] z^{-2m}$$

let $2m = n$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{-2m}$$

$$= X(z^2)$$

BLOCK DIAGRAM REPRESENTATION.

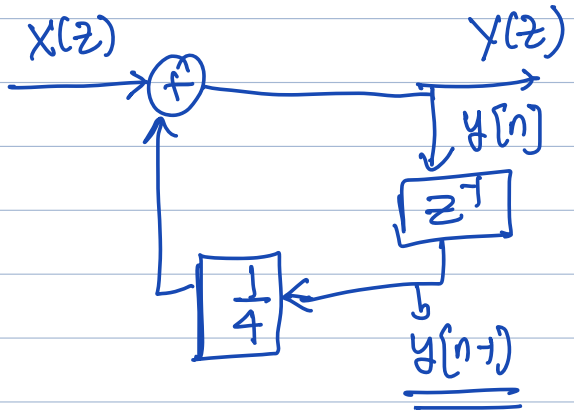
$h[n] = \left(\frac{1}{4}\right)^n u[n]$



$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$X(z) = Y(z) - \frac{1}{4}z^{-1}Y(z)$$

$$\Rightarrow Y(z) = X(z) + \frac{1}{4}z^{-1}Y(z)$$



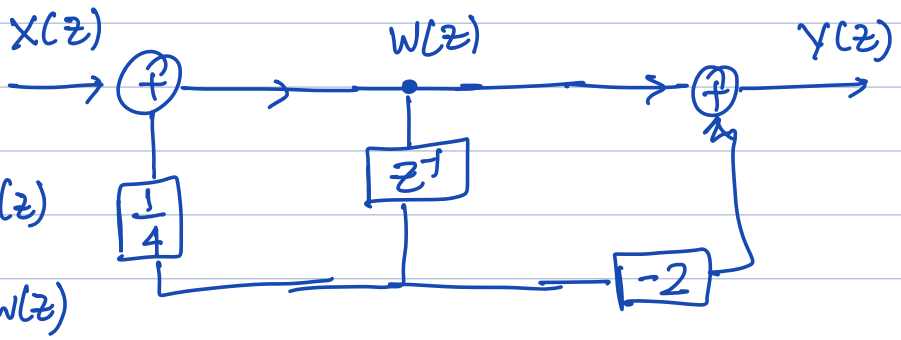
inverse z-transform yields

$$y[n] = x[n] + \frac{1}{4}y[n-1].$$



Example:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \frac{Y(z)}{X(z)} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

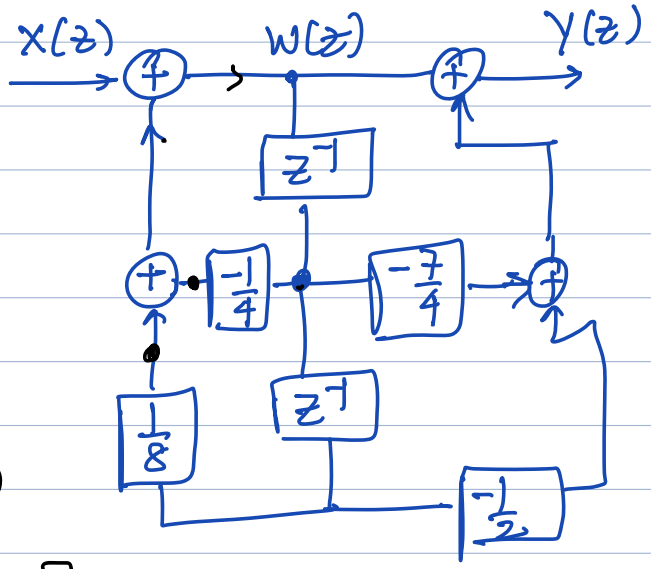


$$Y(z) = (1-2z^{-1})W(z)$$

$$= W(z) - 2z^{-1}W(z)$$

Ex

Determine $H(z) = \frac{Y(z)}{X(z)}$ and a difference eqn relating $y[n]$ and $x[n]$.



$$W(z) = X(z) + \left[-\frac{1}{4}z^{-1}W(z) + \frac{1}{8}z^{-2}W(z)\right]$$

$$\Rightarrow X(z) = W(z)\left[1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right]$$

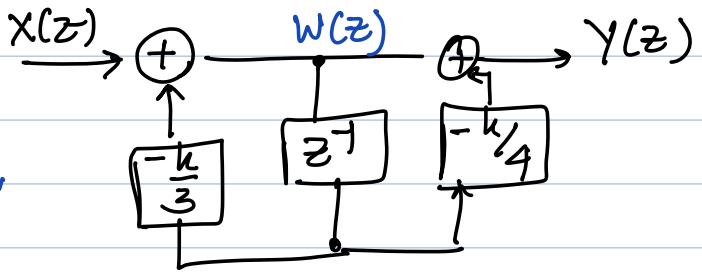
$$Y(z) = W(z) + \left[-\frac{7}{4}z^{-1}W(z) - \frac{1}{2}z^{-2}W(z)\right]$$

$$= W(z)\left[1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$\Rightarrow y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{7}{4}x[n-1] - \frac{1}{2}x[n-2].$$

Q



System is causal.

1) Find range of $k \in \mathbb{R}$, s.t. system is stable.

2) Let $k=1$. $x[n] = \left(\frac{2}{3}\right)^n \; \forall n$. Determine $y[n]$.

$= H\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^n$

Solution: $W(z) = X(z) - \frac{k}{3} z^{-1} W(z)$

$Y(z) = W(z) - \frac{k}{4} z^{-1} W(z)$

$$\frac{Y(z)}{X(z)} = \frac{1 - \frac{k}{4} z^{-1}}{1 + \frac{k}{3} z^{-1}}, \quad \text{pole: } -\frac{k}{3}.$$

$$ROC = \left\{ |z| > \frac{|k|}{3} \right\}$$

$$\left|\frac{k}{3}\right| < 1 \Rightarrow k \in (-3, 3)$$

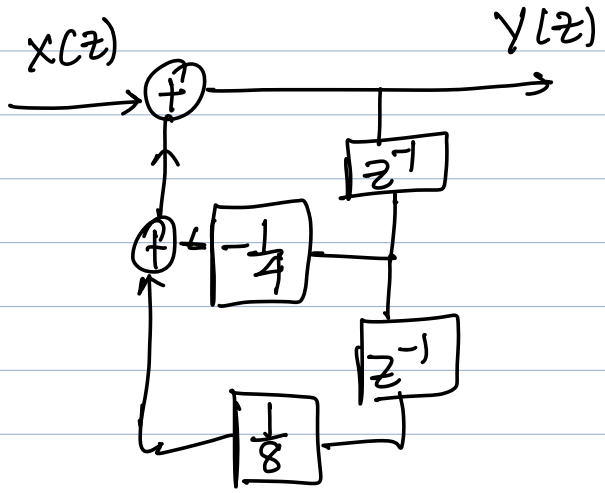
(ii) $$\frac{Y(z)}{X(z)} = H(z) = \frac{1 - \frac{1}{4} z^{-1}}{1 + \frac{1}{3} z^{-1}}, = \frac{1 - \frac{1}{4} \cdot \frac{3}{2}}{1 + \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2}}$$

$$= \frac{\left(\frac{8-3}{8}\right)}{\left(\frac{3}{2}\right)}$$

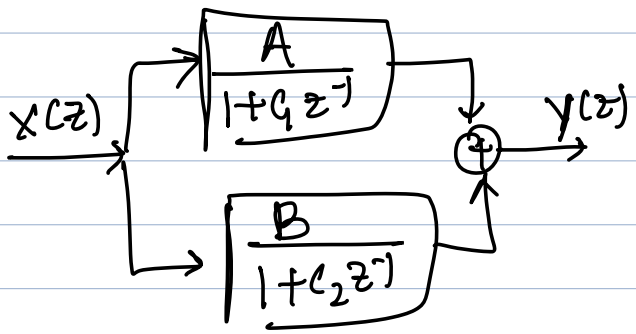
$$= \frac{5 \times 2}{3 \times 8} = \frac{5}{12}.$$

Q) Find any two different block diagram representations of $H(z) = \frac{1}{1 + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{Y(z)}{X(z)} = \frac{A}{1 + C_1 z^{-1}} + \frac{B}{1 + C_2 z^{-1}}$

Case-1 : $Y(z) + \frac{1}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) = X(z)$

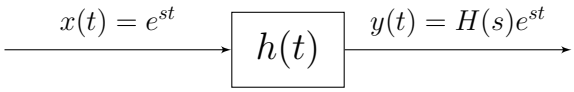


Case-2 :



30th October 2026

Laplace Transform



$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

For any CT signal $x(t)$, it's **Bilateral** Laplace-transform is the function:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}.$$

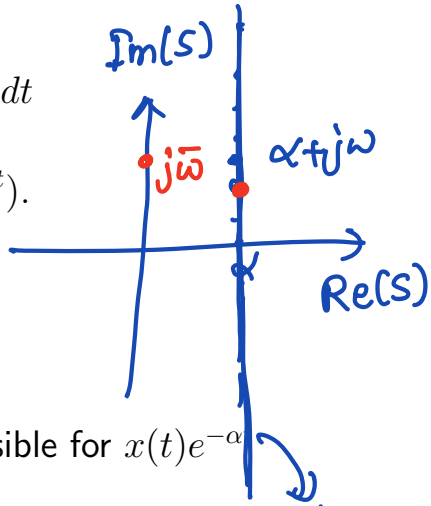
The Region of Convergence of $X(s)$ is defined to be

$$\texttt{ROC} = \{\texttt{s} \mid \int_{-\infty}^{\infty} |\texttt{x(t)}||\texttt{e}^{-\texttt{st}}|\texttt{dt} = \int_{-\infty}^{\infty} |\texttt{x(t)}|\texttt{e}^{-\texttt{Re(s)t}}\texttt{dt} < \infty\}.$$

- Letting $\alpha = \texttt{Re}(s)$ and $\omega = \texttt{Im}(s)$, we have: i.e., $s = \alpha + j\omega$

$X(j\bar{\omega}) = X(s)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-\alpha t}e^{-j\omega t}dt$$
$$= \int_{-\infty}^{\infty} (x(t)e^{-\alpha t})e^{-j\omega t}dt = \texttt{FT}(x(t)e^{-\alpha t}).$$



- Therefore, if $s = \alpha + \omega j \in \texttt{ROC}$, $X(s) = \texttt{FT}(x(t)e^{-\alpha t})$.
- If $\alpha = 0$, $X(s) = \texttt{FT}(x(t))$.
- Even when $x(t)$ is not absolutely integrable, it may be possible for $x(t)e^{-\alpha}$ to be absolutely integrable if α is sufficiently large.
- Notation: $x(t) \longleftrightarrow X(s)$

this entire line either belongs to ROC or it does not.

$$\int_{-\infty}^{\infty} |x(t)|e^{-\alpha t}e^{j\omega t}|\,dt = \int_{-\infty}^{\infty} |x(t)|e^{-\alpha t}|\,dt < \infty$$

if $s = \alpha + j\omega \in \text{ROC}$, then $\bar{s} = \alpha + j\bar{\omega} \in \text{ROC}$.

* let $x(t) = e^{2t}u(t)$

$ROC = \{s \in \mathbb{C} \mid Re(s) > 2\}$

$\int_{-\infty}^{\infty} |x(t)|\,|e^{-\alpha t}|\,dt = \int_0^{\infty} e^{2t} e^{-\alpha t}\,dt$

$= \int_0^{\infty} e^{-(\alpha-2)t}\,dt$

Laplace Transform

Special case: $x(t) = u(t) \Rightarrow X(s) = 1/s$, ROC: $Re(s) > 0$

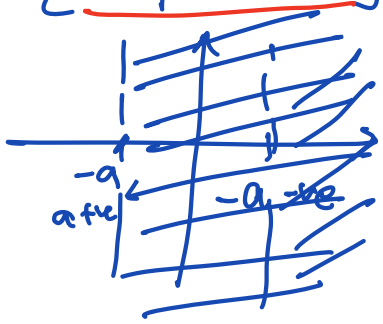
- Example 1: What is the Laplace Transform and ROC of $x(t) = e^{-at}u(t)$ where $a \in \mathbb{R}$?

$X(s) = \int_0^{\infty} e^{-at} e^{-st}\,dt = \int_0^{\infty} e^{-(a+s)t}\,dt \rightarrow$ $a + Re(s) > 0$

$\Rightarrow \{s \in \mathbb{C} \mid Re(s) > -a\}$

$= \frac{-1}{a+s} e^{-(a+s)t}\Big|_0^{\infty}$

$= \frac{1}{s+a}$, when $s \in ROC$.



- Example 2: What is the Laplace Transform and ROC for $x(t) = -e^{-at}u(-t)$?

$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt = \int_{-\infty}^{0} -e^{-at}e^{-st}\,dt$

let $\tau = -t$, $d\tau = -dt$

$= \int_{\infty}^{0} e^{a\tau} e^{s\tau}\,d\tau = \int_{\infty}^{0} e^{(a+s)\tau}\,d\tau$

$= \frac{1}{a+s}[1-0] = \frac{1}{a+s}$,

$Re(s) + a < 0$ → ROC.

$\Rightarrow Re(s) < -a$

- Moral: Laplace-Transform without ROC is meaningless!
- Moral: ROC consists of vertical lines.

Poles and Zeros

- We almost exclusively work with rational Laplace transforms:

$$X(s) = \frac{a_0(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)},$$

  where $z_1, \ldots, z_m \in \mathbb{C}$ are the zeros and $p_1, \ldots, p_n \in \mathbb{C}$ are the poles.

- Location of the poles are important for ROC and stability
- Example: What are the poles and zeros of $X(s) = \frac{2s-1}{s^2+2s-1}$? one zero at $s = \frac{1}{2}$, poles: two poles.

- Determine the Laplace Transform of $x(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$ with its ROC.

$$x(t) = e^{-2t}u(t) + \frac{1}{2}e^{-t}e^{j3t}u(t) + \frac{1}{2}e^{-t}e^{-j3t}u(t) \qquad \cos(3t) = \frac{1}{2}(e^{j3t} + e^{-j3t})$$

$$= e^{-2t}u(t) + \frac{1}{2}e^{-(1-3j)t}u(t) + \frac{1}{2}e^{-(1+3j)t}u(t)$$

$$X(s) = \underbrace{\int_0^\infty e^{-(2+s)t}\,dt}_{\mathrm{Re}(s) > -2} + \frac{1}{2}\underbrace{\int_0^\infty e^{-(1-3j+s)t}\,dt}_{\mathrm{Re}(s) > -1} + \frac{1}{2}\underbrace{\int_0^\infty e^{-(1+3j+s)t}\,dt}_{\mathrm{Re}(s) > -1}$$

ROC: $\mathrm{Re}(s) > -1$

$$= \frac{1}{s+2} + \frac{1}{s+1-3j} + \frac{1}{s+1+3j}$$

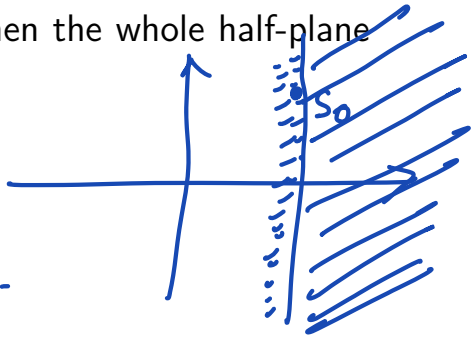
poles: $-2, -1+3j, -1-3j$

Properties of ROC for Laplace Transform

- Property 1: ROC does not contain any poles.
- Property 2: If $x(t)$ is of finite duration and $x(t)$ is absolutely integrable, i.e., $\int_a^b |x(t)|dt < \infty$, then ROC is the entire $\mathbb{C}$.
  - To show this, let $s \in \mathbb{C}$ with $\mathrm{Re}(s) = \alpha$. Then:

$$\int_{-\infty}^{\infty} |x(t)|e^{-\alpha t}dt = \int_a^b |x(t)|e^{-\alpha t}dt$$
$$\leq \max(e^{-\alpha t} : t \in [a,b]) \int_a^b |x(t)|dt < \infty.$$

- Property 3: If $x(t)$ is right-sided and $s_0 \in$ ROC, then the whole half-plane $\{s : \mathrm{Re}(s) \geq \mathrm{Re}(s_0)\}$ is in ROC.



$\exists\, \tau$ s.t. $x(t)=0$ for $t<\tau$

if $s_0 \in$ ROC, $\int_\tau^\infty |x(t)|\, e^{-\mathrm{Re}(s_0)t}\, dt < \infty$.

let $\bar{s} \in \mathbb{C}$ s.t. $\mathrm{Re}(\bar{s}) > \mathrm{Re}(s_0)$.

if $\tau > 0$, then

$\mathrm{Re}(\bar{s})t > \mathrm{Re}(s_0)t \quad \forall t \geq \tau$

$$\int_\tau^\infty |x(t)|\, e^{-\mathrm{Re}(\bar{s})t}\, dt$$
$$\leq \int_\tau^\infty |x(t)|\, e^{-\mathrm{Re}(s_0)t}\, dt < \infty$$

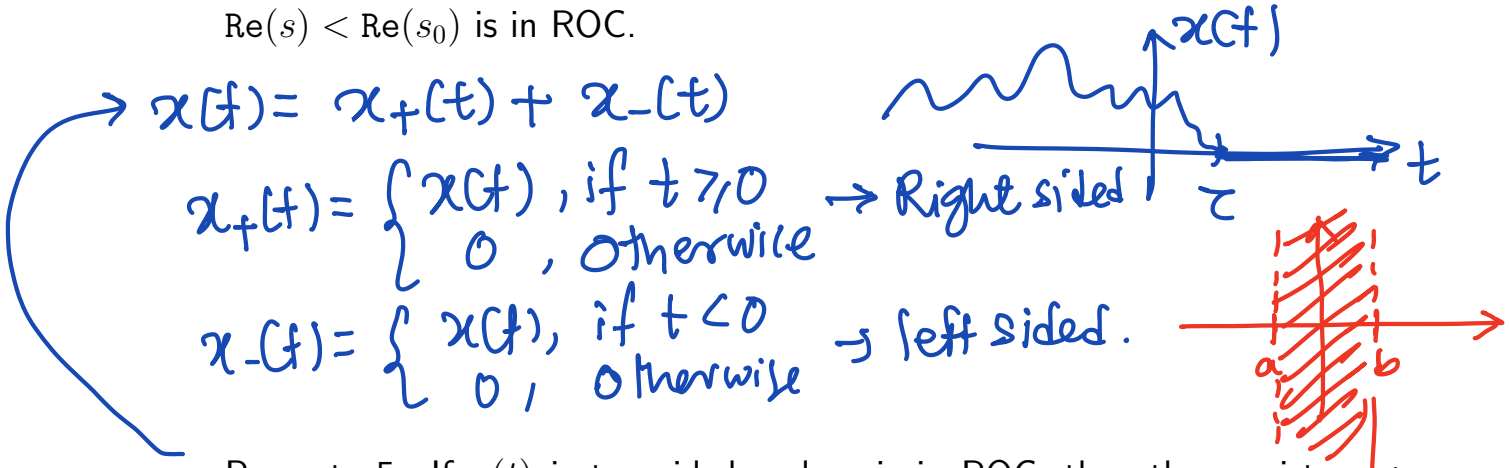
Let $\tau < 0$:

$$\int_\tau^0 |x(t)|\, e^{-\mathrm{Re}(\bar{s})t}\, dt + \int_0^\infty |x(t)|\, e^{-\mathrm{Re}(\bar{s})t}\, dt$$
$$\leq \int_0^\infty |x(t)|\, e^{-\mathrm{Re}(s_0)t}\, dt < \infty.$$

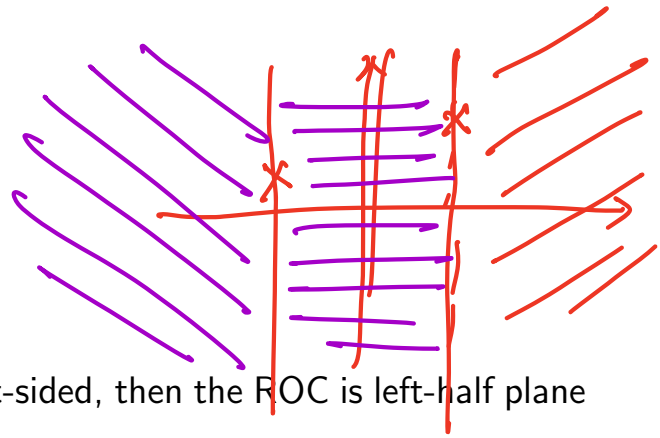
$$= \int_\tau^0 |x(t)|\, e^{-\mathrm{Re}(s_0)t}\, e^{(\mathrm{Re}(s_0)-\mathrm{Re}(\bar{s}))t}\, dt$$
$$\leq e^{(\mathrm{Re}(s_0)-\mathrm{Re}(\bar{s}))\tau}.$$ Hence proved.

Properties of ROC for Laplace Transform

- Property 4: If $x(t)$ is left-sided and $s_0 \in$ ROC, then the half-plane s : $\mathrm{Re}(s) < \mathrm{Re}(s_0)$ is in ROC.



- Property 5: If $x(t)$ is two-sided and s_0 is in ROC, then there exist $a < \mathrm{Re}(s_0) < b$ such that the strip $s : a < \mathrm{Re}(s) < b$ is in ROC.
- Property 6: If $X(s)$ is rational, then ROC is either bounded between two poles or extends to infinity.
- Property 7: If $X(s)$ is rational and right-sided, then the ROC is right-half plane to the right-most pole.



- Property 8: If $X(s)$ is rational and left-sided, then the ROC is left-half plane to the left-most pole.

Laplace Transform

$b \in \mathbb{R}$.

Determine the Laplace transform of $x(t) = e^{-b|t|}$ for both $b > 0$ and $b < 0$.

$$\int_{-\infty}^{0} e^{+bt} e^{-st}\, dt + \underbrace{\int_{0}^{\infty} e^{-(b+s)t}\, dt}_{\{s \in \mathbb{C} \mid Re(s) > -b\} = S_1}$$

let $\tau = -t$

$$\int_{\infty}^{0} e^{-b\tau + s\tau}\, -d\tau = \underbrace{\int_{0}^{\infty} e^{(s-b)\tau}\, d\tau}_{\{s \in \mathbb{C} \mid Re(s) < b\} = S_2}$$

if $b > 0$, then $S_1 \cap S_2 = \{s \in \mathbb{C} \mid -b < Re(s) < b\}$

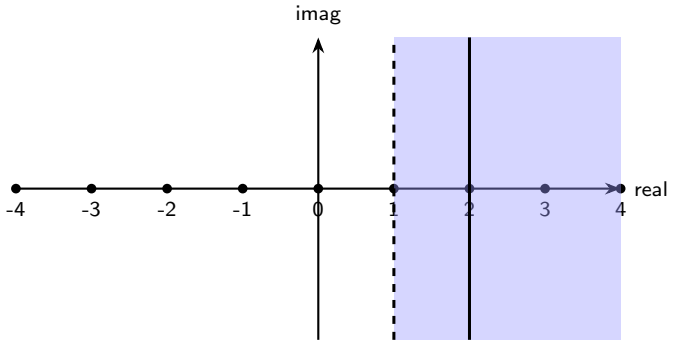
if $b < 0$, $S_1 \cap S_2 = \emptyset$, $X(s)$ is not defined.

$$\rightarrow X(s) = \frac{1}{s+b} - \frac{1}{s-b}$$

$$= \frac{s-b-s-b}{(s+b)(s-b)} = \frac{-2b}{s^2-b^2} = \frac{2b}{b^2-s^2}.$$

Inverse Laplace Transform

- Suppose that the line $\{s|\mathtt{Re}(s) = \sigma_1\} \in ROC$



- We know that along this line $X(\sigma + j\omega) = \mathtt{FT}(e^{-\sigma t}x(t))$.
- Therefore,

$$x(t) = e^{\sigma t}\mathtt{IFT}(X(\sigma + j\omega)) = e^{\sigma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X((\sigma + j\omega))e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds.$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, s\, e^{st}\, ds$$

→ inverse laplace transform of $sX(s)$

Inverse Laplace Transform

- For rational transfer functions $X(s) = \frac{q(s)}{p(s)}$, we proceed as in the case of inverse Fourier transform:

  a. We perform partial fraction on

  $$X(s) = \frac{q(s)}{p(s)} = \frac{A}{s+p_1} + \frac{B}{s+p_2} + \cdots$$

  b. If pole p_i is to the left side of ROC, we use $e^{-p_i t}u(t) \longleftrightarrow \frac{1}{s+p_i}$

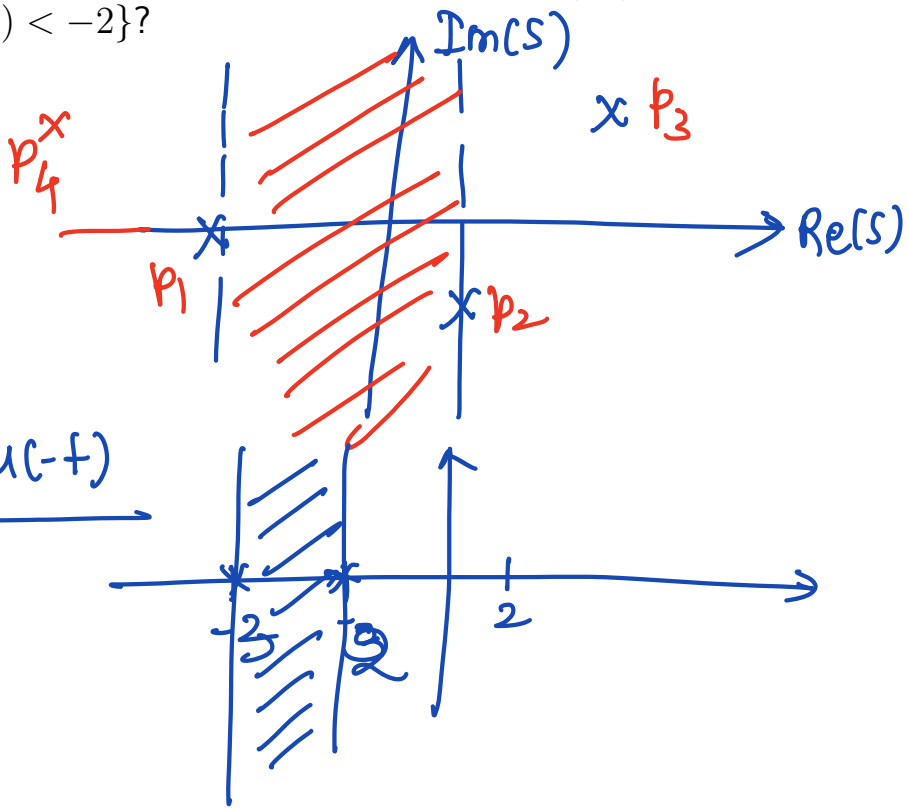
  c. If pole p_i is to the right side of ROC, we use $-e^{-p_i t}u(-t) \longleftrightarrow \frac{1}{s+p_i}$

- Example: What is the Laplace Transform inverse of $X(s) = \frac{s}{s^2+5s+6}$ given that ROC$= \{s \mid -3 < \mathrm{Re}(s) < -2\}$?

$X(s) = \frac{s}{(s+2)(s+3)}$

$= \frac{A}{s+2} + \frac{B}{s+3}$

$x(t) = Ae^{-3t}u(t) - Be^{-2t}u(-t)$



Causality and Stability

- For **causality**, we know that the impulse response $h(t)$ is right-sided, therefore, ROC is to the right-side of the right-most pole
- For **stability** of such a system, we need $\int_{-\infty}^{\infty} |h(t)|dt < \infty$
- This means that the $j\omega$ axis, i.e., $\{s \mid \text{Re}(s) = 0\}$, be in the ROC of $H(s)$.
- A causal system with rational $H(s)$ is stable if and only if all the poles are to the left of the $j\omega$ axis.
- Determine causality and stability of

$$H(s) = \frac{s-1}{(s+1)(s-2)}.$$

poles: $2, -1$

ROC_1: $\text{Re}(s) > 2$ $\longrightarrow$ causal, but not stable

ROC_2: $\text{Re}(s) \in (-1, 2)$ $\longrightarrow$ not causal, stable

ROC_3: $\text{Re}(s) < -1$ $\longrightarrow$ neither.

Properties of Laplace Transform

- Linearity: $ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$, ROC contains the intersection.

$d\tau = dt$
$\tau = t - t_0$

- Time-Shift: $x(t - t_0) \longleftrightarrow e^{-st_0}X(s)$. ROC is not affected.

$$\int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-s(t-t_0)}\, e^{-st_0}\, dt = e^{-st_0}\int_{-\infty}^{\infty} x(\tau) e^{-s\tau}\, d\tau$$

$$= e^{-st_0} X(s)$$

↱ will be clarified next class.

- Frequency Shift: $e^{s_0 t}x(t) \longleftrightarrow X(s - s_0)$. ROC now includes $\mathrm{Re}(s_0)$.

$$\int_{-\infty}^{\infty} x(t)\, e^{s_0 t}\, e^{-st}\, dt = \int_{-\infty}^{\infty} x(t) e^{-(s-s_0)t}\, dt = X(s - s_0)$$

- Convolution: $x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s)$. ROC contains the intersection.

Let $x(t) \longleftrightarrow X(s)$

$$\int_{-\infty}^{\infty} x(t)^*\, e^{-st}\, dt = \int_{-\infty}^{\infty} x(t)^* \left(e^{-s^* t}\right)^* dt$$

$$= X(s^*)^*$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt \quad , \quad \frac{d}{ds}X(s) = \int_{-\infty}^{\infty} (-t\, x(t))\, e^{-st}\, dt$$

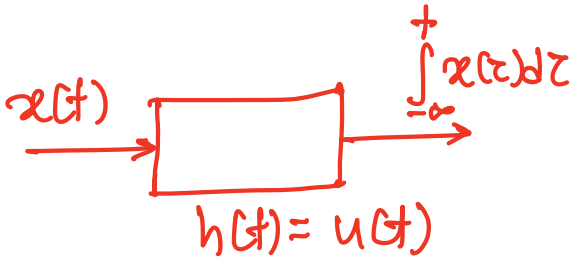
5th NOV.

Properties of Laplace Transform

- Differentiation: $\frac{dx(t)}{dt} \longleftrightarrow sX(s)$.

ROC will remain unchanged except possible inclusion of imaginary axis when $s=0$ is a pole of $X(s)$.

- Differentiation: $-tx(t) \longleftrightarrow \frac{dX(s)}{ds}$.



- Integration: $\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{s}X(s)$.

$x(t) * u(t)$

$X(s)\,U(s) = \frac{1}{s}X(s)$

- Initial Value Theorem: For a causal signal $x(t)$ which does not contain an impulse at the origin, we have

$$x(0^+) = \lim_{s\to\infty} sX(s),$$

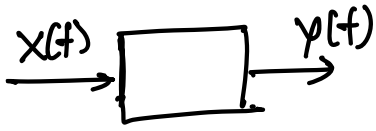
where $X(s)$ is the Laplace transform of $x(t)$.

- Final value theorem: For a causal signal $x(t)$ for which $\lim_{t\to\infty} x(t)$ is finite, we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

Finding Impulse Response of Differential Equations

- Consider a LTI system satisfying $\frac{d}{dt}y(t) + 3y(t) = x(t)$.



Can we uniquely determine its impulse response?

if we apply Laplace transform to both sides of the equation,

$$sY(s) + 3Y(s) = X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}.$$

Howevere, we cannot conclude what the ROC is from the above expression, and need additional information (e.g; system is causal/stable) to infer the ROC, and thereby compute $h(t)$.

Q. For a LTI system, suppose $x(t) = e^{-3t}u(t)$ is applied as input. The output $y(t) = (e^{-t} - e^{-2t})u(t)$.

Determine if system is stable & causal?

$$X(s) = \mathcal{L}[x(t)] = \frac{1}{s+3}, \quad ROC = \{s \in \mathbb{C} \mid Re(s) > -3\}$$

$$Y(s) = \mathcal{L}[y(t)] = \frac{1}{s+1} - \frac{1}{s+2}, \quad ROC = \{s \in \mathbb{C} \mid Re(s) > -1\}$$

$$= \frac{1}{(s+2)(s+1)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+2)(s+1)}, \quad ROC = \{s \in \mathbb{C} \mid Re(s) > -1\}$$

↓ subset of complex plane to the right of the rightmost pole.

$\Rightarrow h(t)$ is causal.

Stability: $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ is true.

if origin $\in$ ROC, then ↑

For $H(s)$, origin & the entire imaginary axis belongs to ROC.

$\Rightarrow$ the system is stable.

Laplace transform of some common signals.

(1) $e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$, $ROC = \{s \in \mathbb{C} \mid Re(s) > -Re(a)\}$

(2) $te^{-at}u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right)$

$= \frac{1}{(s+a)^2}$, $ROC = \{s \in \mathbb{C} \mid Re(s) > -Re(a)\}$

(3) $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^n}$

$t^{n-1}e^{-at}u(t) \longleftrightarrow \frac{(n-1)!}{(s+a)^n}$ ✓

(4) $x_1(t) = \cos(\omega_0 t)\,u(t)$

$= \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

$X(s) = \frac{1}{2}\cdot\left(\frac{1}{s+j\omega_0}\right) + \frac{1}{2}\cdot\left(\frac{1}{s-j\omega_0}\right)$, $ROC = \{Re(s) > 0\}$

$= \frac{\frac{1}{2}[s-j\omega_0 + s+j\omega_0]}{(s+j\omega_0)(s-j\omega_0)} = \frac{s}{s^2+\omega_0^2}$.

$x_2(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$\int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-st}\,dt = \int_{-\infty}^{\infty} e^{-(s-j\omega_0)t}\,dt$

does not converge to any finite value.

Laplace transform does not exist.

6th NOV

Practice problems on Laplace transform

9.2, 9.4, 9.7, 9.8, 9.16, 9.13, 9.22, 9.26, 9.31, 9.33

(9.6) $x(t)$ is absolutely integrable. $X(s)$ has a pole at 2.

(i) can $x(t)$ be right sided ? NO

(ii) can $x(t)$ be left " ? yes

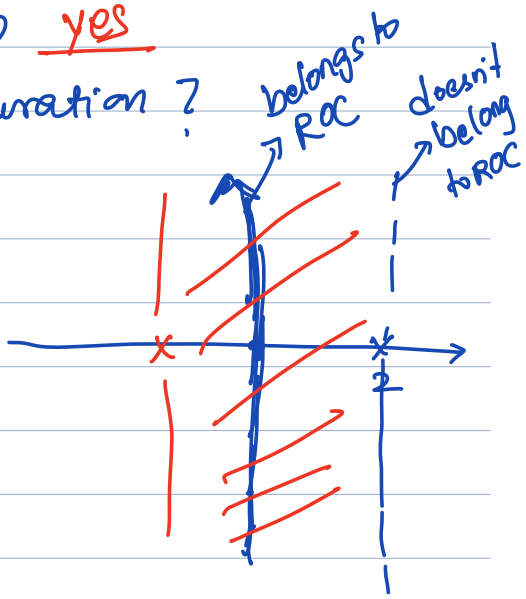
(iii) can $x(t)$ be two-sided ? yes

(iv) can $x(t)$ be of finite duration ? NO

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

⇓

imaginary axis is part of ROC.



(9.14) A signal $x(t)$ satisfies:

(i) $x(t)$ is real and even

(ii) $X(s)$ has four poles & no zeros

(iii) $X(s)$ has a pole at $\frac{1}{2}e^{j\pi/4}$

(iv) $\int_{-\infty}^{\infty} x(t)\,dt = 4 \Rightarrow X(0) = 4$

Determine $X(s)$ and its ROC.

Solution: $x(t) = x(-t)$, $X(-s) = X(s)$

$x(t)$ is either two-sided or has finite duration.

$x(t) = (x(t))^*$, $(X(s^*))^* = X(s)$

$$X(s) = \frac{(\quad)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

$$= \frac{(\quad)}{(-s+p_1)(-s+p_2)(-s+p_3)(-s+p_4)}$$

✓ if p is a pole, $-p$ is also a pole.

$$= \frac{(\quad)}{(s^*+p_1)^*(s^*+p_2)^*(s^*+p_3)^*(s^*+p_4)^*}$$

$$= \frac{(\quad)}{(s+p_1^*)(s+p_2^*)(s+p_3^*)(s+p_4^*)}$$

✓ If p is a pole, then p^* is also a pole.

Given: $\frac{1}{2}e^{j\frac{\pi}{4}}$ is a pole.

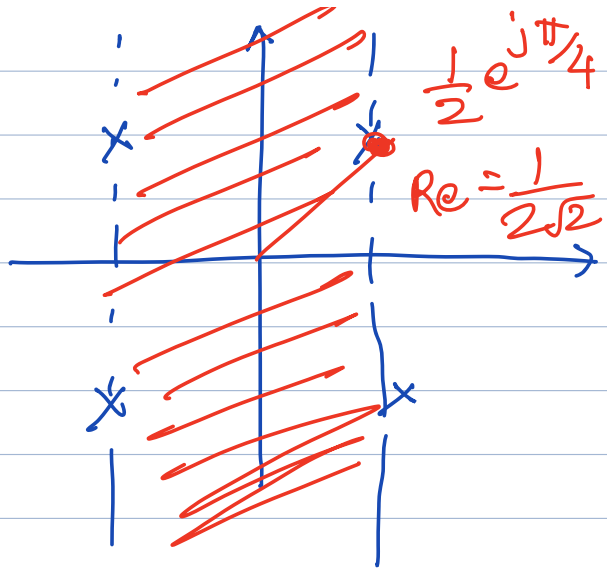
$\Rightarrow -\frac{1}{2}e^{j\pi/4}$, $\frac{1}{2}e^{-j\pi/4}$, $-\frac{1}{2}e^{-j\pi/4}$

$$4 = X(0) = \frac{NUM}{\left(\frac{1}{2}e^{j\pi/4} \cdot \frac{1}{2}e^{-j\pi/4}\right)}$$

$$\times \frac{NUM}{\frac{1}{2}e^{j\pi/4}\,\frac{1}{2}e^{-j\pi/4}}$$

$$4 = \frac{NUM}{\frac{1}{4} \cdot \frac{1}{4}}$$

$$\Rightarrow NUM = \frac{1}{4}$$



$$ROC = \left\{ s \in \mathbb{C} \;\middle|\; -\frac{1}{2\sqrt{2}} < Re(s) < \frac{1}{2\sqrt{2}} \right\}$$

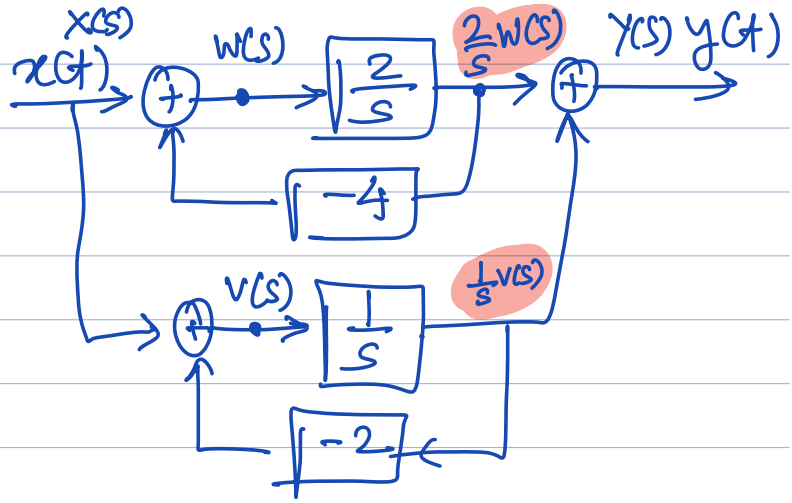
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt = 4$$

$$X(s) = \frac{NUM}{\left(s + \frac{1}{2}e^{j\pi/4}\right)\left(s + \frac{1}{2}e^{-j\pi/4}\right)\left(s - \frac{1}{2}e^{j\pi/4}\right)\cdot \left(s - \frac{1}{2}e^{-j\pi/4}\right)}$$

(9.17)

Causal LTI system. Determine a differential eqn relating $x(t)$ and $y(t)$.



$$W(s) = X(s) - 4 \cdot \frac{2}{s} \cdot W(s)$$

$$\Rightarrow W(s)\left[1 + \frac{8}{s}\right] = X(s)$$

$$\Rightarrow W(s) = \frac{s}{s+8} X(s)$$

$$V(s) = X(s) - 2\frac{1}{s}V(s)$$

$$\Rightarrow V(s)\left[1 + \frac{2}{s}\right] = X(s)$$

$$\Rightarrow V(s) = \frac{s}{s+2} X(s)$$

$$Y(s) = \frac{1}{s}V(s) + \frac{2}{s}W(s)$$

$$= \frac{X(s)}{s+2} + \frac{2X(s)}{s+8}$$

$$\Rightarrow (s^2+10s+16)Y(s) = (s+8)X(s) + (2s+4)X(s)$$

$$= (3s+12)X(s)$$

$$\Rightarrow \frac{d^2y(t)}{dt^2} + 10\frac{d}{dt}y(t) + 16\,y(t) = 3\frac{d}{dt}x(t) + 12x(t)$$

(9.15) $x(t)$ and $y(t)$ are two right sided signals.

$$\frac{d}{dt}x(t) = -2y(t) + \delta(t) \quad -$$

$$\frac{d}{dt}y(t) = 2x(t) \quad -$$

Determine $X(s)$, $Y(s)$ & their ROCs.

Solution:

$$sX(s) = -2Y(s) + 1$$

$$sY(s) = 2X(s) \Rightarrow X(s) = \frac{s}{2}Y(s)$$

$$\frac{s^2}{2}Y(s) + 2Y(s) = 1$$

$$\Rightarrow Y(s)\left(\frac{s^2+4}{2}\right) = 1 \Rightarrow Y(s) = \frac{2}{s^2+4}$$

poles: $\underline{+j2, -j2}$

$$X(s) = \frac{s}{s^2+4}$$

ROC $= \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$.

9.34 causal & stable LTI system with impulse respone $h(t)$, $\mathcal{L}[h(t)] = H(s)$. $H(s)$: rational fn.

(i) $H(1) = 0.2$

(ii) when input is $u(t)$, output is absolutely integrable.

(iii) when input is $tu(t)$, output is not abs. integrable

(iv) The signal $\frac{d^2h(t)}{dt^2} + 2\frac{dh(t)}{dt} + 2h(t)$ is of finite duration.

(v) $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ & its ROC.

From (v):

deg of den = deg of num + 1

and there is no term other than the term with highest degree.

$$H(s) = \frac{s}{s^2+4s+3}$$

$$= \frac{\lim_{s\to\infty}\left(\frac{1}{s}\right)}{\lim_{s\to\infty}\left(1+\frac{4}{s}+\frac{3}{s^2}\right)}$$

$= 0 \Rightarrow$ there is a single zero at $s=\infty$.

$(s^2+2s+2)H(s)$

(ii) $\frac{1}{s}H(s)$ is integrable

$\frac{1}{s^2}H(s)$ is not integrable.

(iv) $(s^2+2s+2)H(s)$ does not have any pole at finite complex numbers.

$\Rightarrow H(s) = \dfrac{N(s)}{s^2+2s+2}$

$= \dfrac{cs}{s^2+2s+2}$ /

$H(1) = \dfrac{c}{1+2+2} = 0.2$

$\Rightarrow \underline{c = 1}$

(9.43) Let $h(t)$ impulse response of causal, stable LTI system, $H(s)$ is a rational fn.

(i) ✓ System with impulse response $\frac{d}{dt}h(t)$ is causal & stable. (True) $sH(s)$

(ii) system with impulse response $y(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$ is causal & stable. (unclear / insufficient information)

Determine true of false.

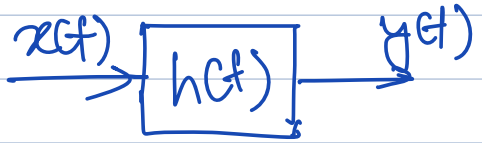
solution: $y(t) \longleftrightarrow \frac{1}{s} H(s)$

$\Rightarrow$ new pole possibly added at $s=0$.

all the poles of $H(s)$ had -ve real part.

9.45

$X(s) = \frac{s+2}{s-2}$, $\mathrm{Re}(s) < 2$



$x(t) = 0, \quad t > 0$

$y(t) = -\frac{2}{3} e^{2t} u(-t) + \frac{1}{3} e^{-t} u(t)$

$Y(s) = \frac{1}{3} \cdot \frac{1}{s+1} + \frac{2}{3} \cdot \frac{1}{s-2}$

Determine

(a) $H(s)$ & its ROC

(b) $h(t)$

(c) find $y(t)$ when input $x(t) = e^{3t}$

$-1 < \mathrm{Re}(s) < 2$

$H(s) = \frac{Y(s)}{X(s)}$, ROC: $-1 < \mathrm{Re}(s) < 2$ contains

$= \frac{s-2}{s+2} \cdot \left(\frac{1}{3s+3} + \frac{2}{3s-6} \right)$

$= \frac{(s-2)2/3\,s}{3(s+2)(s+1)(s-2)}$

$= \frac{2s}{(s+2)(s+1)}$

$$= \underbrace{\frac{A}{s+2}}_{} + \underbrace{\frac{B}{s+1}}_{}$$

$$h(t) = \left(Ae^{-2t} + Be^{-t}\right)u(t)$$

Practice problems on Z-transform

10.2, 10.4, 10.7, 10.8, 10.11, 10.17, 10.18, 10.20
10.26, 10.34, 10.37, 10.38, 10.47
