LECTURE - 40, 29th October

Properties of z-transform

Suppose that $x_1[n] \longleftrightarrow X_1(z)$ with ROC R_1 and $x_2[n] \longleftrightarrow X_2(z)$ with ROC R_2

- Linearity: $a_1x_1[n] + a_2x_2[n] \longleftrightarrow a_1X_1(z) + a_2X_2(z)$ with ROC containing $R_1 \cap R_2$

$$\sum_{n=-\infty}^{\infty} x_1[n-n_0] z^{-n} = \sum_{n=-\infty}^{\infty} x_1[n-n_0] z^{-(n-n_0)} z^{-n_0}$$
$$= z^{-n_0} \sum_{n=-\infty}^{\infty} x_1[n-n_0] z^{-(n-n_0)}$$

- Time-shifting: $x_1[n-n_0] \longleftrightarrow z^{-n_0}X_1(z)$ with ROC R_1 except possible addition or removal of 0 or ∞

let $m = n - n_0$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x_1[m] z^{-m}$$
$$= z^{-n_0} X_1(z)$$

Suppose $n_0 = 2$, $X_1(z) = \text{const.}$

$z^{-n_0} \cdot \text{const.} = z^{-2}\,\text{const}$

- Frequency Scaling: $z_0^n x_1[n] \longleftrightarrow X_1\left(\frac{z}{z_0}\right)$ with ROC $|z_0|R_1$.

$$\sum_{n=-\infty}^{\infty} z_0^n x_1[n] z^{-n} = \sum_{n=-\infty}^{\infty} x_1[n] \left(\frac{z}{z_0}\right)^{-n} = X_1\left(\frac{z}{z_0}\right)$$

Let ROC of $z_0^n x_1[n]$ be $\overline{R}$. Then, $\overline{z} \in \overline{R} \Rightarrow X_1\left(\frac{\overline{z}}{z_0}\right)$ is finite.

$\frac{\overline{z}}{z_0} \in R_1 \Rightarrow \overline{z} \in R_1 z_0$

$|z_0| R_1$

- Time-reversal: $x_1[-n] \longleftrightarrow X_1(\frac{1}{z})$ with ROC $\frac{1}{R_1}$

$$\sum_{n=-\infty}^{\infty} x_1[-n] z^{-n} = \sum_{m=-\infty}^{\infty} x_1[m] z^{m}$$
$$= \sum_{m=-\infty}^{\infty} x_1[m] \left(\frac{1}{z}\right)^{-m} = X_1\left(\frac{1}{z}\right).$$

$\{ z \mid z^{-1} \in R_1 \}$.

- Conjugation: $x_1^*[n] \longleftrightarrow X_1^*(z^*)$ with ROC R_1

HW: derive what happens when $x[n]$ is real, even, odd, imaginary & their combinations.

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n}$$

$$(X_1(z))^* = \sum_{n=-\infty}^{\infty} (x_1[n])^* (z^*)^{-n}$$

$$(X_1(z^*))^* = \sum_{n=-\infty}^{\infty} x_1[n]^* z^{-n}$$

z-transform of $x_1[n]$

Properties of z-transform

Suppose that $x_1[n] \longleftrightarrow X_1(z)$ with ROC R_1 and $x_2[n] \longleftrightarrow X_2(z)$ with ROC R_2

- **Convolution**: $x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$ with ROC containing $R_1 \cap R_2$.

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k].$$

$$\sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \right] z^{-n} z^{k} z^{-k} = \left[\sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-(n-k)} \right] \left[\sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \right] = X_1(z)X_2(z).$$

- Example: For the DT integrator system, $\boxed{Y(z) = \frac{1}{1-z^{-1}} X(z)}$

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$

impulse response $\quad h[n] = u[n]$

$$H(z) = \frac{1}{1-z^{-1}}$$

- **Differentiation in z-domain**: $nx_1[n] \longleftrightarrow -z\frac{d}{dz}X_1(z)$ with ROC R_1.

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad \frac{d}{dz}X_1(z) = \sum_{n=-\infty}^{\infty} x[n](-n)\, z^{-n-1}$$

$$\Rightarrow \frac{d}{dz}X_1(z) = \left(-\frac{1}{z}\right) \sum_{n=-\infty}^{\infty} x[n]\, n z^{-n}$$

$$\Rightarrow -z\frac{d}{dz}X_1(z) \longleftrightarrow n\, x_1[n]$$

- Example: If $X(z) = \log(1 + az^{-1})$ with $|z| > |a|$, then determine $x[n]$.

$$\frac{d}{dz}X(z) = \frac{1}{1+az^{-1}} \cdot \frac{d}{dz}\left(1 + \frac{a}{z}\right)$$

$$= \frac{1}{1+az^{-1}} \cdot \frac{-a}{z^2}$$

$$\Rightarrow -z\frac{d}{dz}X(z) = \frac{a}{z(1+az^{-1})} = \frac{az^{-1}}{1+az^{-1}}$$

$$\frac{1}{1+az^{-1}} \longleftrightarrow (-a)^n u[n], \quad \frac{az^{-1}}{1+az^{-1}} \longleftrightarrow a(-a)^{n-1} u[n-1] = n\, x[n]$$

$\Rightarrow x[n] = -\frac{(-a)^n u[n-1]}{n}, \quad n \geq 1,$ $x[0]$ was uniquely determined using the series expansion stated earlier.

- **Initial value theorem:** For a causal signal, $x[n] = 0$ for $n < 0$, we have $x[0] = \lim_{z\to\infty} X(z)$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$
$$= x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \cdots$$

- For this class of signals, the number of finite zeros cannot be greater than the number of finite poles.

- For this class of signals, degree of numerator of $X(z)$ is at most the degree of denominator of $x(z)$.

$$X(z) = \frac{(z-1)(z-2)}{(z-3)}$$

the underlying $x[n]$ cannot be causal.

z-transform properties

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	R R_1 R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \to \infty} X(z)$			

HW

z-transform pairs

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

$$\frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right)$$

$$r^n\,\frac{1}{2}\left[e^{j\omega_0 n}+e^{-j\omega_0 n}\right]u[n] = \left(\underline{\frac{1}{2}\left(re^{j\omega_0}\right)^n} + \underline{\frac{1}{2}\left(re^{-j\omega_0}\right)^n}\right)u[n]$$

Evaluation of DTFT from Pole-Zero Plots

- From the pole-zero plot of the z-transform, we can graphically evaluate the DTFT when the unit circle contained in the ROC.
- Let $h[n] = a^n u[n]$ with $|a| < 1$.

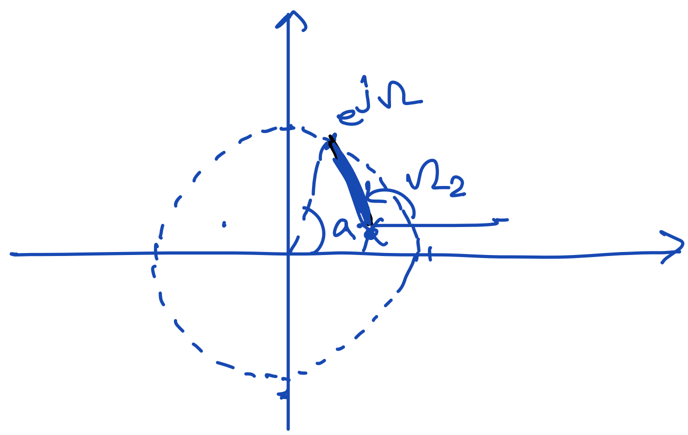
$$H(z) = \frac{1}{1 - az^{-1}}$$

$$H(e^{j\Omega}) = \frac{1}{1 - a e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

$$|H(e^{j\Omega})| = \frac{1}{|e^{j\Omega} - a|}$$

$$\measuredangle H(e^{j\Omega}) = \measuredangle e^{j\Omega} - \left(\measuredangle (e^{j\Omega} - a)\right)$$

$$= \Omega - \Omega_2$$



More generally:

if $H(z) = \dfrac{(z-a_1)(z-a_2)\cdots(z-a_k)}{(z-b_1)\cdots(z-b_m)}$

$$|H(e^{j\Omega})| = \frac{\prod_{j=1}^{k} |e^{j\Omega} - a_j|}{\prod_{j=1}^{m} |e^{j\Omega} - b_j|}$$

- What happens when there are multiple zeros and poles?

System Properties through ROC

- **Causality:** an LTI system is causal when its impulse response $h[n] = 0$ for $n < 0$. In other words, $h[n]$ is a causal signal.
- Thus, the z-transform of $h[n]$, denoted $H(z)$ has ROC as the exterior of a circle including ∞.
- The radius of this circle is larger than the maximum value of the magnitude of its poles.
- Since the ROC includes ∞, the order of the numerator is less than or equal to the order of the denominator.

ZεROC $\xRightarrow{H(z)} \leq \sum_{n=-\infty}^{\infty} |h[n] z^{-n}| = \sum_{n=-\infty}^{\infty} |h[n]| \, |z|^{-n} < \infty.$

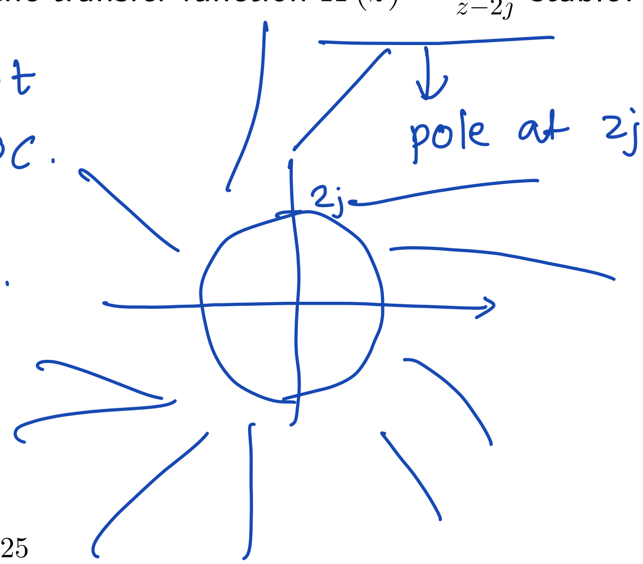
- **Stability:** an LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
- In other words, the ROC of the z-transform of $h[n]$ includes the unit circle.

> A causal system with rational $H(z)$ is stable if and only if all the poles are inside the unit circle.

- Example: Is a causal system with the transfer function $H(z) = \frac{1}{z-2j}$ stable?

unit circle is not part of ROC.

$\Rightarrow$ system is not stable.



Examples

Consider the following z-transforms of impulse responses of several LTI systems. Determine if these systems are stable, causal, both or neither for different possible ROCs.

$$H_1(z) = \frac{z^3 - 2z^2 + z}{z^2 + 0.25z + 1/8}$$

→ deg(num) > deg(den)

→ system is not causal

$$H_2(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

$$H_3(z) = \frac{1}{1 - 2r\cos(\theta)z^{-1} + r^2z^{-2}}$$

$H_1(z)$: poles: $\frac{-1 \pm \sqrt{7}j}{8}$, $\left|\frac{-1 \pm \sqrt{7}j}{8}\right| = \frac{1}{\sqrt{8}} < 1$

↘ located inside unit circle

$ROC_1 = \{z \mid |z| > \frac{1}{\sqrt{8}}\}$ ⟶ stable, not causal

$ROC_2 = \{z \mid |z| < \frac{1}{\sqrt{8}}\}$ ⟶ not stable, "

$H_2(z)$: poles: 0.5, 2

$ROC_1 = \{z \mid |z| > 2\}$ ⟶ unstable, causal

$ROC_2 = \{z \mid |z| \in (0.5, 2)\}$ → stable, not causal

$ROC_3 = \{z \mid |z| < 0.5\}$ → neither stable, nor causal.

$H_3(z)$: poles: $re^{j\theta}, re^{-j\theta}$

$ROC_1 = \{z \mid |z| > r\}$, $ROC_2 = \{z \mid |z| < r\}$

For $r>1$, $ROC_1 \Rightarrow$ causal, but not stable, ROC_2: stable, but not causal.

For $r<1$, $ROC_1 \Rightarrow$ stable & causal, ROC_2: neither.

LTI Systems - Linear Constant Coefficient Difference Equations

- Consider the following general form of linear difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where $x[n]$ is the input and $y[n]$ is the output.

applying z-transform on both sides, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} =$$

- Then, we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

The ROC is determined from whether the system is stable, causal, etic.

Examples

- Consider a LTI system whose input and output satisfy

$$y[n] - 0.5y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

Determine its impulse response for different possible ROCs.

$$Y(z)\left[1-0.5z^{-1}\right] = X(z)\left[1+\frac{1}{3}z^{-1}\right]$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}. \qquad \text{pole: } \frac{1}{2}.$$

$$= \frac{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= 1 + \left(\frac{5}{6}\right)\cdot\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

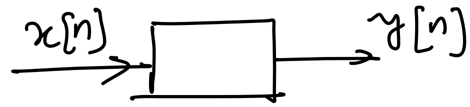
Case 1: $ROC_1 = \{z \mid |z| > \frac{1}{2}\}$: $h[n] = \delta[n] + \frac{5}{6}\left(\frac{1}{2}\right)^{n-1}u[n-1]$

Case 2: $ROC_2 = \{z \mid |z| < \frac{1}{2}\}$: $h[n] = \delta[n] + \frac{5}{6}\left[-\left(\frac{1}{2}\right)^{n-1}\cdot\right]. \; u[-(n-1)-1]$

$$= \delta[n] - \frac{5}{6}\left(\frac{1}{2}\right)^{n-1}u[-n]$$

Example 10.26

Q) If $x_1[n] = \left(\frac{1}{6}\right)^n u[n]$, $y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n\right] u[n]$.



If $x_2[n] = (-1)^n$, $y_2[n] = \frac{7}{4}(-1)^n$ $\quad z^n \longrightarrow H(z)z^n$

Determine $H(z)$, value of a and whether the system is causal and stable.

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solution:

$$X_1(z) = \frac{1}{1 - \frac{1}{6}z^{-1}}, \quad ROC = \left\{z \,\middle|\, |z| > \frac{1}{6}\right\}$$

$$Y_1(z) = a \cdot \frac{1}{1 - \frac{1}{2}z^{-1}} + 10 \cdot \frac{1}{1 - \frac{1}{3}z^{-1}}, \quad ROC: |z| > \frac{1}{2}$$

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \left(1 - \frac{1}{6}z^{-1}\right) \cdot \left[\frac{a}{1 - \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{3}z^{-1}}\right],$$

– system is stable & causal. $\quad$ ROC: $|z| > \frac{1}{2}$.

From x_2, y_2, we have $H(-1) = 7/4$

$$= \left(1 + \frac{1}{6}\right)\left[\frac{a}{1 + \frac{1}{2}} + \frac{10}{1 + \frac{1}{3}}\right]$$

$$\frac{7}{4} = \frac{7}{6} \cdot \left[\frac{2a}{3} + \frac{30}{4}\right]$$

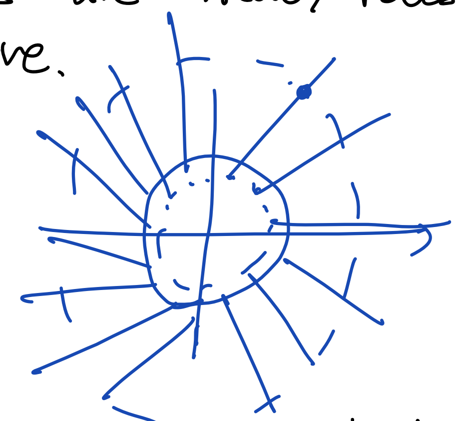
$$\Rightarrow \frac{3}{2} - \frac{30}{4} = \frac{2a}{3}$$

$$\Rightarrow a = \frac{3}{2}\left[\frac{6 - 30}{4}\right] = -\frac{72}{8}$$

$$= -9.$$

Example 10.27

A discrete-time system is causal and stable. $H(z)$ has a pole at $\frac{1}{2}$ and zero on the unit circle. Other poles & zeros may exist. | Determine if the following statements are true / false / inconclusive.



(a) DTFT of $2^{-n} h[n]$ exists. yes.

(b) $H(e^{j\omega}) = 0$ for some ω. yes

(c) $h[n]$ has finite duration. False

(d) $h[n]$ is real. — inconclusive.

(e) $g[n] = n\left[h[n] * h[n]\right]$ is impulse response of a stable system. Yes.

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solution :

a) $DTFT(2^{-n} h[n]) = H(z)$, for $|z| = 2$

system is causal & stable.

c) If $h[n]$ has finite duration, ROC is entire complex plane except 0 & ∞.
But we have a pole at $\frac{1}{2}$, which must not be part of ROC.

d) For $h[n]$ to be real, $H(z) = (H(z^*))^*$

(e) $h[n] \circledast h[n] \longleftrightarrow (H(z))^2$

$n(h[n] \circledast h[n]) \longleftrightarrow -z \frac{d}{dz}(H(z))^2 = -2z H(z) \frac{d}{dz} H(z)$

$\Downarrow$

ROC will extend outwards from within unit circle.

$\downarrow$

poles of $G(z)$ $\subseteq$ poles of $H(z)$ with higher orders.

LECTURE 42 & 43: 31st October

(Q1) Let $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$, $x_2[n] = \left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \dfrac{1}{1-\frac{1}{3}z^{-1}}, |z| > \frac{1}{3}$

$y[n] = x_1[n+3] \circledast x_2[-n+1]$.

Determine $Y(z)$.

Solution: $X_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$, ROC: $|z| > \frac{1}{2}$

$$x_1[n+3] \longleftrightarrow \frac{z^3}{1-\frac{1}{2}z^{-1}}, \quad \text{ROC: } \left\{z \,\middle|\, |z| > \tfrac{1}{2}, z \neq \infty\right\}$$

$$x_2[n+1] \longleftrightarrow \frac{z}{1-\frac{1}{3}z^{-1}}, \quad \left\{|z| > \tfrac{1}{3}, z \neq \infty\right\}.$$

$$x_2[-n+1] \longleftrightarrow \frac{z^{-1}}{1-\frac{1}{3}z}, \quad \text{ROC: } \left\{z \,\middle|\, |z| < 3, z \neq 0\right\}$$

$$Y(z) = \frac{z^3}{1-\frac{1}{2}z^{-1}} \cdot \frac{z^{-1}}{1-\frac{1}{3}z}, \quad \text{ROC: } \left\{z \,\middle|\, |z| \in \left(\tfrac{1}{2}, 3\right)\right\}$$

$$= \frac{3z^3}{(z-1/2)(3-z)}.$$

(Q2) Let $x[n] \longleftrightarrow X(z)$, and $x_1[n] := \begin{cases} x\left[\frac{n}{2}\right], & \text{when } n \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$

Determine $X_1(z)$ in terms of $X(z)$.

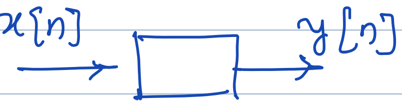
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{\substack{n=-\infty \\ n:\text{even}}}^{\infty} x_1[n] z^{-n} = \sum_{m=-\infty}^{\infty} x_1[2m] z^{-2m} \quad (n = 2m)$$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{-2m}$$

$$= \sum_{m=-\infty}^{\infty} x[m] (z^2)^{-m} = H(z^2)$$

BLOCK DIAGRAM REPRESENTATION

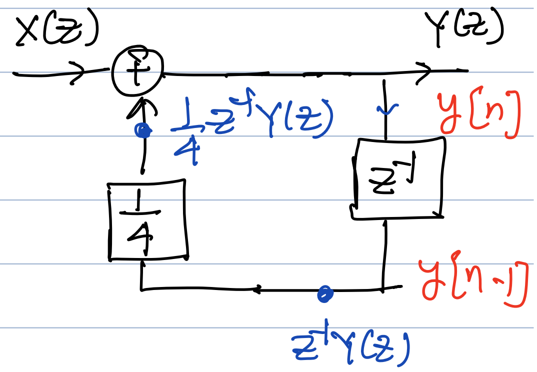
Example ① Let $h[n] = \left(\frac{1}{4}\right)^n u[n]$



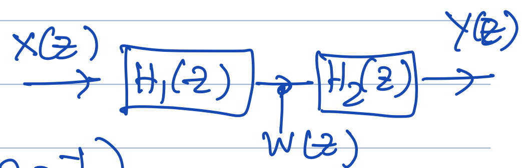
$$H(z) = \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{Y(z)}{X(z)}$$

$$\Rightarrow Y(z) - \frac{1}{4}z^{-1}Y(z) = X(z)$$

$$\Rightarrow Y(z) = X(z) + \frac{1}{4}z^{-1}Y(z)$$

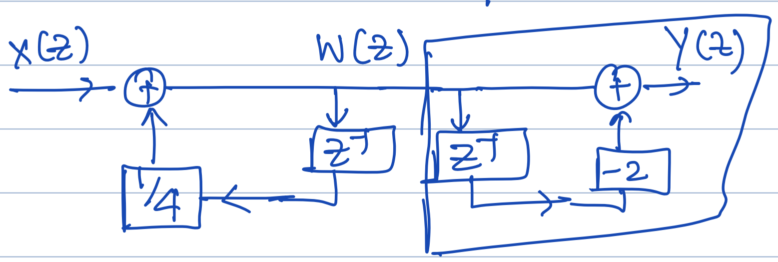


Example 2 : $H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}$.

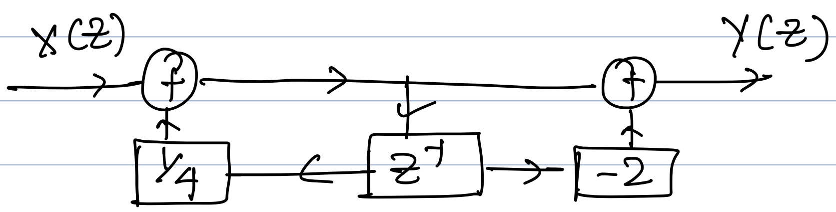


$$= \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

$$W(z) = X(z) + \frac{1}{4}z^{-1}W(z)$$

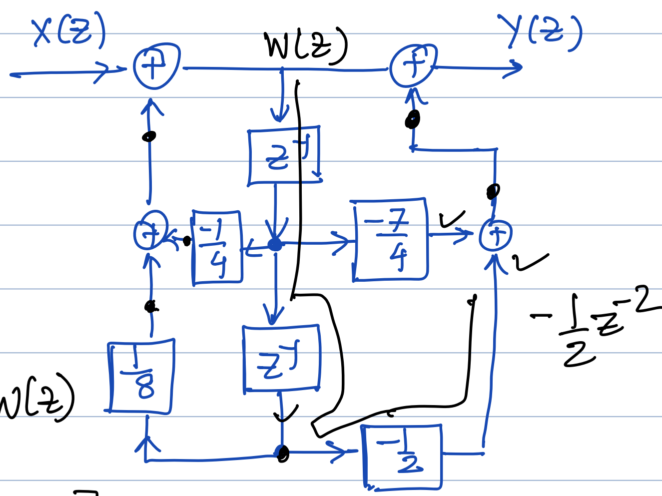


$$Y(z) = (1-2z^{-1})\,W(z)$$



Example :

Determine $H(z)$ and relation between $x[n]$ and $y[n]$ as a difference equation.



$$W(z) = X(z) - \frac{1}{4}z^{-1}W(z) + \frac{1}{8}z^{-2}W(z)$$

$$\Rightarrow X(z) = W(z)\left[1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right]$$

$$Y(z) = W(z)\left[1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{\left(1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}\right)}{\left(1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right)}$$

$$\Rightarrow Y(z)\left[1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right] = X(z)\left[1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}\right]$$

$$\Rightarrow y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}z[n-2] = x[n] - \frac{7}{4}x[n-1] - \frac{1}{2}x[n-2]$$

For a given $H(z)$, its block diagram representation is not unique.

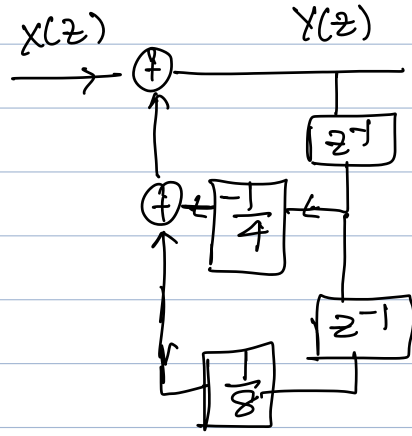
$$H(z) = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = \frac{Y(z)}{X(z)} = \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

$$\Rightarrow X(z) = Y(z)\left(1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right) \qquad = \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)} \cdot \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)}$$

$$\Rightarrow X(z) - \frac{1}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = Y(z)$$

$$Y(z) = \frac{W(z)}{1 - \frac{1}{4}z^{-1}}$$

$$Y(z) = \underbrace{\left[\frac{X(z)}{\left(1+\frac{1}{2}z^{-1}\right)}\right]}_{W(z)} \cdot \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$$

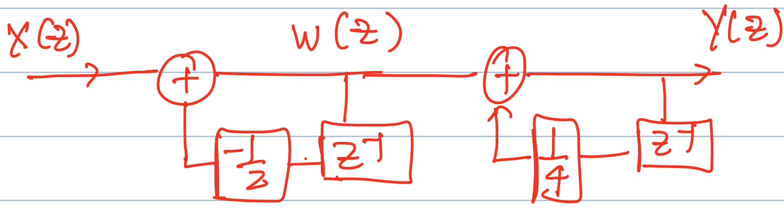


$$W(z) = \frac{X(z)}{1+\frac{1}{2}z^{-1}}$$

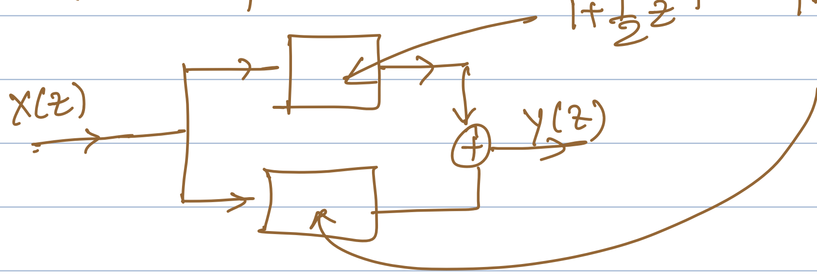
$$\Rightarrow W(z) = X(z) - \frac{1}{2}z^{-1}W(z)$$

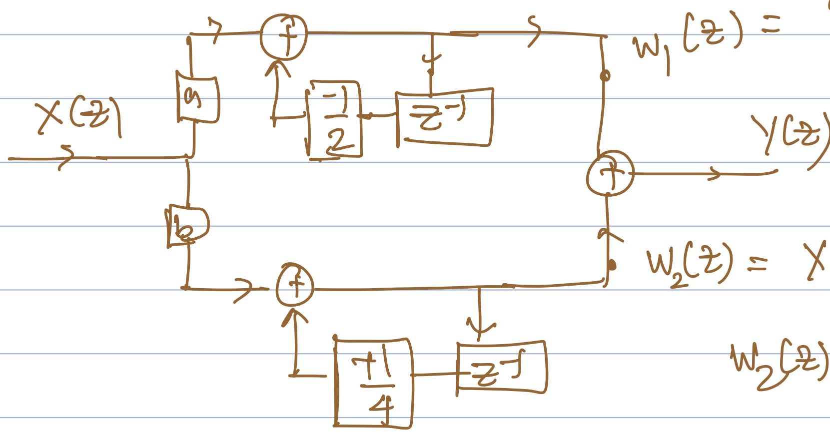
$$Y(z) = \frac{W(z)}{1-\frac{1}{4}z^{-1}}$$

$$= W(z) + \frac{1}{4}z^{-1}Y(z)$$



We can also express $H(z) = \frac{a}{1+\frac{1}{2}z^{-1}} + \frac{b}{1-\frac{1}{4}z^{-1}}$.





$$w_1(z) = aX(z) - \frac{1}{2}z^{-1}w(z)$$

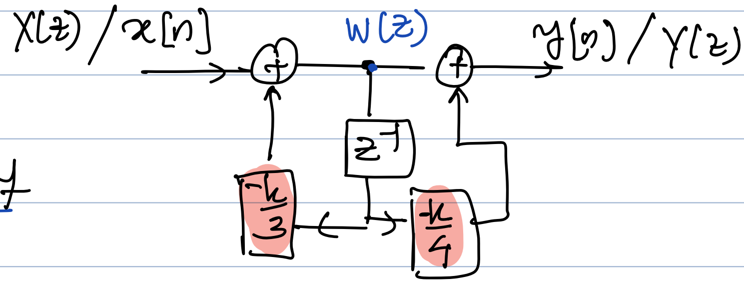
$$\Rightarrow \frac{W_1(z)}{X(z)} = \frac{a}{1+\frac{1}{2}z^{-1}}.$$

$$W_2(z) = X(z) + \frac{1}{4}z^{-1}W_2(z)$$

$$W_2(z)\left[1-\frac{1}{4}z^{-1}\right] = X(z)$$

Practice Problems

10.59



The system is causal.

a) Determine the range of K such that the system is stable.

b) Let $K=1$, $x[n] = \left(\frac{2}{3}\right)^n \;\forall n$. Determine $y[n]$.

$y[n] = H\left(\frac{2}{3}\right)\cdot\left(\frac{2}{3}\right)^n$

solution:

$$W(z) = X(z) - \frac{K}{3} z^{-1} W(z)$$

$$\Rightarrow W(z)\left[1 + \frac{K}{3} z^{-1}\right] = X(z)$$

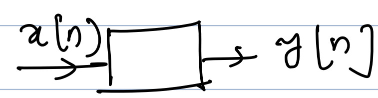
$$Y(z) = W(z) - \frac{K}{4} z^{-1} W(z) = W(z)\left[1 - \frac{K}{4} z^{-1}\right]$$

$$H(z) = \frac{1 - \frac{K}{4} z^{-1}}{1 + \frac{K}{3} z^{-1}}, \quad \text{ROC}: \left\{ z \;\middle|\; |z| > \left|\frac{K}{3}\right| \right\}$$

For the system to be stable, unit circle $\in$ ROC

$\Rightarrow |K|/3 < 1 \Rightarrow |K| < 3.$

10.47 If $x[n] = (-2)^n$, $y[n] = 0 \;\forall n$.



If $x[n] = \left(\frac{1}{2}\right)^n u[n]$, then

$$y[n] = \delta[n] + a\left(\frac{1}{4}\right)^n u[n].$$

a) Determine the value of a.

b) Determine $y[n]$ when $x[n] = 1 \;\forall n.$ $= (1)^n$

solution: $y[n] = (-2)^n H(-2) = 0 \Rightarrow H(-2) = 0.$

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$y[n] = H(1).$$

$$1$$

$$\delta[n] + a\left(\frac{1}{4}\right)^n u[n] \longleftrightarrow 1 + \frac{a}{1-\frac{1}{4}z^{-1}}$$

$$H(z) = \frac{\left(1-\frac{1}{4}z^{-1}+a\right)}{\left(1-\frac{1}{4}z^{-1}\right)} \cdot \left(1-\frac{1}{2}z^{-1}\right).$$

$$H(-2) = 0 \Rightarrow \boxed{a = -9/8}$$