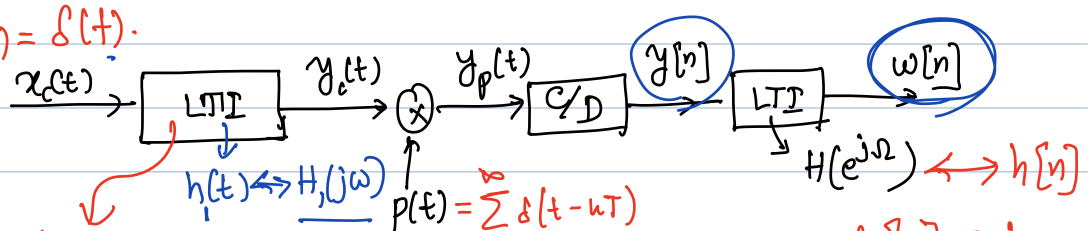
Problem 7.30

LECTURE 36 : 22nd OCT

Suppose $x_c(t) = \delta(t)$.



$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t)$$

Determine $h[n]$ s.t. $w[n] = \delta[n]$.

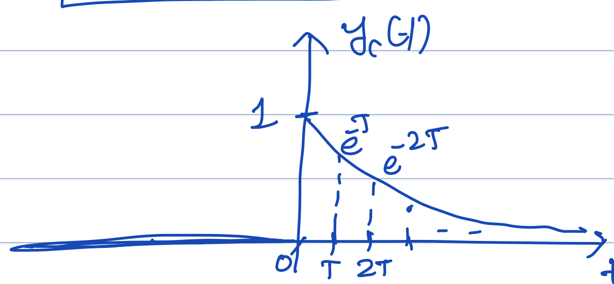
Solution:

$$H_1(j\omega) = \frac{1}{1+j\omega} \qquad W_n(e^{j\Omega}) = 1$$

$$Y_c(j\omega) = \frac{1}{1+j\omega} \Rightarrow \boxed{y_c(t) = e^{-t}u(t)}$$

Let T be the sampling time.

$$y[n] = e^{-nT}u[n].$$



$$y[n] = \begin{cases} 0 & \text{if } n < 0 \\ e^{-nT} & \text{if } n \geq 0 \end{cases}$$

$$Y(e^{j\Omega}) = \frac{1}{1 - e^{-T}e^{-j\Omega}} = \frac{1}{1 - e^{-(T+j\Omega)}}$$

For $w[n] = \delta[n]$ or $W(e^{j\Omega}) = 1$, we need

$$Y(e^{j\Omega})H(e^{j\Omega}) = 1$$

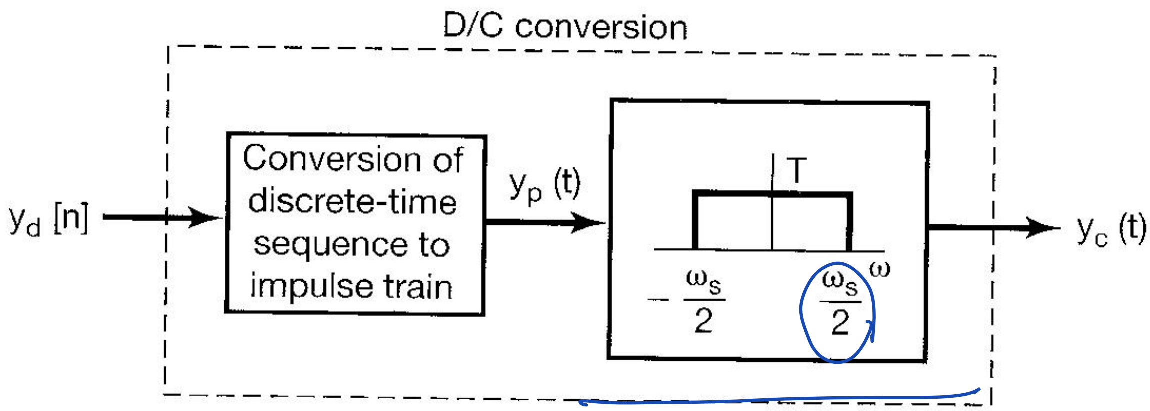
$$\Rightarrow H(e^{j\Omega}) = 1 - e^{-(T+j\Omega)} = 1 - e^{-T}e^{-j\Omega}$$

$$\Rightarrow h[n] = \delta[n] - e^{-T}\,\underline{DTFT^{-1}(e^{-j\Omega})} = ?$$

$$h[n] = \delta[n] - e^{-T}\delta[n-1]$$

DT Processing of CT Signals

$$x_c(t) \longrightarrow \boxed{\text{C\D Conversion}} \xrightarrow{x_d[n] = x(nT)} \boxed{\text{DT System } H_d(e^{j\Omega})} \xrightarrow{y_d[n] = y(nT)} \boxed{\text{D\C Conversion}} \longrightarrow y_c(t)$$



where $\omega_s = \frac{2\pi}{T}$.

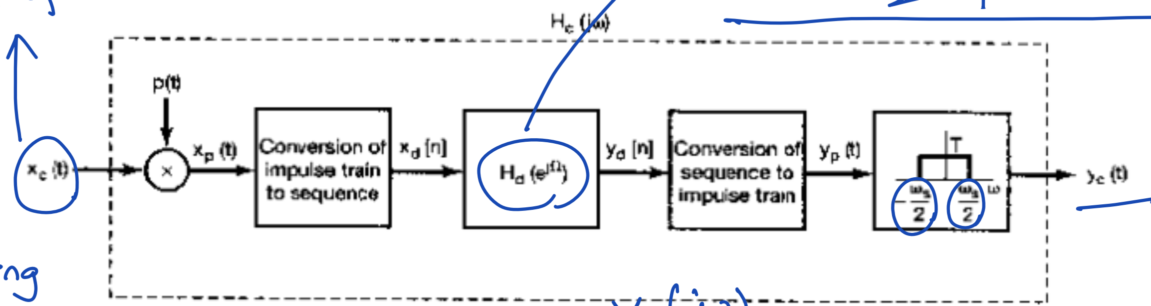
$X_c(j\omega) \longrightarrow X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$, $X_d(e^{j\Omega}) = X_p(j\frac{\Omega}{T})$

$Y_d(e^{j\Omega}) = X_d(e^{j\Omega})\, H_d(e^{j\Omega})$

DT Processing of CT Signals

band limited with $\omega_M < \frac{\pi}{T}$,

$y_d[n] = \frac{1}{2} y_d[n-1] + x_d[n]$



T: sampling time

- No aliasing

Determine $\frac{Y_c(j\omega)}{X_c(j\omega)}$:

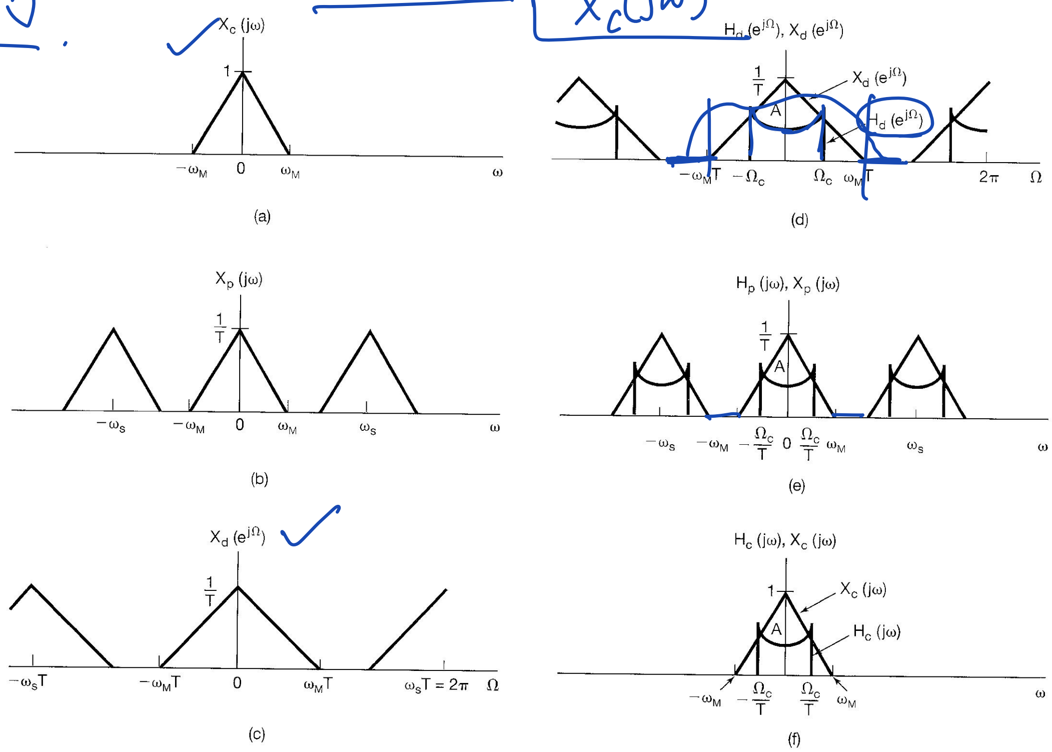


Figure 7.25 Frequency-domain illustration of the system of Figure 7.24: (a) continuous-time spectrum $X_c(j\omega)$; (b) spectrum after impulse-train sampling; (c) spectrum of discrete-time sequence $x_d[n]$; (d) $H_d(e^{j\Omega})$ and $X_d(e^{j\Omega})$ that are multiplied to form $Y_d(e^{j\Omega})$; (e) spectra that are multiplied to form $Y_p(j\omega)$; (f) spectra that are multiplied to form $Y_c(j\omega)$.

Example 7.31

$$\frac{Y_c(j\omega)}{X_c(j\omega)} = \frac{1}{T} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega T}}.$$

$$y_d[n] = \frac{1}{2} y_d[n-1] + x_d[n]$$

applying DTFT $\Rightarrow Y_d(e^{j\Omega}) = \frac{1}{2} e^{-j\Omega} Y_d(e^{j\Omega}) + X_d(e^{j\Omega})$

$$\Rightarrow Y_d(e^{j\Omega}) = \frac{X_d(e^{j\Omega})}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\Rightarrow Y_p(j\omega) = Y_d(e^{j\omega T})$$

$$= \frac{X_d(e^{j\omega T})}{1-\frac{1}{2}e^{-j\omega T}}$$

$$= \underline{\frac{1}{T} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega T}}} \cdot \underbrace{\left[\sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))\right]}_{\Downarrow \; X_c(j\omega).}$$

$$Y_c(j\omega) = Y_p(j\omega)\, H(j\omega)$$

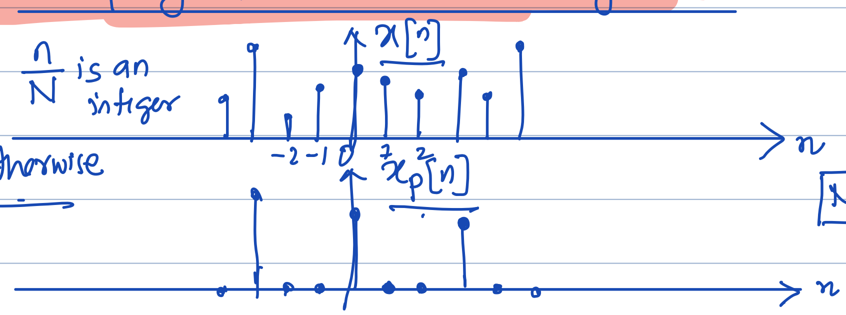
$$= \underbrace{\frac{1}{T} \cdot \frac{1}{1-\frac{1}{2}e^{-j\omega T}}}\cdot X_c(j\omega).$$

LECTURE 37

23rd Oct

Sampling a discrete-time signal.

$$x_p[n] = \begin{cases} x[n], & \text{if } \frac{n}{N} \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$



$N=3$

Let N be the sampling time.

$$x_p[n] = x[n] \times \underbrace{\left(\sum_{K=-\infty}^{\infty} \delta[n-KN]\right)}_{p[n]}$$

Let us determine $X_p(e^{j\Omega})$.

Let $X(e^{j\Omega})$ be the DTFT of $x[n]$.

$$p[n] = \sum_{K=0}^{N-1} a_K e^{j\frac{2\pi K}{N}n}$$

$$a_K = \frac{1}{N}\sum_{l=0}^{N-1} p[l]e^{-j\frac{2\pi}{N}l} = \frac{1}{N}, \quad \forall K$$

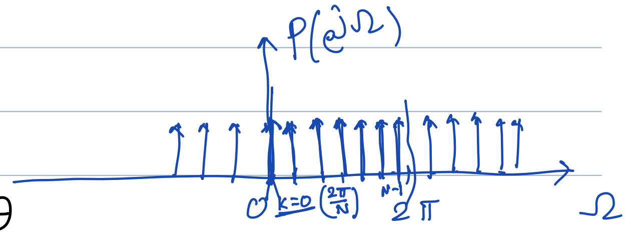
$$P(e^{j\Omega}) = \sum_{K=0}^{N-1} a_k \, \mathrm{DTFT}\left(e^{j\frac{2\pi}{N}Kn}\right)$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} \mathrm{DTFT}\left(e^{j\frac{2\pi}{N}Kn}\right)$$

Recall that $\mathrm{DTFT}(e^{j\theta n}) = 2\pi \sum_{K=-\infty}^{\infty} \delta(\omega - \theta - 2\pi k)$

$$\Rightarrow P(e^{j\Omega}) = \frac{2\pi}{N}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{N-1} \delta\left(\Omega - \frac{2\pi k}{N} - 2\pi l\right)$$

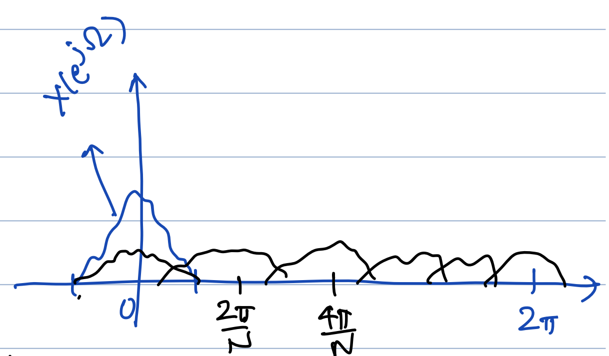
$$= \frac{2\pi}{N}\sum_{K=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi}{N}k\right)$$



$$X_p(e^{j\Omega}) = \frac{1}{2\pi}\int_0^{2\pi} P(e^{j\theta})X(e^{j(\Omega-\theta)})\,d\theta$$

$$= \frac{1}{N}\sum_{K=-\infty}^{\infty}\int_0^{2\pi}\delta\left(\theta - \frac{2\pi}{N}k\right)X(e^{j(\Omega-\theta)})\,d\theta$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} X\left(e^{j\left(\Omega - \frac{2\pi}{N}k\right)}\right).$$



For exact recovery, we need

$\omega_s = \frac{2\pi}{N} > 2\omega_M$

$\Rightarrow N < \frac{\pi}{\omega_M}$, For discrete-time sampling, $N \geq 2$

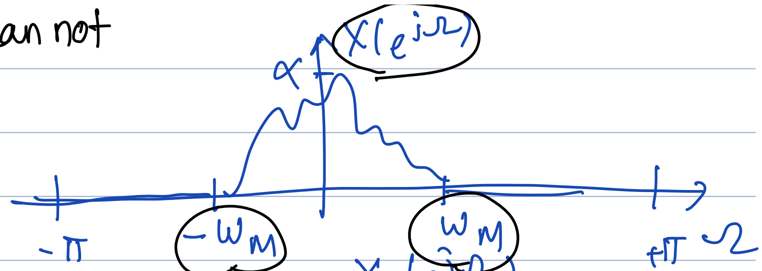
$\Rightarrow \omega_M \leq \frac{\pi}{2}$.

- If ω_M is too large, we can not avoid aliasing.

$x[n]$

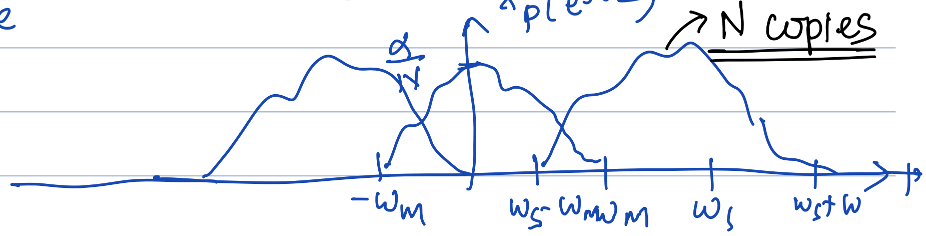
$\downarrow$

$x_p[n]$ with Sampling time N.



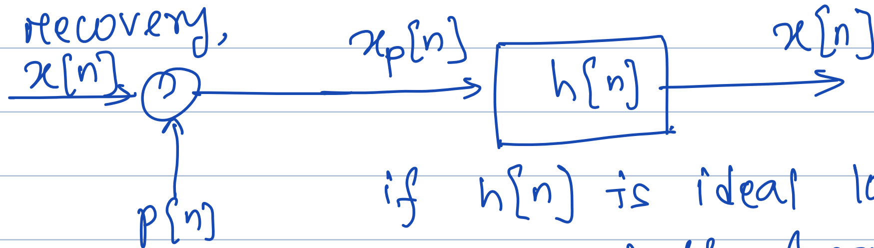
Bandwidth

$\omega_M < \pi$



[if $\omega_s > 2\omega_M$, there will be no overlap.

For recovery,

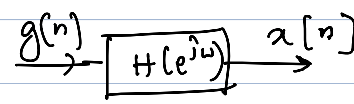


if $h[n]$ is ideal low pass filter with cutoff frequency $= \frac{\omega_s}{2}$.

Recovery is exact if N is chosen such that

$$\omega_s = \frac{2\pi}{N} > 2\omega_M.$$

Problems: 7.32 For a signal $x[n]$, its DTFT $X(e^{j\Omega}) = 0$ for $\frac{\pi}{4} \le |\Omega| \le \pi$.



Let $g[n] = x[n] \cdot \boxed{\sum_{k=-\infty}^{\infty} \delta[n-1-4k]} = \{\text{impulses at } 1, 5, 9, \ldots, -3, -7, \ldots\}$

$= p[n-1]$

Find $H(e^{j\omega})$ s.t. if $g[n]$ is applied as input a LTI system with transfer function $H(e^{j\omega})$, then the output is $x[n]$.

$\omega_M = \frac{\pi}{4}$, $\omega_s = \frac{2\pi}{4} = \pi/2$

$$g[n] = x[n]\cdot \underline{p[n-1]},$$

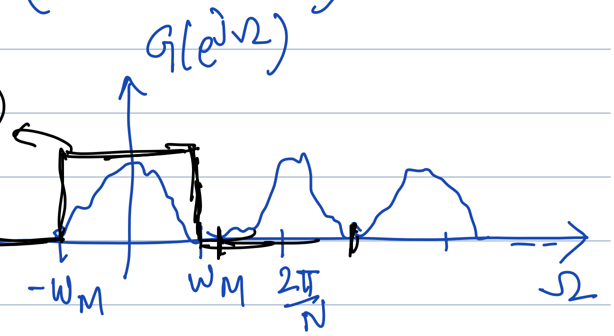
$$\text{DTFT}(p[n-1]) = e^{-j\Omega}\,\text{DTFT}(p[n]) = e^{-j\Omega}\,\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{2\pi}{N}k\right).$$

$$\underline{G(e^{j\Omega})} = \frac{1}{2\pi}\int_0^{2\pi}\left[e^{-j\theta}\cdot\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\theta-\frac{2\pi}{N}k\right)\right]X\left(e^{j(\Omega-\theta)}\right)d\theta$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty}\int_0^{2\pi} e^{-j\theta}\,\delta\left(\theta-\frac{2\pi}{N}k\right)X\left(e^{j(\Omega-\theta)}\right)d\theta$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k}\,X\left(e^{j\left(\Omega-\frac{2\pi}{N}k\right)}\right).$$

$$\text{if } H(e^{j\Omega}) = \begin{cases} N, & \Omega\in[-W_M, W_M] \\ 0, & \text{otherwise}\end{cases}$$



Then $\underline{G(e^{j\Omega})\,H(e^{j\Omega}) = X(e^{j\Omega})}.$

Z-Transform

- Recall that eigenfunction property was satisfied for general complex exponential signals, rather than exponential signals with an imaginary exponent.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z)z^n$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$\underline{H(z)} = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$= z^n \underline{\sum_{k=-\infty}^{\infty} h[k] z^{-k}}$$

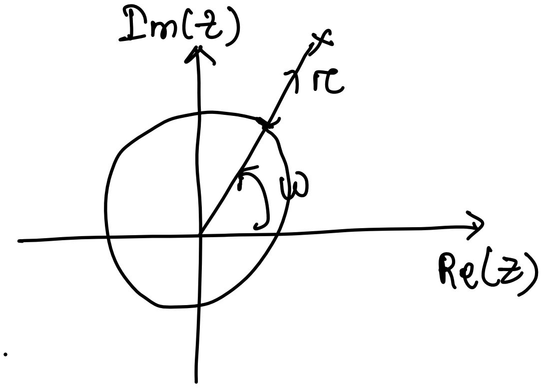
For any DT signal $x[n]$, its z-transform is the function:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

The Region of Convergence of $x[n]$ is defined to be $\{z \mid X(z) \text{ exists, i.e., is finite}\}$ which is a subset of the complex plane.

Let $z_1 = re^{j\omega_1} \in \text{ROC}$. Then $z_2 = re^{j\omega_2} \in \text{ROC}$ $\forall \omega_2 \in [0, 2\pi]$

z-Transform



- For a complex number z, let $z = re^{j\omega}$. We have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$
$$= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}.$$

- Therefore, $X(z) = X(re^{j\omega}) = \mathtt{DTFT}(x[n]r^{-n})$.
- In particular, for $|z| = r = 1$, $z = e^{j\omega}$, which implies $X(z) = \mathtt{DTFT}(x[n])$.
- Thus, when $r = 1$ and we vary ω, z moves on the unit circle in the complex plane. The unit circle plays a critical role in z-transform of discrete-time signals.
- Since $X(z) = \mathtt{DTFT}(x[n]r^{-n})$, then a point in the complex plane $\bar{z}$ lies in the ROC if its magnitude $\bar{r}$ is such that the $x[n]\bar{r}^{-n}$ is absolutely summable.
- Thus, the ROC is determined by the magnitude $\bar{r}$ rather than the phase. Hence, the ROC mainly consists of circular regions and/or rings in the complex plane.
- DTFT of $x[n]$ exists only when the ROC of the z-transform of $x[n]$ contains the unit circle.

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$,

$\sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1$

$\Rightarrow$ ROC of $\delta[n]$ is the entire complex plane

Examples

$\sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = \delta[0] z^{-1} = \frac{1}{z}$, ROC $= \mathbb{C} \setminus \{0\}$

Example 1

- What is the z-transform of $x[n] = \delta[n] + \delta[n-1]$? What is its ROC?

$X(z) = 1 + \frac{1}{z}$, with ROC being entiree complex plane $\setminus \{0\}$.

Example 2

- What is the z-transform of $x[n] = a^n u[n]$ where a is a complex number? What is its ROC?

$$\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

converges when $\left|\frac{a}{z}\right| < 1$

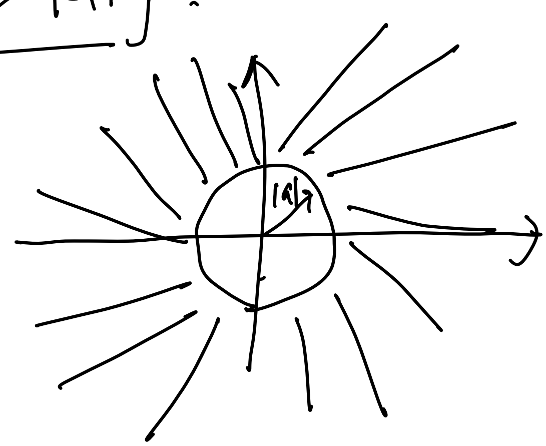
$\Rightarrow |z| > |a|$.

$$= \frac{1}{1 - \frac{a}{z}}$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}.$$

ROC $= \{ z \in \mathbb{C} \mid |z| > |a| \}$.

DTFT is defined only when $|a| < 1$. Otherwise, it is not defined.



$$u[-n-1] = \begin{cases} 1 & , \quad -n-1 \geq 0 \Rightarrow \underline{n \leq -1} \\ 0 & , \quad \text{otherwise} \end{cases}$$

Example 3

- What is the z-transform of $x[n] = \underline{-a^n u[-n-1]}$ where a is a complex number? What is its ROC?

$$\sum_{n=-\infty}^{\infty} -a^n u[-n-1]\, z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= \sum_{K=1}^{\infty} -a^{-K} z^{K} = 1 - \sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^K \quad \text{is finite when } \left|\frac{z}{a}\right| < 1$$

$$\Rightarrow \underline{|z| < |a|}$$

$$= 1 - \frac{1}{1 - \frac{z}{a}}$$

$$= 1 - \frac{a}{a-z}$$

$$= \frac{a - z - a}{a - z} = \underline{\frac{z}{z-a}}.$$

z-transform without ROC is meaningless

- What is the $X(z)$ for $x[n] = -a^n u[-n-1]$?
- Let's investigate:

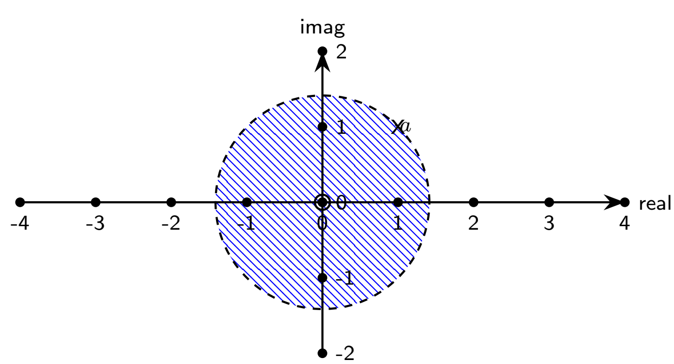
$$
\begin{aligned}
X(z) &= -\sum_{n=-\infty}^{\infty} x[n] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= -\sum_{n=-\infty}^{1} (\frac{a}{z})^n = -\sum_{k=1}^{\infty} (\frac{z}{a})^k \\
&= -\frac{z/a}{1-\frac{z}{a}},
\end{aligned}
$$

where the last equality holds for $|\frac{z}{a}| < 1$. The ROC is sketched below.

- Therefore,

$$
-a^n u[-n-1] \longleftrightarrow \frac{z}{z-a}.
$$

- But $X(z) = \frac{z}{z-a}$ for $x[n] = a^n u[n]$!
- Therefore: **z-transform without ROC is meaningless!**
- Since it is possible that z-transforms of distinct signals would have the same algebraic expression, it is unable to determine the inverse z-transform without the ROC.



ROC and Pole-Zero

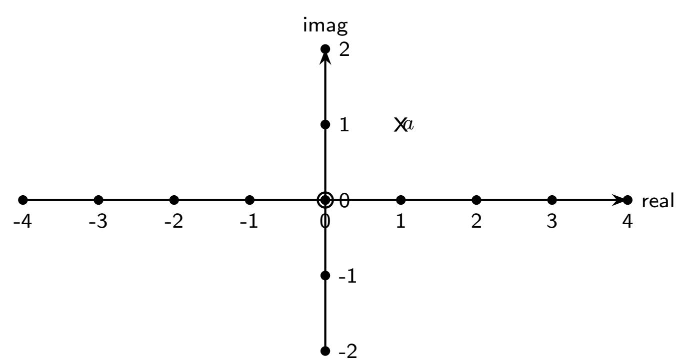
- Most of the time, we will encounter z-transforms that are rational functions of z. For a z-transform of the form:

$$H(z) = a\frac{(z-p_1)\cdots(z-p_m)}{(z-q_1)\cdots(z-q_n)},$$

  the roots of the numerator $p_1, \ldots, p_m$ are called the **zeros** and the roots of the denominator $q_1, \ldots, q_n$ are called the **poles** of $H(z)$.

- On the complex ~~plain~~ plane, the poles are marked by (x) and zeros are marked by (○).

- It is more convenient to express $X(z)$ as polynomials in z^{-1} rather than z.



- Determine the poles and zeros of $X_1(z) = \frac{z}{z-a}$. zeros: 0 poles: a

- Determine the poles and zeros of $X_2(z) = 1 + \frac{1}{z}$. zeros: -1 poles: 0

  $= (z+1)/z$

- Determine the poles and zeros of $X_3(z) = \frac{1-3z^{-1}}{(1-5z^{-1})(1-0.5z^{-1})}$. poles: $5, 0.5$ zeros: $0, 3$

$$= \frac{1-\frac{3}{z}}{\left(1-\frac{5}{z}\right)\left(1-\frac{1}{2z}\right)}$$

$$= \frac{z(z-3)}{(z-5)(z-1/2)}$$

Practice Problem

a. What is the ROC for the z-transform of $x[n] = (\frac{1}{2})^n u[n] + 2^n u[n]$?

$$\underbrace{(\tfrac{1}{2})^n u[n]}_{\frac{1}{1-\frac{1}{2}z^{-1}}} \qquad \underbrace{2^n u[n]}_{\frac{1}{1-2z^{-1}}}$$

ROC: $|z| > \frac{1}{2}$ $\leftarrow \frac{1}{1-\frac{1}{2}z^{-1}}$ $\qquad$ $\frac{1}{1-2z^{-1}}$, ROC: $|z| > 2$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad \text{ROC: } |z| > 2.$$

b. Determine the z-transform with ROC as well as the location of poles and zeros of the signal

$$x[n] = (\frac{1}{3})^n \sin(\frac{\pi}{4}n) u[n].$$

Does the above signal admit DTFT?

$$x[n] = \left(\frac{1}{3}\right)^n \frac{1}{2j}\left[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right] u[n]$$

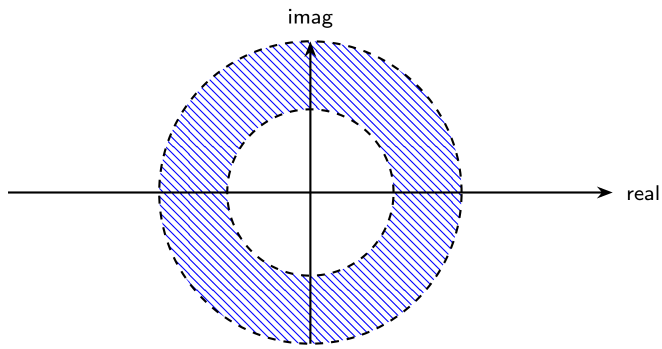
$$= \frac{1}{2j}\left(\frac{e^{j\pi/4}}{3}\right)^n u[n] - \frac{1}{2j}\left(\frac{e^{-j\pi/4}}{3}\right)^n u[n]$$

$$= \frac{1}{2j} \cdot \underbrace{\frac{1}{1-\frac{e^{j\pi/4}}{3}z^{-1}}}_{|z| > \left|\frac{e^{j\pi/4}}{3}\right| = \frac{1}{3}} - \frac{1}{2j}\underbrace{\frac{1}{1-\frac{e^{-j\pi/4}}{3}z^{-1}}}_{|z| > \frac{1}{3}.},$$

ROC: $|z| > \frac{1}{3}$

Properties of ROC

- **Property 1: ROC is a ring on the complex ~~plain~~ plane.**



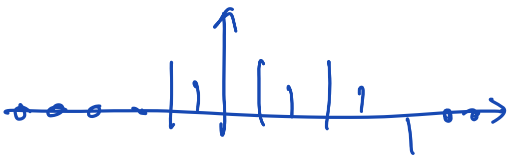
  - The inner circle's radius can be 0 (e.g., $x[n] = -a^n u[-n-1]$)
  - The outer circle's radius can be ∞ (e.g., $x[n] = a^n u[n]$)

- **Property 2: ROC does not contain any pole.**
  Simply because at a pole, the value of $X(z)$ would be infinity!

$$X(z) = \frac{1}{1 - \frac{a}{z}}$$

Properties of ROC

- **Property 3: If $x[n]$ is finite duration, then ROC is the whole complex ~~plain~~ plane, except possibly $z = 0$ and $z = \infty$.**
  - If $x[n]$ is only non-zero for $n = N_1, \ldots, N_2$ for $N_1 \leq N_2$. Then:



$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}.$$

$$x[n] z^{-N_1} = x[n]\, z^{(+ve)} \Rightarrow \infty \notin \text{ROC}$$

$$x[n] \cdot \frac{1}{z^{(+ve)}} \Rightarrow 0 \notin \text{ROC}.$$

- Determine whether 0 and ∞ belong to the ROC in the following cases.
  - $N_1 < 0, N_2 > 0$: neither 0 nor ∞ belong to ROC
  - $N_1 \geq 0$: $\infty \in$ ROC, $0 \notin$ ROC
  - $N_2 \leq 0$: $\infty \notin$ ROC, $0 \in$ ROC.

For a right-sided signal, $x[n] = 0$ for $n < N$.

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=N}^{\infty} x[n] z^{-n}$$

Since $z_0 \in$ ROC, we have $\underline{\sum_{n=N}^{\infty} |x[n] z_0^{-n}|} < \infty$.

show that z with $|z| > |z_0|$ is an element of ROC.

$$\underbrace{\sum_{n=N}^{-1}|x[n]z^{-n}|}_{\text{finite}} + \sum_{n=0}^{\infty}|x[n]z^{-n}| \leq \sum_{n=0}^{\infty}|x[n]||z^{-n}|$$

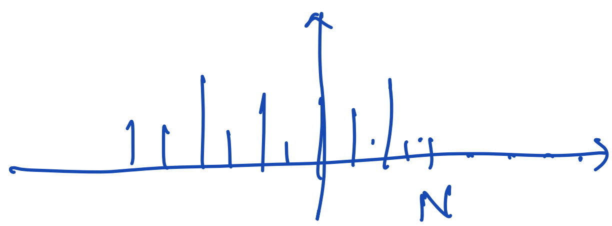
$$\leq \underbrace{\sum_{n=0}^{\infty}|x[n]||z_0^{-n}|}_{\text{finite since } z_0 \in \text{ROC.}}$$

Properties of ROC

- Definition: We say that a signal $x[n]$ is right-sided if $x[n] = 0$ for all $n < N$ and some N.
- **Property 4: If $x[n]$ is right-sided and z_0 is in ROC, then z is in ROC for all $|z| > |z_0|$, possibly excluding ∞.**
- The main reason is:

$$\sum_{n=0}^{\infty}|x[n]||z|^{-n} \leq \sum_{n=0}^{\infty}|x[n]||z_0|^{-n} < \infty$$

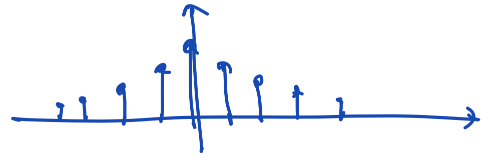
and if $N < 0$, then $\sum_{n=N}^{-1}|x[n]||z|^{-n} < \infty$.



- Definition: We say that a signal $x[n]$ is left-sided if $x[n] = 0$ for all $n > N$ and some N.
- **Property 5: If $x[n]$ is left-sided and z_0 is in ROC, then z is in ROC for all $0 < |z| < |z_0|$.**
- The main reason is:

$$\sum_{n=-\infty}^{0}|x[n]||z|^{-n} \leq \sum_{n=-\infty}^{0}|x[n]||z_0|^{-n} < \infty$$

and if $N > 0$, then $\sum_{n=1}^{N}|x[n]||z|^{-n} < \infty$.



Properties of ROC

- Definition: We say that a signal $x[n]$ is double-sided if $x[n] \neq 0$ for arbitrarily large and small n.
- **Property 6: If $x[n]$ is double-sided and z_0 is in ROC, then ROC is a ring containing z_0.**
- Example: What is the z-transform of $x[n] = b^{|n|}$ for $|b| < 1$?

$$\sum_{n=-\infty}^{\infty} b^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} (bz)^{-n} + \sum_{n=0}^{\infty} (b/z)^{n}$$

$$= \sum_{k=1}^{\infty} (bz)^{k} + \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^{n}$$

$$= \sum_{k=0}^{\infty} (bz)^{k} + \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^{n} - 1 = \frac{1}{1-bz} + \frac{1}{1-\frac{b}{z}} - 1$$

  – We have:

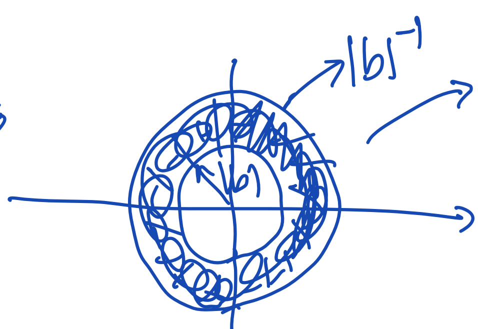
$$x[n] = b^n u[n] + b^{-n} u[-n-1].$$

  – $b^n u[n] \longleftrightarrow \frac{1}{1-bz^{-1}}$ with ROC $|z| > |b|$
  – $b^{-n} u[-n-1] \longleftrightarrow -\frac{1}{1-b^{-1}z^{-1}}$ with ROC $|z| < \frac{1}{|b|}$
  – Combining the two:

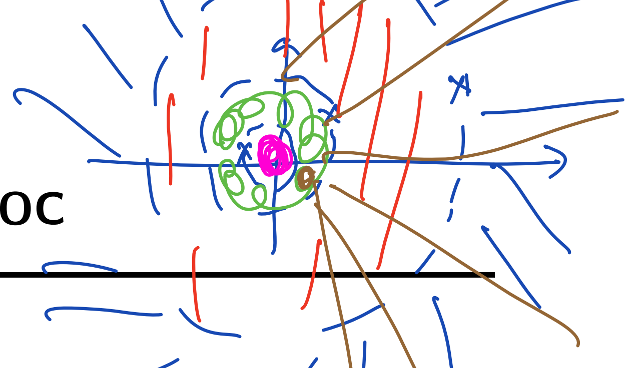
$$X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}} = \frac{b^2-1}{b} \frac{z}{(z-b)(z-b^{-1})},$$

for $|b| < |z| < \frac{1}{|b|}$.

  – If $|b| > 1$, there is no overlap in the ROCs of two parts, hence the z-transform is not defined.



unit circle is part of ROC.



Properties of ROC

- **Property 7: If $X(z)$ is rational, then ROC is always bounded by poles or extend to infinity.**

- Definition: We say that $x[n]$ is a **causal signal** if $x[n] = 0$ for $n < 0$.

- **Property 8: If $X(z)$ is rational and $x[n]$ is right-sided, then ROC is the region from the outermost pole to infinity. Furthermore, if $x[n]$ is causal, $X(\infty)$ exists.**

  - if $z_0 \in$ ROC, then all z with $|z| > |z_0|$ belong to ROC.
  - However, ROC must not contain any pole.
  - Hence $|z_0|$ must be larger than $|\text{poles}|$.

- Definition: We say that $x[n]$ is a **anti-causal signal** if $x[n] = 0$ for $n > 0$.

- **Property 9: If $X(z)$ is rational and $x[n]$ is left-sided, then ROC is the region from the innermost pole to zero. Furthermore, if $x[n]$ is anti-causal ROC includes $z = 0$.**

Question

Suppose that a we know that a signal is causal and its z-transform is:

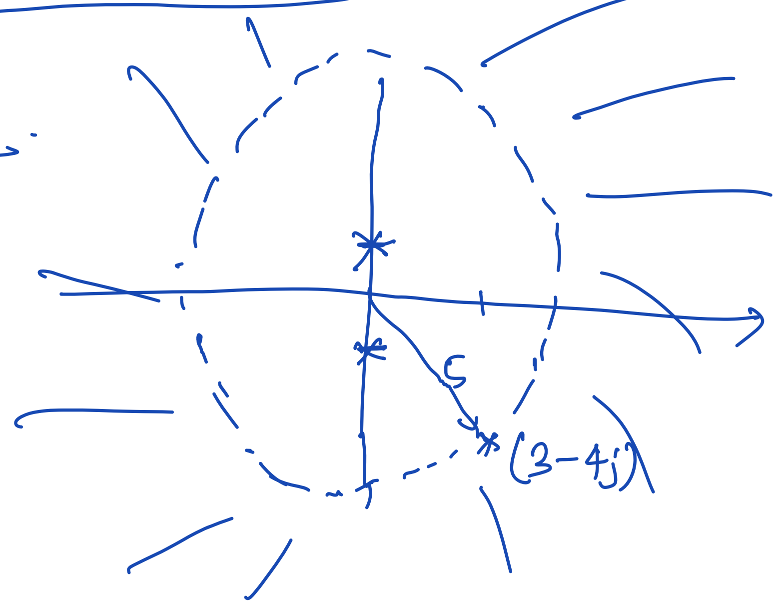
$$X(z) = \frac{2z}{(z+j)(z-j)(z-(3-4j))}.$$

Determine the ROC of $X(z)$.

poles : $-j,\ j,\ 3-4j$

absolute values of poles : $1, 1, 5$.

ROC : $|z| > 5$



Solution: $\{z \mid |z| > 5\}$

$$X(z) = \frac{A}{1-\frac{1}{3}z^{-1}} + \frac{B}{1-2z^{-1}} = \frac{A - 2Az^{-1} + B - \frac{1}{3}z^{-1}B}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})},$$

$A + B = 1$

$2A + \frac{B}{3} = 0$

$\Downarrow$

$B = -6A$

$A - 6A = 1$

$\Rightarrow A = -1/5, \quad B = 6/5$

Question

poles : $\frac{1}{3}, 2$

For the following z-transform

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}.$$

Determine all possible ROCs of $X(z)$ and find the corresponding $x[n]$ for each of those cases.

case 1 : $|z| < \frac{1}{3}$

case 2 : $\frac{1}{3} < |z| < 2$

case 3 : $|z| > 2$

$$X(z) = \frac{-1}{5} \cdot \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{6}{5} \cdot \frac{1}{1-2z^{-1}}.$$

$$x_1[n] = \left(-\frac{1}{5}\right) \cdot -\left(\frac{1}{3}\right)^n u[-n-1] + \frac{6}{5} - (2)^n u[-n-1]$$

$$= \left[\frac{1}{5}\left(\frac{1}{3}\right)^n - \frac{6}{5}2^n\right] u[-n-1].$$

$$x_2[n] = -\frac{1}{5}\left(\frac{1}{3}\right)^n u[n] - \frac{6}{5}2^n u[-n-1]$$

$$x_3[n] = -\frac{1}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{6}{5}2^n u[n].$$

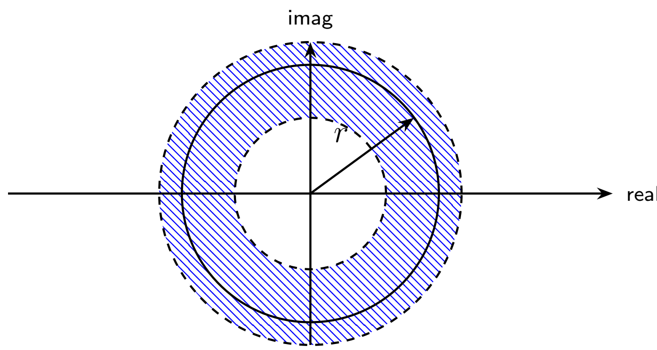
z-transform, Fourier Transform, and Inverse z-transform

- We know that if $z_0 = re^{j\omega}$ is in the ROC of the z-transform of $x[n]$, then

$$X(re^{j\omega}) = \mathtt{DTFT}(r^{-n}x[n]).$$

- Therefore, if the z-transform exists on a circle of radius r, then:

$$r^{-n}x[n] = \mathtt{DTFT}^{-1}(X(re^{j\omega})).$$



- Therefore, $x[n] = r^n \mathtt{DTFT}^{-1}(X(re^{j\omega}))$

z-transform, Fourier Transform, and Inverse z-transform

- Example: We know that the z-transform of a signal $x[n]$ is $X(z) = \frac{1}{1-z^{-1}}$ and the ROC is $\{z \mid |z| > 1\}$. What is $x[n]$?

$u[n]$

- Here are the steps to find $x[n]$:

1. Pick a circle of radius r in ROC. We pick $r = 2$.
2. Calculate $\texttt{DTFT}^{-1}(X(re^{j\omega}))$: $X(2e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$, but:

$$a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\omega}},$$

therefore: $\texttt{DTFT}^{-1}(X(re^{j\omega})) = (\frac{1}{2})^n u[n]$

3. Finally: $x[n] = r^n \texttt{DTFT}^{-1}(X(re^{j\omega})) = 2^n (\frac{1}{2})^n u[n] = u[n]$

Inverse z-transform: Partial Fraction Expansion

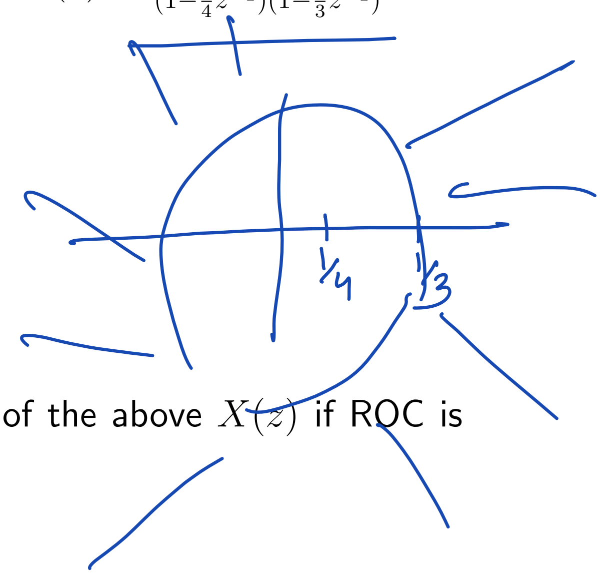
- Try to write $X(z) = \sum_{i=1}^{n} \frac{b_i}{1-a_i z^{-1}}$
- If ROC is outside the circle with radius $|a_i|$, then use the fact that

$$a_i^n u[n] \longleftrightarrow \frac{1}{1-a_i z^{-1}}, \quad \text{for} \quad |z| > |a_i|$$

- If ROC is inside the circle with radius $|a_i|$, then use the fact that

$$-a_i^n u[-n-1] \longleftrightarrow \frac{1}{1-a_i z^{-1}}$$

- Example 1: What is the inverse z-transform of $X(z) = \frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$ with ROC $\{z \mid |z| > \frac{1}{3}\}$?



- Example 2: What is the inverse z-transform of the above $X(z)$ if ROC is $\{z \mid \frac{1}{4} < |z| < \frac{1}{3}\}$?

Inverse z-transform: Power Series Expansion

- If we can express $X(z) = \sum_k a_k z^{-k}$, then we can express $x[n] = \sum_k a_k \delta[n-k]$.

- Example 1: What is the inverse z-transform of $X(z) = 4z^2 + 2 + 3z^{-1}$ with ROC $\{z \mid |0 < |z| < \infty\}$?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

- Example 2: Recall that $\log(1+\nu) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\nu^n}{n}$ for $|\nu| < 1$. Using this fact, determine the inverse z-transform of $X(z) = \log(1 + az^{-1})$ with ROC $|z| > |a|$.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(az^{-1})^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n}{n} \cdot z^{-n}$$

$$x[n] = \begin{cases} \frac{(-1)^{n+1}a^n}{n}, & n \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$