LECTURE 1: 17^{th} July 2026

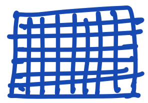
EE21201: Signals and Systems
Instructor: Prof. Ashish R. Hota

- Class Hours: E4 Slot. Wednesday: 12pm - 12:55pm, Thursday: 11am - 11:55am, Friday: 9am - 10:55am
- Venue: NR 321
- Grading Scheme: 30 % Endsem, 30 % Midsem, 10 % Attendance, 30 % Class Tests. → two, one before midsem, one after midsem.
- Preferred Mode of Contact: Send email to ashish.hota@ieee.org with subject containing [EE21201]. Do not forget to write your name and roll no. Any email with a blank subject and without name and roll no. will be ignored.
- Class handouts and other materials will be uploaded on my website at https://facweb.iitkgp.ac.in/~ahota/signals_systems.html
- Reference Books:
  1. **Primary Reference:** Signals and Systems 2nd Edition, by Oppenheim, Willsky and Nawab
  2. Principles of Linear Systems and Signals, by B. P. Lathi
- General expectation: spend 3-4 hours every week to keep up with the lecture, spend time solving problems yourself rather than watch videos online
- TAs: Aanchal, Sai Praveen

Overview

- A **signal** represents information or data about a phenomenon of interest.
- A **system** accepts *input* signals and produces *output* signals in response.
- Examples can be found in many disciplines.
  - Electrical: voltage, current
  - Mechanical: force, torque, velocity.
  - Chemical and Biological:
  - Environmental: concentration of pollutant in atmosphere.
  - Economic: interest rate, GDP, ..
  - Social:
  - Audio/Visual:
  - Computing:
- In this subject, we will learn about a set of mathematical tools to represent, analyze and design a class of signals and systems.

What is a signal?

- Mathematically, a signal can be regarded as a function of one or more independent variables.
- Example:
  - Voltage signal: $v(t) = 5\ V$, $v(t) = V_0 \sin(\omega t)$: domain : $\mathbb{R}$, range : $\mathbb{R}$.
  - Image signal:  domain : $\binom{x}{y}$, range : $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$
  - Video signal:
- In this course: we are dealing with signals that are functions of a single independent variable (typically time).
- Time-dependent signals can be classified into the following two sub-classes:
  - Continuous Time (CT)
  - Discrete Time (DT)

$||$

domain $\begin{bmatrix} x \\ y \\ t \end{bmatrix}$, range: $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$

Continuous-time Signals

domain: set of real numbers

- **Continuous-time Signals**: Any mapping from real-numbers (time variable t) to complex numbers

$x(t)$

- We use notation: $x(t), y(t), z(t), \ldots$:
  - reserve time variable t for continuous-time signals' independent variable
  - use parenthesis $(\cdot)$
- Typically, physical quantities or variables are examples of continuous-time signals.
- Examples: ?

$x(5)$: value, real number

→ $x(\tau)$: value of the signal at time τ,

→ $x(t)$: signal

Discrete-time Signals

domain

- **Discrete-time Signals**: Any mapping from integer-numbers (time variable n) to complex numbers

$\{\ldots, -2, -1, 0, 1, 2, \ldots$

- We use notation: $x[n], y[n], z[n], \ldots$:
  - reserve time variable n for discrete-time signals' independent variable
  - use square bracket $[\cdot]$

$x[5]$: value at index $n=5$

$x[n]$: signal itself

- Examples: ?

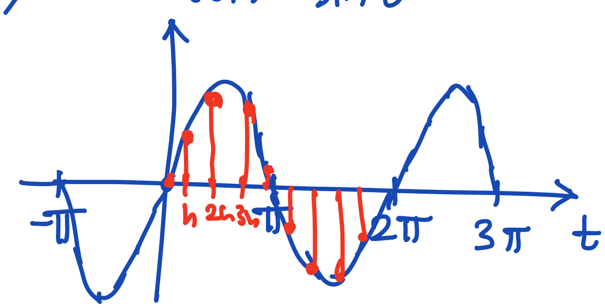
- One can obtain discrete-time signals by sampling continuous-time signals.

→ video : discrete-time, but with other independent variables in the domain.

→ sheet music.

→ stock price of a company, ...

$x(t) = \sin t$



Let h: sampling time/resolution

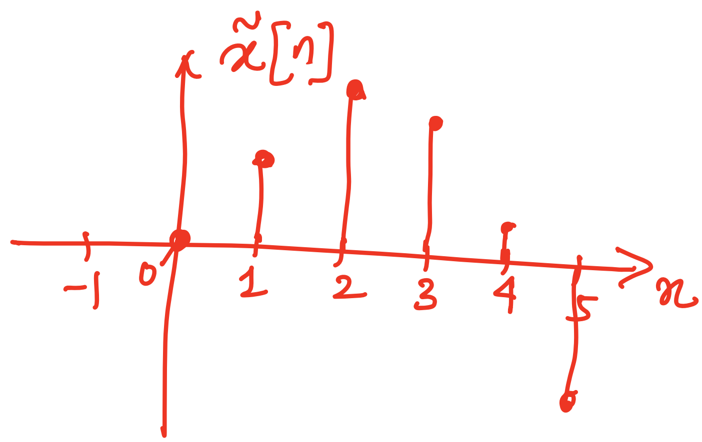
$$\tilde{x}[n] = x(nh)$$

$$\tilde{x}[0] = x(0)$$

$$\tilde{x}[1] = x(h)$$

$$\vdots$$

$$\tilde{x}[k] = x(kh)$$



deposit/spending in a day $\rightarrow$ $\boxed{\text{Bank account}}$ $\rightarrow$ amount accumulated at the end of the day.

voltage $\rightarrow$ $\boxed{\text{Fan}}$ $\rightarrow$ angular speed

continuous-time system

Systems

- Continuous-time system : input is continuous-time signal, output is " " " , ex: fan
- Discrete-time system : both input & output are discrete-time signals.
- Hybrid system: input & output can be both CT & DT signals.
- Dynamical system: any system with an internal state and a rule govering how the state evolves in time
- Example: a car with input being acceleration $a(t)$ and output being velocity $v(t)$. The evolution of speed is given by

$$\frac{dv}{dt} = a(t).$$

  The output is measured by the speedometer sensor. This is an example of a continuous-time dynamical system.

- The field of *signal processing* studies how to extract desirable features from given signals, often via design of *filters*.
- The field of *control systems* studies how to design systems (*controllers*) which produce suitable signals to steer the output of a given dynamical system in a desirable manner.

Outline of this subject

- Classifications, transformations and basic properties of signals and systems
- Time domain analysis of *Linear Time-Invariant* (LTI) systems
- Frequency domain analysis of signals and LTI systems: decomposition and eigenfunctions.
- Sampling, quantization and associated challenges

The other broad classification of a signal is as follows.

(A) Analog signal : range is a subset of the set of real numbers.

e.g: $x(t) = \sin t$
range: $[-1, 1]$

(B) Digital signal: $\downarrow$ range has finite cardinality.

In this course, unless specified otherwise, the signals are going to be analog.

$$x \longrightarrow \boxed{A} \longrightarrow y = Ax$$

$A \in \mathbb{R}^{n \times n}$

if x is an eigenvector v corresponding to eigenvalue λ, then $y = \lambda v$

$x(t) = \sum_{i \in I} \alpha_i v_i(t)$, where $v_i(t)$: is an eigenfunction with eigenvalue λ_i

then output $y(t) = \sum_{i \in I} \alpha_i \lambda_i v_i(t)$

LECTURE-2 : 22^{nd} July

Basic Notations

- Notations:

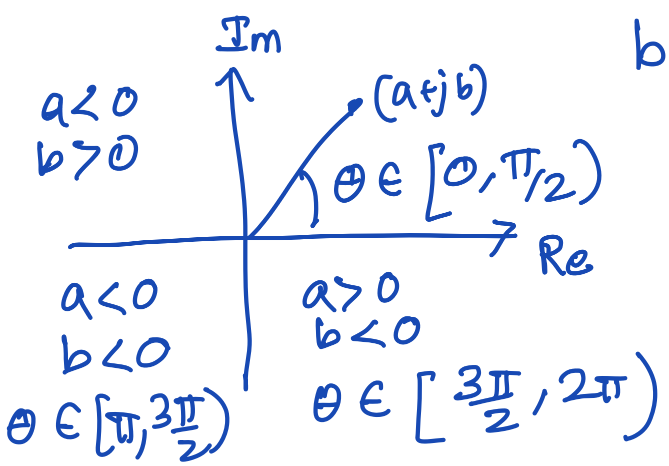
  a. $\mathbb{Z}$: the set of integers $\ldots, -2, -1, 0, 1, 2, \ldots$

  b. $\mathbb{R}$: the set of real numbers

  c. $\mathbb{C} := \{(a + jb) \mid a, b \in \mathbb{R}\}$ the set of complex numbers

- A complex number can be identified as a vector on the $\mathbb{R}^2$ plane.
- Notation $j := \sqrt{-1}$
- $x = a + jb$ is Cartesian coordinate representation
- $x = re^{j\theta}$, where $r = \sqrt{a^2 + b^2}$ and $\theta = ?$ is the polar coordinate representation.

$\tan^{-1}\left(\frac{b}{a}\right) \in [0, 2\pi)$ and the sign of a and b determine its exact value.



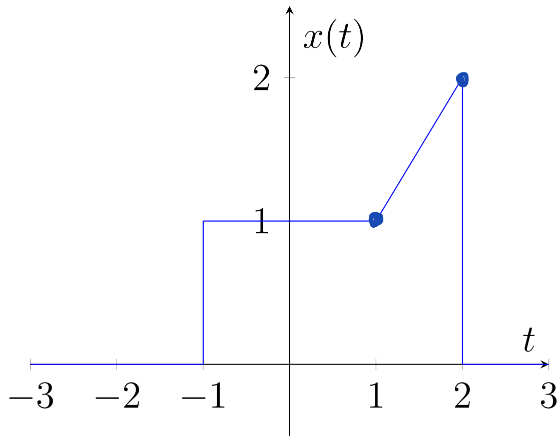
Transformation of Signals: Time Shift

- For any $t_0 \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$, **time shift** is defined as

$$\begin{array}{lll} x(t) \longrightarrow x(t-t_0) & & \text{CT} \\ x[n] \longrightarrow x[n-n_0] & & \text{DT} \end{array}$$

- For $t_0 > 0$ ($n_0 > 0$), it is called **delay** and for $t_0 < 0$ ($n_0 < 0$) it is called **advance**.

- Example: Suppose that $x(t)$ is given by:



- What would be $x(t-1)$?

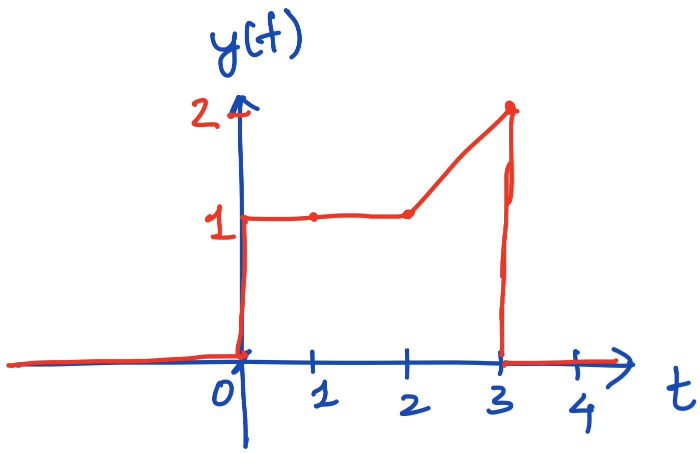
$y(t) := x(t-1)$

$y(3) = x(3-1) = x(2) = 2$

$y(2) = x(1) = 1$

$y(1) = x(0) = 1$

and so on.

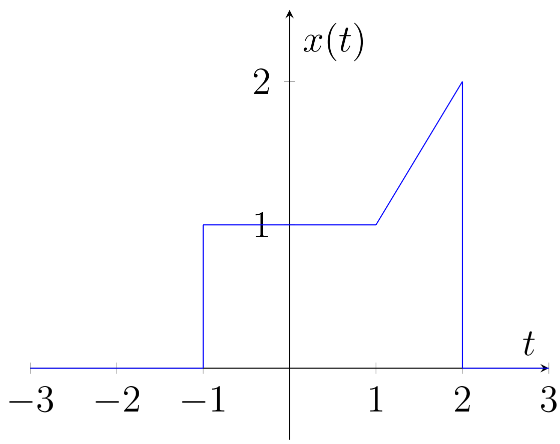


Transformation of Signals: Time Reversal

- Time reversal is defined as

$$\begin{aligned} x(t) &\longrightarrow x(-t) \\ x[n] &\longrightarrow x[-n], \end{aligned}$$

- Interpretation: Flipping over y-axis
- How does $x(-t)$ look like in this example?



Let $y(t) := x(-t)$

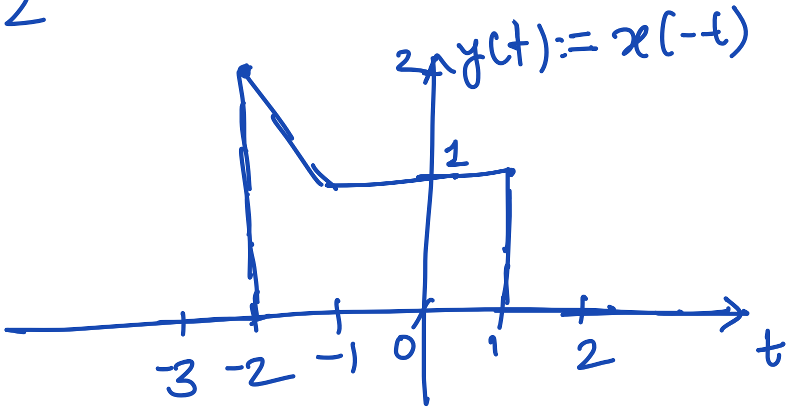
$y(1) = x(-1) = 1$

$y(2) = x(-2) = 0$

$y(-1) = x(1) = 1$

$y(-2) = x(2) = 2$

and so on.



Transformation of CT Signals: Time Scaling

- For CT Signals, time-scaling by a factor $a > 0$ is defined as

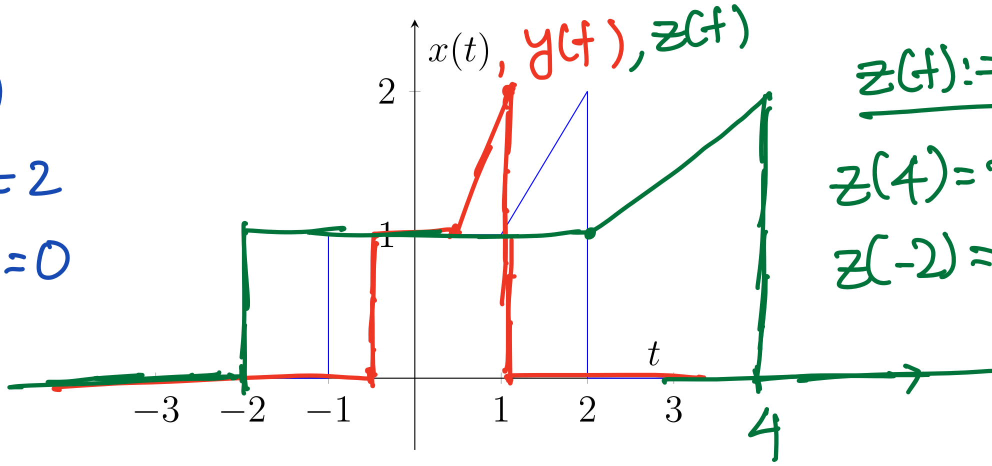
$$x(t) \longrightarrow x(at), \quad a > 0.$$

- If $a > 1$ it is called **Decimation** (squeezing/compression)
- If $1 > a > 0$, it is called **Expansion** (enlarging)
- If $a < 0$, we get a reversed compressed/expanded signal
- Class activity: How does $x(2t)$ and $x(t/2)$ look like in the above example?

$y(t) := x(2t)$

$y(1) = x(2) = 2$

$y(-1) = x(-2) = 0$



$z(t) := x\left(\frac{t}{2}\right)$

$z(4) = x(2) = 2$

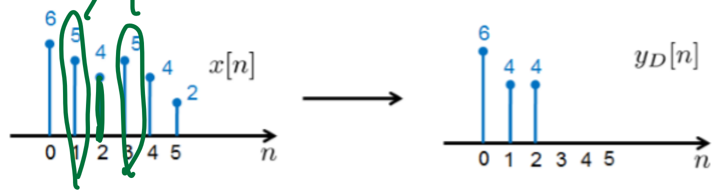
$z(-2) = x(-1) = 1$

Time Scaling: Discrete Time

- **Decimation**: For a DT Signal, and an **integer** factor $M \geq 1$, the decimated signal is defined by:

$$y_D[n] = x[Mn].$$

lost



$M=2$

$y_D[0] = x[0]$

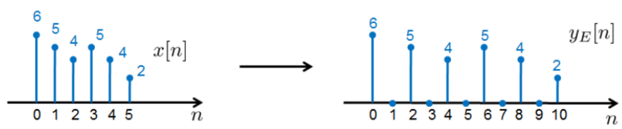
$y_D[1] = x[2]$

we lose information at different intermediate indices.

- **Expansion**: For an integer $L \geq 1$, the discrete-time expanded signal (by a factor L) is:

$$y_E[n] := \begin{cases} x\left[\frac{n}{L}\right], & n = \text{integer multiple of } L \\ 0, & \text{otherwise.} \end{cases}$$

if $L=3$,



$y_E[2] = x\left[\frac{2}{3}\right]$

which is not defined, so value of y_E (in such cases) needs to be specified in the definition itself.

Linear-Time Transformation

- Often we are interested in the transformation $\boxed{x(t) \to x(at-b)}$
- It is indeed a combination of time shift and time-scaling (order matters):
  1. Define $v(t) = x(t-b)$, → shift first
  2. Define $y(t) = v(at) = x(at-b)$. → scale later
- So $x(t) \to x(at-b)$ is equivalent to time-delay then scale!

- If you scale first, and then do the delay, what do you obtain?

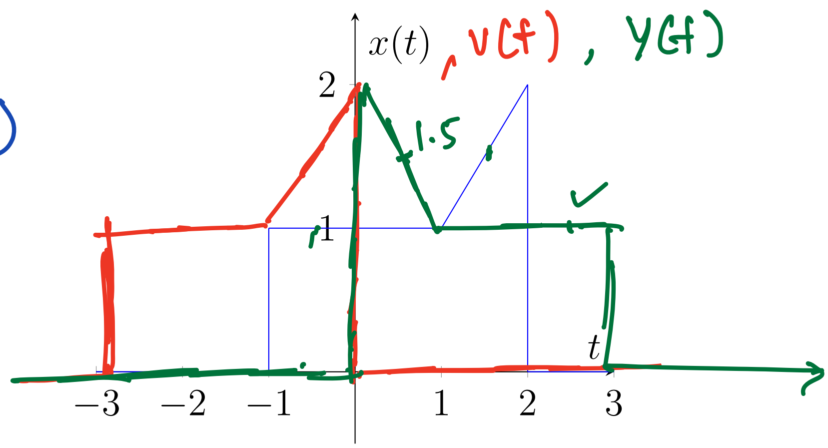
scale first: $\tilde{v}(t) = x(at)$

shift later: $\tilde{y}(t) = \tilde{v}(t-b) = x(a(t-b)) \neq x(at-b)$

- Example: Plot the signal $x(-t+2)$ for the signal $x(t)$ as bellow,

step-1:

$v(t) = x(t+2)$



step 2:

$y(t) = v(-t)$

$= x(-t+2)$

$y(2.5) = x(-2.5+2)$

$= x(-0.5) = 1$

$y(0.5) = x(-0.5+2)$

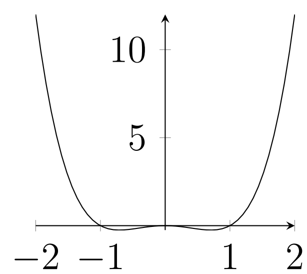
$= x(1.5)$

Signal Categories: Even/Odd Signals

Definition 1. *A CT (DT) signal $x(t)$ $(x[n])$ is* even *if*

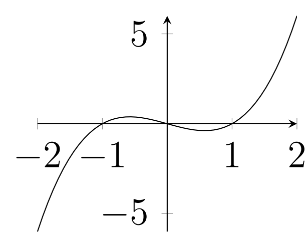
$$x(-t) = x(t) \qquad (x[-n] = x[n]),$$

for all t (n).



Definition 2. *A CT (DT) signal $x(t)$ $(x[n])$ is* odd *if*

$$x(-t) = -x(t) \qquad (x[-n] = -x[n]).$$



Examples of Even/Odd Signals

- Is $x[n] = n^3$ even or odd?
- Is $x[n] = n^3 - n^2 - 1$ even or odd?

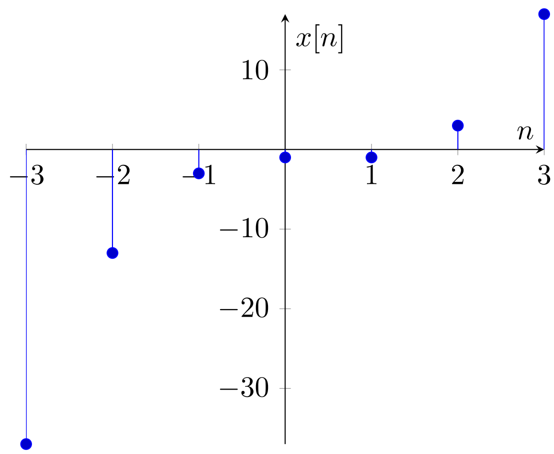


Figure 1: Plot of signal $x[n] = n^3 - n^2 - 1$

- But $x[n] = \underbrace{n^3}_{x_{odd}[n]} + \underbrace{(-n^2 - 1)}_{x_{even}[n]}$.
- Can we do this for any signal, i.e. write down any signal as $x(t) = x_{even}(t) + x_{odd}(t)$?
- Answer: yes!

$x_o(-t) = \frac{x(-t) - x(t)}{2} = -x_o(t)$

Even/Odd Decomposition: For any CT signal $x(t)$ we can write:

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t) = \frac{x(t)+x(-t)}{2}$ is even and $x_o(t) = \frac{x(t)-x(-t)}{2}$ is odd.
For any DT signal $x[n]$ we can write:

$$x[n] = x_e[n] + x_o[n],$$

where $x_e[n] = \frac{x[n]+x[-n]}{2}$ is even and $x_o[n] = \frac{x[n]-x[-n]}{2}$ is odd.

$x_e(-t) = \frac{x(-t) + x(-(-t))}{2} = x_e(t)$

$x(t) = \sin t$, $T = 2\pi$ (fundamental period), $\omega_0 = 1$ rad/sec

Periodic and Aperiodic Signals

$x(\tau + 2T) = x(\tau + T) = x(\tau)$

- Continuous Time: $x(t)$ is *periodic* with period $T > 0$ if $x(t + T) = x(t)$ for all time t.

  can be any real numbers

- **Fundamental period and frequency:**
  - smallest $T > 0$ that satisfies this is called the fundamental period
  - for the fundamental period T, $\omega_0 = \frac{2\pi}{T}$ is called the fundamental angular frequency. Unit: radian/second
  - frequency $f = 1/T$ has unit second^{-1} or Hertz (Hz)
- Discrete Time: $x[n]$ is *periodic* with period $N > 0$ if $x[n + N] = x[n]$ for all integer n. N needs to be an integer.
- **Fundamental period and frequency:**
  - smallest $N > 0$ that satisfies this is called the fundamental period
  - for the fundamental period N_0, $\omega_0 = \frac{2\pi}{N_0}$ is called the fundamental frequency.
- If a signal is not periodic, we call it an **aperiodic** signal.
- Examples: $x(t) = \sin(\omega t)$ and $x[n] = \sin(\omega n)$. → is it periodic?

LECTURE -3: 23rd July.

Periodic and Aperiodic Signals

Clicker Question: What can we say about the periodicity of the following signals:

$$x(t) = \cos\left(\frac{\pi t^2}{8}\right),$$
$$x[n] = \cos\left(\frac{\pi n^2}{8}\right).$$

Let $x(t)$ be periodic with $T>0$ being the period. Then,

$$x(t) = x(t+T) \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \cos\left(\frac{\pi t^2}{8}\right) = \cos\left(\frac{\pi}{8}(t^2+T^2+2tT)\right)$$

$$= \cos\left(\frac{\pi t^2}{8} + \underbrace{\frac{\pi T^2}{8} + \frac{2\pi t T}{8}}\right)$$

which would require $\frac{\pi T}{8}\left[T+2t\right]$ to be an integer multiple of 2π.

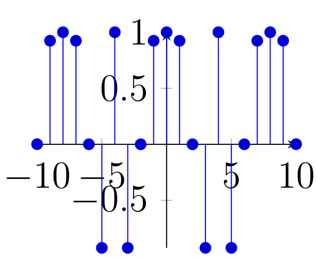
However, this would not be true since t is a continuous variable, irrespective of the choice of T.

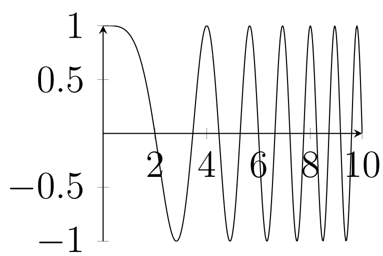
a. $x(t)$ is periodic, but $x[n]$ is aperiodic.

b. $x(t)$ is aperiodic, and $x[n]$ is aperiodic.

c. $x(t)$ is aperiodic, but $x[n]$ is periodic.

d. $x(t)$ is periodic, but $x[n]$ is periodic.





For discrete-time signal, a similar hypothesis results in

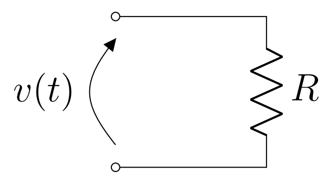
$$\cos\left(\frac{\pi n^2}{8}\right) = \cos\left(\frac{\pi}{8}n^2 + \underbrace{\frac{\pi}{8}N^2 + \frac{\pi}{8}2nN}\right)$$

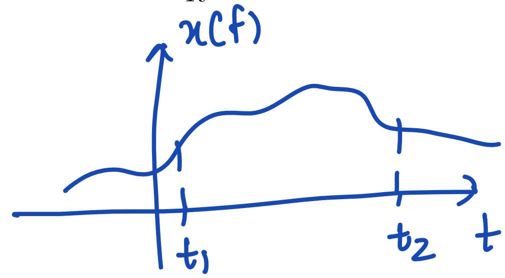
Can we find N s.t. these two terms become integer multiple of 2π?

at N=8 : $\frac{\pi}{8}\cdot 8^2 + \frac{\pi}{8}\cdot 2n\cdot 8 = \underbrace{2\pi n + 8\pi}$ always an integer multiple of 2π when $n \in \mathbb{Z}$.

Energy and Power of a Signal

- Remember: Instantaneous power of a resistor: $p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$





- Energy: power consumption over time $= \int_{t_1}^{t_2} p(t)dt$

- Motivated by these: We **DEFINE** the energy of a CT signal $x(t)$ over $[t_1, t_2]$ interval by:

$$E_{[t_1,t_2]} = \int_{t_1}^{t_2} |x(t)|^2 dt, \geq 0$$

  where $|x(t)|$ is the magnitude of the (complex-valued) signal at time t.

- We define the energy of a DT signal $x[n]$ over $[n_1, n_2]$ interval by:

$$E_{[n_1,n_2]} = \sum_{n=n_1}^{n_2} |x[n]|^2.$$

- We define *average power* of a signal:
  - Continuous-time: $x(t)$ over continuous interval (t_1, t_2) is:

$$P = \frac{1}{t_2 - t_1} E_{[t_1,t_2]} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt.$$

  - Discrete-time: $x[n]$ over discrete interval $[n_1, n_2]$ is:

$$P = \frac{1}{n_2 - n_1 + 1} E_{[n_1,n_2]} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2.$$

- The above definitions may not always correspond to the physical notion of energy or power, rather represent the *size* of the signal.

Total Energy of a Signal

- We define the total energy and average power of
  - a continuous-time signal $x(t)$ to be

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt,$$
$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt,$$

  - a discrete-time signal $x[n]$ to be

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- We say that a signal is finite energy if $E_\infty < \infty$.
- We say that a signal has finite average power if $P_\infty < \infty$.
- There could also be signals which have neither finite energy nor finite power.
- Example: Find the total energy and power of the signals $x_1(t) = 1$, $x_2(t) = t$, $x_3(t) = e^{-t}$ for $t \geq 0$ and $x_3(t) = 0$ for $t < 0$, and $x_4(t) = e^{-(1+j)t}$?

$x_1(t)$: $\int_{-T}^{T} |x_1(t)|^2 dt = \int_{-T}^{T} 1 dt = 2T$, $E^1_\infty = \lim_{T\to\infty} 2T = \infty$

$\frac{1}{2T} \int_{-T}^{T} |x_1(t)|^2 dt = \frac{1}{2T} \cdot 2T = 1$, $P^1_\infty = \lim_{T\to\infty} 1 = 1$

$x_3(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $\int_{-T}^{T} (e^{-t})^2 dt = \int_{0}^{T} e^{-2t} dt$

$= \frac{1}{2}(1 - e^{-2T})$

$E^3_\infty = \lim_{T\to\infty} \frac{1}{2}(1 - e^{-2T}) = \frac{1}{2}$.

Show that $P^3_\infty = 0$ in this case.

Discrete-Time: Unit Step

- Discrete-time case: $u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

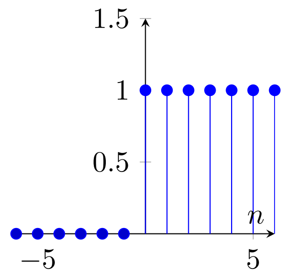


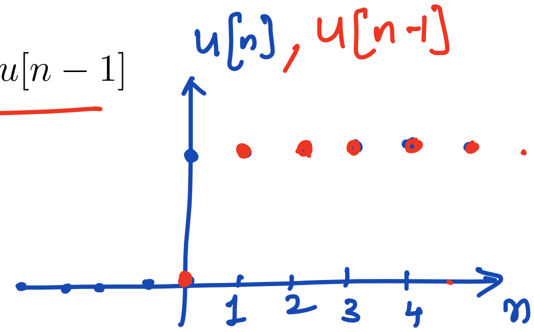
Figure 2: Plot of step signal $u[n]$

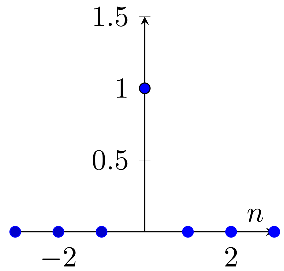
- How does $\sum_{k=0}^{\infty} u[n-k]$ look like?

Discrete-Time: Unit Impulse

- We define the discrete-time unit impulse as:

$$\delta[n] = \begin{cases} 0 & , \quad n \neq 0 \\ 1 & , \quad n = 0 \end{cases} = u[n] - u[n-1]$$





$z[n] := u[n] - u[n-1]$

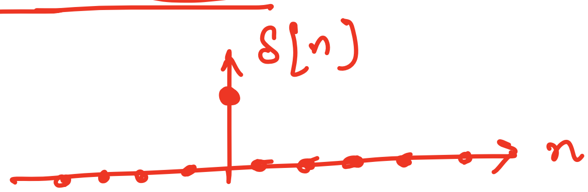
$z[0] = u[0] - u[0-1] = 1$

$z[1] = u[1] - u[0] = 0$

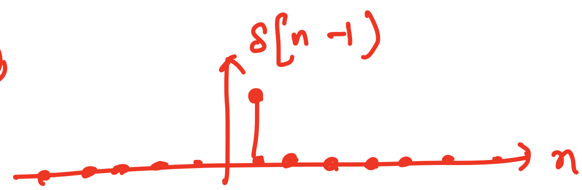
$\vdots$

- Show that $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$.

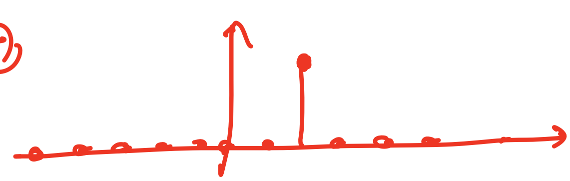
$k=0$: $\delta[n]$



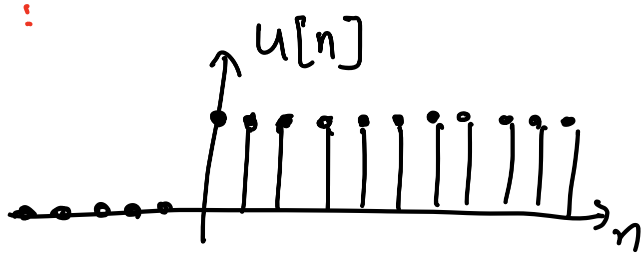
$k=1$: $\delta[n-1]$ $+$



$k=2$: $\delta[n-2]$ $+$

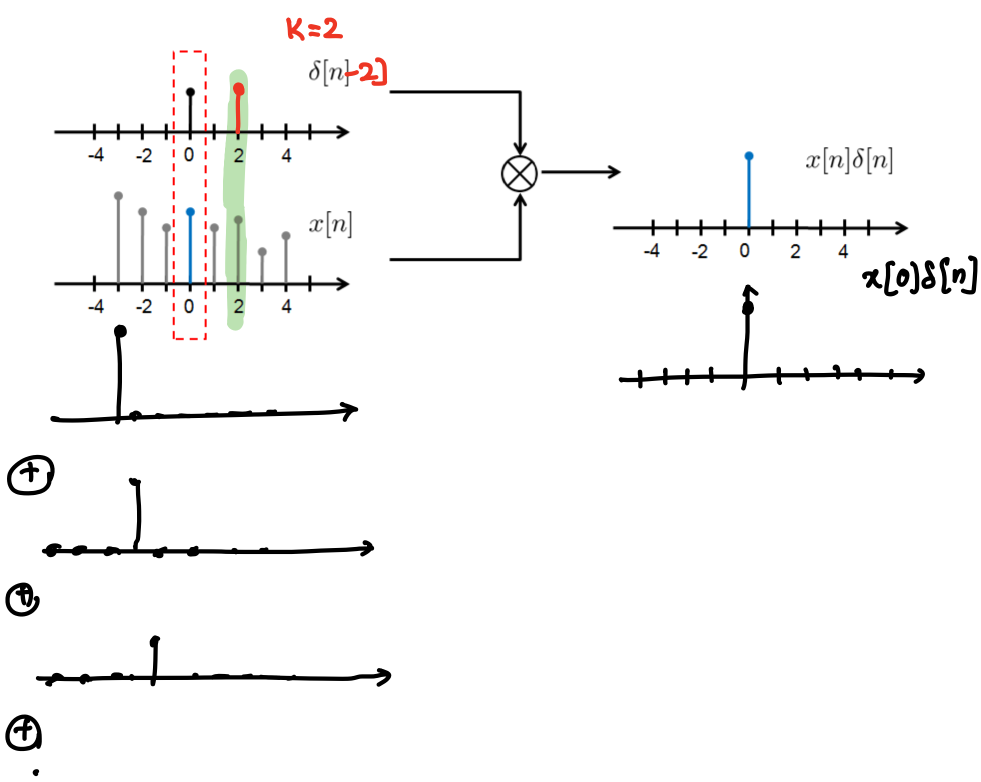


$\vdots$



Sampling Property of $\delta[n]$

- Convention: $x[n]$ represents the complete signal. $x[k]$ denotes the value of the signal at time k.
- Sampling Property
$$\underbrace{x[n]\delta[n-k]}_{y[n]} = \underbrace{x[k]\delta[n-k]}_{z[n]}.$$
- How to show this?
- By the definition of $\delta[n]$, $\delta[n-k] = 1$ if $n = k$, and 0 otherwise. Therefore,
$$\begin{aligned} x[n]\delta[n-k] &= \begin{cases} x[n], & n = k \\ 0, & n \neq k \end{cases} \\ &= x[k]\delta[n-k]. \end{aligned}$$
- Special case sampling property: When $k = 0$, we have $x[n]\delta[n] = x[0]\delta[n]$!



Shifting and Representation Properties of $\delta[n]$

Show that for any signal $x[n]$, we have:

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = x[k].$$

Proof. Follows from the sampling property:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]\delta[n-k] &= \sum_{n=-\infty}^{\infty} x[k]\delta[n-k] \\ &\overset{(a)}{=} x[k]\left(\sum_{n=-\infty}^{\infty} \delta[n-k]\right) \\ &\overset{(b)}{=} x[k].\end{aligned}$$

□

Show that any signal $x[n]$ can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Proof. Consider the sampling property:

$$x[n]\delta[n-k] = x[k]\delta[n-k].$$

Instead of summing over n (shifting property), sum over k:

$$\sum_{k=-\infty}^{\infty} x[n]\delta[n-k] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Left side is $x[n]$! □

Continuous-Time: Unit Step

- Continuous-time case: $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

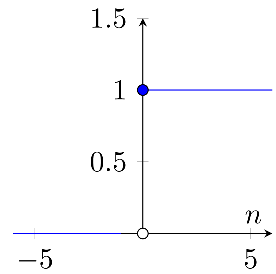
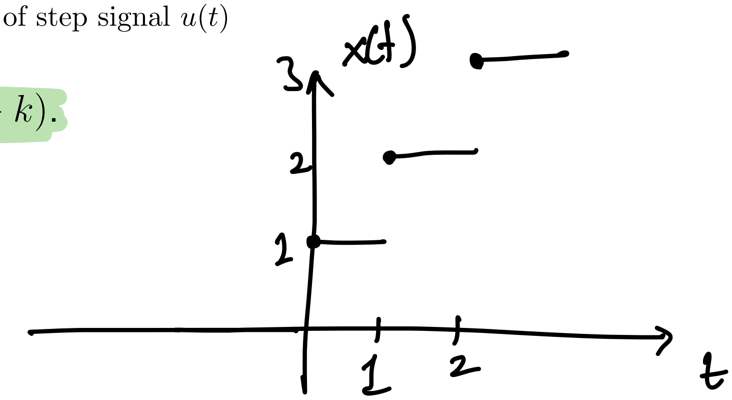


Figure 3: Plot of step signal $u(t)$

- Example: Plot $x(t) = \sum_{k=0}^{\infty} u(t-k)$.



LECTURE 4 & 5 : 24th July

Continuous Unit Impulse: Mathematical formulation

- Define $\delta(t)$ to be the signal that satisfies the following properties:

  a. $\delta(t) = 0$ at any point $t \neq 0$

  b. $\delta(0) = \infty$ such that:

$$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1.$$

  c. All the rules of calculus applies to it.

- a. and b. imply that $\int_{t_1}^{t_2} \delta(\tau)d\tau = 1$ for all $t_1 < 0 < t_2$.

$$\hat{\tau} := t - \tau$$
$$d\hat{\tau} = dt$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)dt = \int_{-\infty}^{\infty} \delta(\hat{\tau})d\hat{\tau} = 1$$

$$\int_{-\varepsilon_1}^{\varepsilon_2} \delta(\tau)d\tau = 1,$$

when $\varepsilon_1 > 0$, $\varepsilon_2 > 0$

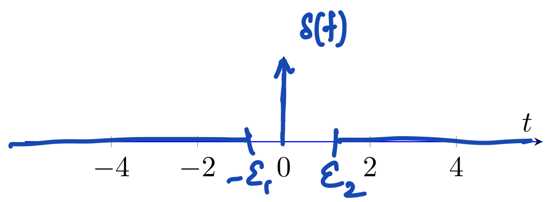
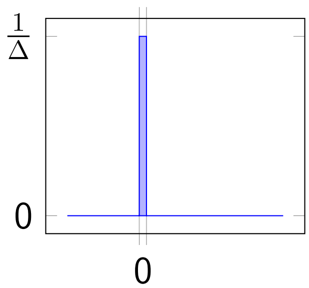


Figure 4: Representing $\delta(t)$

Unit-impulse: really, what is it?

- Define $\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } t \in [0, \Delta] \\ 0 & \text{else.} \end{cases}$

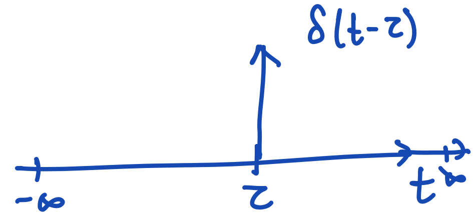


$$\int_{-\infty}^{\infty} \delta_\Delta(t)\, dt = \int_0^{\Delta} \frac{1}{\Delta}\, dt = \frac{1}{\Delta} \cdot \Delta = 1$$

- For any signal $x(t)$ and small enough $\Delta > 0$:

$$\int_{-\infty}^{\infty} \delta_\Delta(\tau) d\tau = \int_0^{\Delta} \delta_\Delta(\tau) d\tau$$
$$= \frac{1}{\Delta} \int_0^{\Delta} d\tau = 1.$$

- We can **think** of $\delta(t)$ as $\lim_{\Delta \to 0} \delta_\Delta(t)$.



Sampling Property of $\delta(t)$

Sampling Property: If $x(t)$ is continuous at τ, then

$$x(t)\delta(t-\tau) = \underline{\underline{x(\tau)\delta(t-\tau)}}.$$

$t^2\,\delta(t-1)$ $\qquad$ $(1)^2\,\delta(t-1)$

Proof.

- First: $\delta(t-\tau) = 0$ whenever $t \neq \tau$. Therefore,

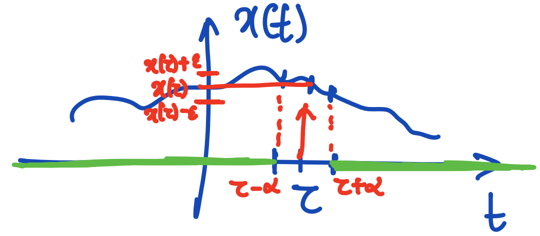
$$x(t)\delta(t-\tau) = \begin{cases} +\infty, & t = \tau \\ 0, & t \neq \tau. \end{cases}$$

- This is the same as $x(\tau)\delta(t-\tau)$ because

$$x(\tau)\delta(t-\tau) = \begin{cases} +\infty, & t = \tau \\ 0, & t \neq \tau. \end{cases}$$

- Next: check the integration of $x(t)\delta(t-\tau)$.

Integrating R.H.S., we get $\displaystyle\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,dt = x(\tau)\int_{-\infty}^{\infty}\delta(t-\tau)\,dt = x(\tau)$



Properties of $\delta(t)$

- Next: check the integration of $x(t)\delta(t-\tau)$. Let $\epsilon > 0$ be a small constant.
- There exists some $\alpha > 0$ such that $|x(t) - x(\tau)| < \epsilon$ for $\tau - \alpha \le t \le \tau + \alpha$.
- Therefore:

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = \underbrace{\int_{-\infty}^{\tau-\alpha} x(t)\delta(t-\tau)dt}_{0} + \int_{\tau-\alpha}^{\tau+\alpha} x(t)\delta(t-\tau)dt + \underbrace{\int_{\tau+\alpha}^{\infty} x(t)\delta(t-\tau)dt}_{0} = \int_{\tau-\alpha}^{\tau+\alpha} x(t)\delta(t-\tau)dt.$$

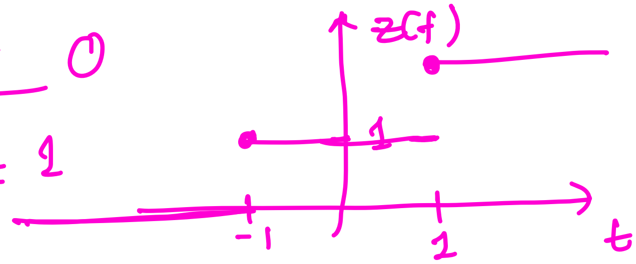
- Since $|x(t) - x(\tau)| < \epsilon$ for $\tau - \alpha \le t \le \tau + \alpha$,

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt - x(\tau)\right| \\ =&\left|\int_{\tau-\alpha}^{\tau+\alpha} x(t)\delta(t-\tau)dt - x(\tau)\right| \\ =&\left|\int_{\tau-\alpha}^{\tau+\alpha} (x(t)-x(\tau))\delta(t-\tau)dt\right| \\ \le& \int_{\tau-\alpha}^{\tau+\alpha} |x(t)-x(\tau)|\delta(t-\tau)dt \\ \le& \epsilon \int_{\tau-\alpha}^{\tau+\alpha} \delta(t-\tau)dt = \epsilon. \end{aligned}$$

$$x(\tau) = \int_{\tau-\alpha}^{\tau+\alpha} x(\tau)\delta(t-\tau)\,dt$$

- As this holds for every $\epsilon > 0$, $\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(\tau)$.
- On the other hand $\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)dt = x(\tau)\int_{-\infty}^{\infty} \delta(t-\tau)dt = x(\tau)$.

$$z(-1+\varepsilon) = \underbrace{\int_{-\infty}^{-1+\varepsilon} \delta(t-1)t^2\,dt}_{=\,0} + \underbrace{\int_{-\infty}^{-1+\varepsilon} \delta(t+1)\,t^2\,dt}_{=\,1}$$

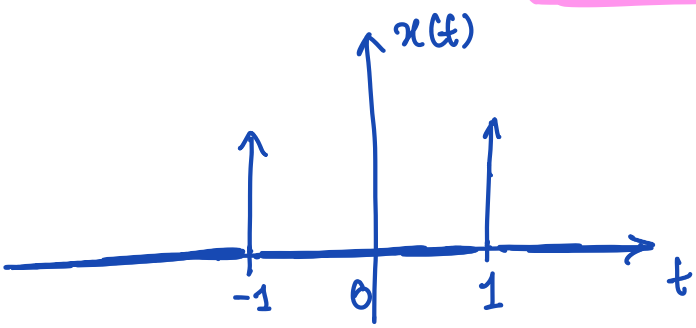


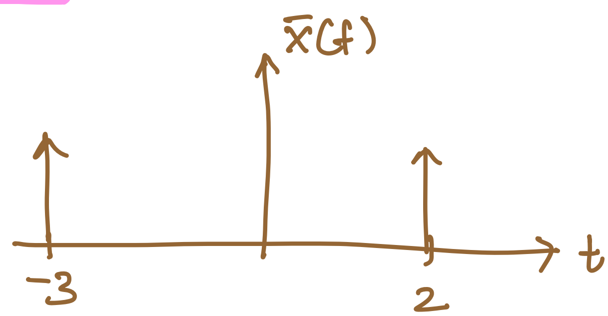
Example

- Example 1: Plot $x(t) = (\delta(t-1) + \delta(t+1))t^2$ and $z(t) = \int_{-\infty}^{t} x(\tau)d\tau$.

$$\bar{x}(t) = \big(\delta(t-2) + \delta(t+3)\big)t^2$$

- Example 2: Show that $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ (or, equivalently, $\frac{d}{dt}u(t) = \delta(t)$).





$$\int_{-\infty}^{\infty} x(t)\,dt = \int_{-\infty}^{\infty} \delta(t-1)\,t^2 dt + \int_{-\infty}^{\infty} \delta(t+1)\,t^2 dt$$

$$= \int_{-\infty}^{\infty} 1.\,\delta(t-1)\,dt + \int_{-\infty}^{\infty} 1.\,\delta(t+1)\,dt$$

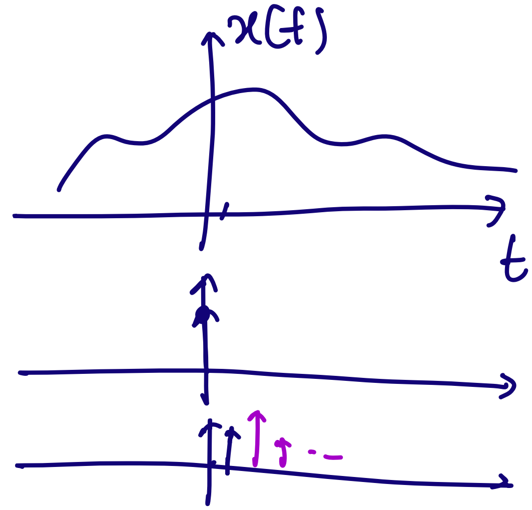
$$= 2.$$

$$\int_{-\infty}^{\infty} \bar{x}(t)\,dt = 4 + 9 = 13.$$

Properties of $\delta(t)$

- **Shifting Property:**

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(\tau).$$



- **Representation Property:**

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau,$$

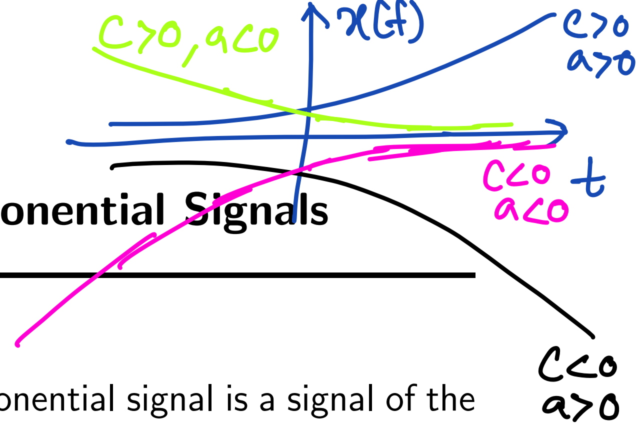
Proof. Using the sampling property:

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau).$$

Integrating both sides:

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t).\end{aligned}$$

$\square$



Special Signals: Complex Exponential Signals

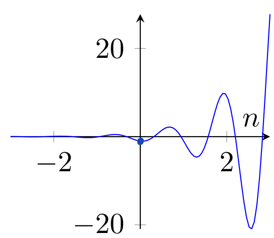
- Definition: A continuous-time (complex) exponential signal is a signal of the form:

$$x(t) = Ce^{at},$$

for $a, C \in \mathbb{C}$.

- Suppose C and a are both real numbers. What are the possible trajectories of this signal?

- Clicker Question 2: The following is the plot of $\text{Re}[x(t)]$ for a complex exponential signal $x(t) = Ce^{at}$ where C is real. Which one of the following set of parameters correspond to this plot?



a. $\text{Re}[C] < 0$, $\text{Re}[a] < 0$

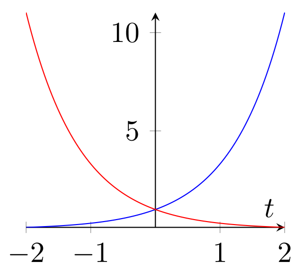
b. $\text{Re}[C] < 0$, $\text{Re}[a] > 0$

c. $\text{Re}[C] > 0$, $\text{Re}[a] < 0$

d. $\text{Re}[C] > 0$, $\text{Re}[a] > 0$

Special Signals: Exponential Signals

- **Real Exponential Signals**:
  - The case where $a = \alpha$ and $C = B$ are real numbers
  - $\alpha > 0$ exponentially increasing, $\alpha < 0$ exponentially decreasing
  - Note: Energy of complex exponentials (except for $C = 0$) is infinity
  - Example: What is the energy of $x(t) = e^{-(2+3j)t}u(t)$?



Exponential and Sinusoidal Signals

- If $C = Ae^{j\theta}$ and $a = \alpha + j\omega_0$, then $x(t) = Ce^{at} = Ae^{\alpha t}e^{j(\omega_0 t+\theta)}$

  $\cos(\omega_0 t + \theta)$
  $+ j\sin(\omega_0 t + \theta)$

- Decaying exponent: real part of a
- Oscillatory behavior: imaginary part of a
- Phase: angle of C
- **Periodic Complex Exponential**:
  - The case where $a = j\omega_0$, i.e., $\alpha = 0$
  - In this case: $x(t) = Ae^{j(\omega_0 t+\theta)} = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta)$
  - In this case, $x(t)$ is periodic with the fundamental period $\frac{2\pi}{|\omega_0|}$
  - What if $\omega_0 < 0$?

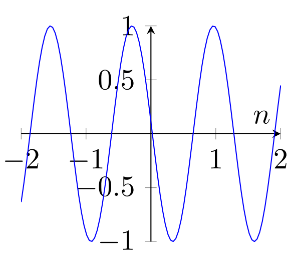


Figure 5: Plot $\mathtt{Re}[e^{(5t+\frac{\pi}{3})j}]$

- If $\alpha \neq 0$, is the signal still periodic?
- Note that periodic signals have infinite energy, but finite average power. Determine the average power of the signal $x(t) = e^{j(\omega_0 t+\phi)}$.

$$\frac{1}{2\pi}\int_0^{2\pi} |e^{j(\omega_0 t+\phi)}|^2 \, dt = 1$$

let $T = \frac{2\pi}{\omega_0}$,

Sinusoids and Harmonics

- Given a complex exponential with fundamental period ω_0, a harmonically related set of complex exponentials are given by

$$\phi_k(t+T) = e^{jk\omega_0(t+T)}$$

$$\underbrace{\phi_k(t) = e^{jk\omega_0 t},}_{} \quad k \in \mathbb{Z}.$$

$$= e^{jk\omega_0 t} e^{jk\cancel{\omega_0}\frac{2\pi}{\cancel{\omega_0}}} = e^{jk\omega_0 t}$$

$\rightarrow$ have T as their period.

- All these signals have period $T_0 = \frac{2\pi}{\omega_0}$.

- We can represent any sinusoid as sum of complex exponentials. E.g.,

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\omega_0 t+\phi} + \frac{A}{2}e^{-j(\omega_0 t+\phi)}.$$

Special Signals: Discrete-Time Exponential Signals

- Similar to CT: a discrete-time complex exponential signal is a signal of the form:

$$x[n] = Ce^{\beta n}.$$

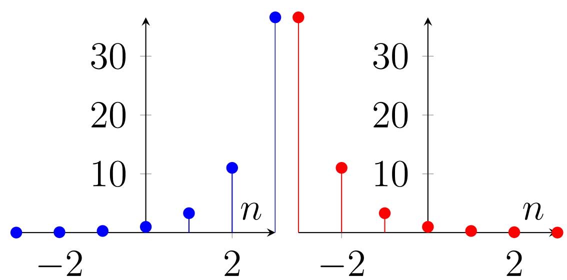
- For technical reasons, it is more convenient to study:

$$x[n] = Ce^{\beta n} = Cz^n,$$

where $z = e^{\beta}$.

Case 1: **Real-valued exponential**:

- Case where C and z (or equivalently β) are real.
- In this case if $|z| < 1$, the signal is exponentially decreasing,and if $|z| > 1$ the signal is exponentially increasing.



Discrete-Time Periodic Exponential Signals

- As before, let us focus on the case where β is imaginary.
- If $x[n] = Ce^{j\Omega_0 n}$ is periodic with period N, then:

$$\begin{aligned} &x[n+N] = x[n] \\ \Rightarrow &Ce^{j\Omega_0(n+N)} = Ce^{j\Omega_0 n}e^{j\Omega_0 N} = Ce^{j\Omega_0 n} \\ \Rightarrow &e^{j\Omega_0 N} = 1. \end{aligned}$$

- When is this true?
- Discrete time complex exponential signals are periodic only when $\Omega_0 = \frac{2\pi k}{N}$ for some integers k, N.
- In contrast, continuous time complex exponential signals were period for any $\Omega_0 \neq 0$.
- Example: Determine the fundamental period of the signal $x[n] = e^{\frac{2\pi}{3}n} + e^{\frac{3\pi}{4}n}$.

$x[n] = \underbrace{e^{j\frac{2\pi n}{3}}} + e^{j\frac{3\pi n}{4}}$

$\Omega_0 = \frac{2\pi}{3}$, find k & N s.t. $\frac{2\pi}{3} = \frac{2\pi k}{N}$

$\Rightarrow N = 3k$, $k=1$ will have smallest N.

$\Omega_0 = \frac{3\pi}{4} = \frac{2\pi k}{N} \Rightarrow N = \frac{8k}{3}$

$N = 8$, is the smallest

$x[n]$ has fundamental period 24.

Discrete-Time Periodic Exponential Signals

- For a CT periodic exponential signal, increasing the frequency reduces the period.

- However, for a DT periodic exponential signal, this is not the case.

$$g[n] = e^{j\Omega_0 n} = e^{j(\Omega_0 + 2\pi)n}.$$

  That is, the signal with frequency $\Omega_0 + 2\pi$ is the same signal with frequency Ω_0, and has the same period.

- Thus, all unique periodic exponential signals have periods confined to the region $[0, 2\pi)$ or $[-\pi, \pi)$.

- As before, the harmonic family of DT complex exponentials are given by

$$\phi_k[n] = e^{jkn\frac{2\pi}{N}}, \quad k \in \mathbb{Z}.$$

  All these signals have a common period N.

Analog and Digital Signals

- A signal is analog if $x(t) \in \mathbb{R}$ or $x[n] \in \mathbb{R}$.
- A signal is digital if $x(t) \in X$ or $x[n] \in X$ where the set X contains a finite number of distinct elements. Example: $X = \{0, 1\}$ or $X = \{-1, 0, 1\}$.
- Continuous-time Analog Signal: $\sin(\omega t)$
- Continuous-time Digital Signal: $\longrightarrow$ unit step signal $u(t)$
- Discrete-time Analog Signal: $\sin(\omega n)$
- Discrete-time Digital Signal: $\longrightarrow$ $u[n]$

Systems

- System: any mapping from (a subclass of) signals to (a subclass of) signals

$$\xrightarrow{x(t)} \boxed{\text{System}} \xrightarrow{y(t)}$$

- Dynamical system: any system with an internal state and a rule govering how the state evolves in time

- Example: a car with input being acceleration $a(t)$ and output being velocity $v(t)$. The evolution of speed is given by

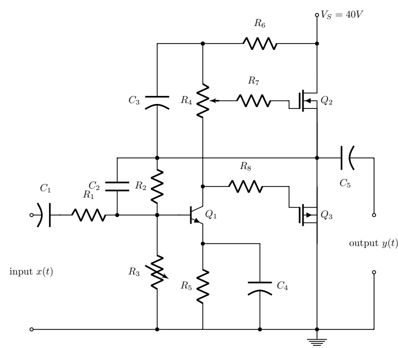
$$\frac{dv}{dt} = a(t).$$

  The output is measured by the speedometer sensor. This is an example of a continuous-time dynamical system.

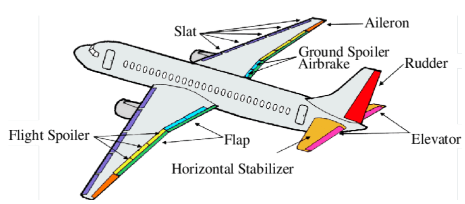
Continuous Time Systems: Examples

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Electrical Circuits:



Flight Control: Multiple Input Multiple Output (MIMO)



Discrete Time Systems: Examples

$$\xrightarrow{x[n]} \boxed{\text{System}} \xrightarrow{y[n]}$$

Economy:

- Input at day n = RBI interest rate
- Output at day n = inflation rate at day n

Sound Processing:

- Input $x[n]$ = input sound
- Output $y[n]$ = output sound

Important Mathematical Systems

$$\xrightarrow{x(t)} \boxed{\text{System}} \xrightarrow{y(t)}$$

- Delay: $x(t) \rightarrow y(t) = x(t - t_0)$ or $x[n] \rightarrow y[n] = x[n - n_0]$
- Scaling: $x(t) \rightarrow y(t) = ax(t)$ or $x[n] \rightarrow y[n] = ax[n]$
- Differentiator: $x(t) \rightarrow y(t) = \frac{d}{dt}x(t)$ or $x[n] \rightarrow y[n] = x[n] - x[n-1]$
- Integrator: $x(t) \rightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ or $x[n] \rightarrow y[n] = \sum_{k=-\infty}^{n} x[k]$
- What is the output of the DT integrator system to the input $x[n] = \delta[n]$?

Properties of Systems: Memoryless

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

- We say that a system is memoryless if output at time t (or n) only depends on the value of the input at time t (or n)
- In words: the mapping is instantaneous
- Example 1: Is $y(t) = x^2(t)$ memoryless? Yes
- Example 2: Is $y[n] = x[n-1]$ memoryless? NO.

Properties of Systems: Causality

$$\xrightarrow{x(t)} \boxed{\text{System}} \xrightarrow{y(t)}$$

- A system is *causal* if output at time t (or n) only depends on the input at time $s \leq t$ (or $k \leq n$)
- Example 1: Is the delay system $y(t) = x(t - t_0)$ causal? when $t_0 \geq 0$.

  otherwise, no.

- Example 2: Is the integrator system $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ causal? yes
- Example 3: Is $y[n] = x[-n]$ causal? No.

  $y[-5] = x[5]$ which violates causality.

Properties of Systems: Invertible

Let $x_1(t)$ produce $y_1(t)$ as output
$x_2(t)$ produce $y_2(t)$ as output.

$$x_1(t) \neq x_2(t) \Rightarrow y_1(t) \neq y_2(t) \xrightarrow{x(t)} \boxed{\text{System}} \xrightarrow{y(t)} \boxed{} \longrightarrow z(t) = y\left(\frac{t}{2}\right)$$

- A system is invertible if distinct inputs produce distinct outputs
- Equivalently: A system is invertible if from the output one knows the input
  - To show that a system is invertible: construct the inversion
  - To show that a system is not invertible: show two signals map to the same output
- Example 1: Is $y(t) = x(2t)$ invertible? yes $z(t) = y\left(\frac{t}{2}\right)$.

  $y[n] = x[2n]$? No.

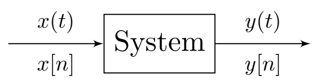
- Example 2: Is the system $y[n] = x[n] + y[n-1]$ invertible? yes

  $x[n] := y[n] - y[n-1]$. $y[n] = \sum_{k=-\infty}^{n} x[k]$

- Example 3: Is $y(t) = x^2(t)$ invertible? NO.
- Example 4: Is $y[n] = x[2n]$ invertible?

Bounded Input Bounded Output

Properties of Systems: BIBO Stable



- Definition: We say that a signal is bounded by $B > 0$ if $|x(t)| \leq B$ (or $|x[n]| \leq B$) for all t (or n)

finite constant that does not depend on t

- Example: Is $x(t) = tu(t)$ bounded? What about $x[n] = e^{jn}$?

No.

Yes.

- BIBO Stability: We say that a system is Bounded-Input Bounded-Output (BIBO) stable if all the bounded inputs result in bounded outputs.

- Example 1: Is $y(t) = 2x^2(t-1) + x(2t)$ bounded? when $x(t)$ is bounded?

$|y(t)| \leq 2\,|x^2(t-1)| + |x(2t)| \leq 2B^2 + B$

- Example 2: What about the integrator system $y[n] = \sum_{k=-\infty}^{n} x[k]$?

$\rightarrow$ Not BIBO Stable.

Properties of Systems: Time-Invariant (TI)

- We say that a system is TI if **for all inputs** $x(t)$ (or $x[n]$) and **all** time-shifts t_0 (or n_0) the output to $x(t-t_0)$ (or $x[n-n_0]$) is $y(t-t_0)$ (or $y[n-n_0]$)

When input is $x(t-t_0)$, the actual output $\sin(x(t-t_0))$

Shifting previous output by t_0 yields $\sin(x(t-t_0))$.

- Example 1: Is $y(t) = \sin(x(t))$ Time-Invarying?

Both are equal.
Hence, time-invariant.

Check output when $x[n] = \delta[n]$.

- Example 2: Is the system $y[n] = nx[n]$ TI?

when input is $x[n-n_0]$, actual output $y[n] = n\,x[n-n_0]$

shifted previous output $(n-n_0)\,x[n-n_0]$ $\neq$

- Example 3: Is the system $y[n] = x[n]x[n-3]$ TI?

- Example 4: Is the system $y(t) = x(5t)$ TI?