## 24th March: Lecture 32

Introduction to LMIs, Schur complement lemma
[on Blackboard]

## 30th March: Lecture 33 & 34

Next class test: 13th April, 5pm - 6pm, closed book

End sem: One A4 sheet allowed per student
endsem will contain questions from the entire course,
but postmidsem portion will be emphasized.

Last class: introduced linear matrix inequalities.

$\{x \in \mathbb{R}^n \mid F_0 + x_1 F_1 + \cdots + x_n F_n \preceq 0\}$

we also saw that if we find $P \succ 0$, $A^T P + PA \prec 0$, then origin is globally asymptotically stable eqm point of the LTI system $\dot{x}(t) = Ax(t)$.

both of these are LMIs in the elements of the matrix $P$.

optimization problems with LMI constraints are special types of convex optimization problems called semidefinite programs (SDPs).

– use suitable solvers such as MOSEK.

Suppose $\dot{x} = Ax + Bu$ has $(A, B)$ stabilizable, but $A$ is not a Hurwitz matrix.

$\Downarrow$

$\exists\, K$ s.t. $A + BK$ is Hurwitz.

so setting $u = Kx$ will stabilize closed-loop system.

Let us try to formulate a LMI to find $K$, (not a $f^n$ of time)

Following the previous approach, we can try to find

$P \succ 0$ s.t. $(A+BK)^T P + P(A+BK) \prec 0$.

$$\Leftrightarrow \quad A^T P + PA + K^T B^T P + PBK \prec 0 \quad \text{---} (A)$$

LMI in $P$ if $K$ is known.

But it is not a LMI when both $P$ and $K$ are unknown.

Note that since $P \succ 0 \Rightarrow P^{-1} \succ 0$, and $(P^{-1})^T = P^{-1}$.

Following congruence transformation property,

(A) is equivalent to

$$P^{-1}\left[A^T P + PA + K^T B^T P + PBK\right] P^{-1} \prec 0$$

$$\Rightarrow \quad P^{-1}A^T P P^{-1} + P^{-1} P A P^{-1} + P^{-1} K^T B^T P P^{-1} + P^{-1} P B K P^{-1} \prec 0$$

$$\Rightarrow \quad P^{-1}A^T + AP^{-1} + P^{-1}K^T B^T + BKP^{-1} \prec 0$$

We define new variables $X = P^{-1}$, $KP^{-1} = W \Rightarrow W^T = P^{-1}K^T$

Then, (A) is equivalent to

$$\boxed{X \succ 0, \quad XA^T + AX + W^T B^T + BW \prec 0}$$

LMI in variables $X$ and $W$.

$$\begin{aligned} \min_{X, W} \quad & 1 \\ \text{s.t.} \quad & X \succ 0 \\ & XA^T + AX + W^T B^T + BW \prec 0 \end{aligned}$$

$\Rightarrow$ Let $X^{OPT}$, $W^{OPT}$ be the optimal sol$^n$.

$\Rightarrow KX^{OPT} = W^{OPT}$

$\Rightarrow K = W^{OPT}(X^{OPT})^{-1}$

# Norms for Signals and systems

<u>Definition</u>: Given a vector space $V$, a function $\pi : V \to \mathbb{R}_+$, is called a norm if it satisfies the following properties.

(i) $\pi(x) \geq 0 \quad \forall x \in V$, $\pi(x) = 0$ if and only if $x = 0$.

(ii) $\pi(\alpha x) = |\alpha| \, \pi(x)$ for all $x \in V$, $\alpha \in \mathbb{R}$ — the homogeneity

(iii) $\pi(x+y) \leq \pi(x) + \pi(y)$ — triangle inequality

<u>Ex</u>: If $V = \mathbb{R}^n$, $\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$ for $p \in [1, \infty)$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$p = 2$ gives us the Euclidean norm.

Similarly, we can define norms on signals.

Let $\mathcal{F}^n$ be the space of signals that are vector-valued.

if $x \in \mathcal{F}^n \Rightarrow \underline{x(t) \in \mathbb{R}^n}$ for all $t \in \mathbb{R}$.

$x : [-\infty, \infty) \to \mathbb{R}^n$.

$\int_0^{2\pi} \sin^2 t \, dt \neq 0$

$$\|x\|_2^2 = \int_{-\infty}^{\infty} \|x(t)\|_2^2 \, dt \Rightarrow \|x\|_2 = \left[ \int_{-\infty}^{\infty} \|x(t)\|_2^2 \, dt \right]^{\frac{1}{2}}.$$

energy of the signal $x(t)$

$$\|x\|_\infty = \sup_{t \in (-\infty, \infty)} \|x(t)\|_\infty .$$

$\mathcal{L}_2$ : set of all signals with finite $\|\cdot\|_2$ norm, i.e., the set of signals with finite energy.

If the signal is defined on $[0, \infty)$, we denote this set as $\mathcal{L}_2[0, \infty)$.

Examples : (1) $x(t) = u(t)$ : unit step signal.

Find $\|x\|_2$ and $\|x\|_\infty = 1$

$\infty$

$$\int_0^\infty 1\,dt = \infty$$

(2) $x(t) = u(t) - u(t-4)$

Find $\|x\|_2$ and $\|x\|_\infty$. $\rightarrow$ $\int_0^4 1\,dt = 4 = \|x\|_2^2$

$\Rightarrow \|x\|_2 = 2$

$\|x\|_\infty = 1$

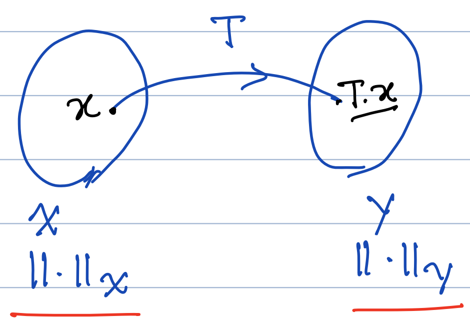
(3) $x(t) = \sin(t)$

$\|x\|_2 = \infty$

$\|x\|_\infty = 1$

## Induced operator norms

Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces with norms $\|\cdot\|_\mathcal{X}$ and $\|\cdot\|_\mathcal{Y}$, respectively.




A mapping

$T : \mathcal{X} \rightarrow \mathcal{Y}$,

Let $\Upsilon$ be the set of all $\underbrace{\text{mappings}}_{\text{operators.}}$ from $\mathcal{X}$ to $\mathcal{Y}$.

Example: $A \in \mathbb{R}^{m \times n}$, then $A$ can be viewed as an operator from $\mathbb{R}^n$ to $\mathbb{R}^m$. $y = Ax$ being the map.

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ s.t. if $x \in \mathbb{R}^n$, $f(x) = Ax \in \mathbb{R}^m$.

$\Upsilon$ : set of all matrices of dimension $m \times n$.

Let $m = 2$, $n = 2$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

To compare between operators, we need to definite suitable norm on the space of operators $\mathcal{T}$.

The induced operator norm of an operator $T \in \mathcal{T}$ is defined as

$$\|T\| = \sup_{\|x\|_{\mathcal{X}} \neq 0} \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}$$

$$= \sup_{\|x\|_{\mathcal{X}} = 1} \|T.x\|_{\mathcal{Y}}$$

– worst-case amplification.

For the set of matrices, we have the induced 2-norm defined as follows.

$$\|A\|_2 = \sup_{\|x\|_2 = 1} \|Ax\|_2 \quad , \quad x \in \mathbb{R}^n, \; Ax \in \mathbb{R}^m$$

$$\max_{\|x\|_2^2 = 1} \|Ax\|_2^2 = \max_{\|x\|_2^2 = 1} \underbrace{x^T A^T A x} = \lambda_{max}(A^T A), \text{ when}$$

$x$ is the unit vector along the eigenvector of $\lambda_{max}(A^T A)$.

Note that $A^T A$ is a symmetric psd matrix.
all its eigenvalues are real & non-negative.
Let the largest eigenvalue of $A^T A$ be $\lambda_{max}(A^T A)$.

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)} = \text{largest singular value of } A.$$

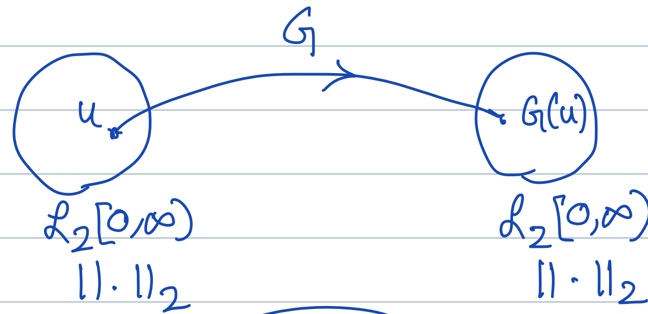
Note: Not all norms that can be defined on matrices are induced norms.

eg, Frobenius norm $\|A\|_F$ = sum of all the singular values of A.

This norm is not an induced norm.

Let us now look at operators defined on signals with finite energy.

$u(t) \longrightarrow \boxed{G} \longrightarrow y(t)$



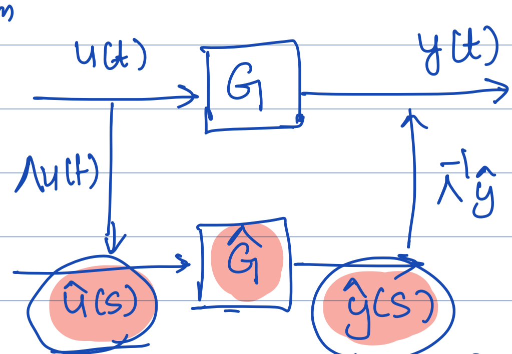

Let $G : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$.

Induced norm: $\|G\|_{\mathcal{L}_2} = \underbrace{\sup_{\substack{u \in \mathcal{L}_2 \\ \|u\|_2 \neq 0}} \frac{\|G(u)\|_2}{\|u\|_2}}_{\text{worst-case amplification of energy of the input signal in the output signal.}} \; . \quad < \frac{\gamma \|u\|_2}{\|u\|_2}$

Let us now specialize to LTI systems.

$\hat{u}(s) = (\Lambda u)(s)$ ← Laplace transform operator

$= \int_0^\infty e^{-st} u(t)\, dt$




$y(t) = (\Lambda^{-1}\hat{u})(t)$

$\underline{\hat{y}(s) = \hat{G}(s)\hat{u}(s)}$

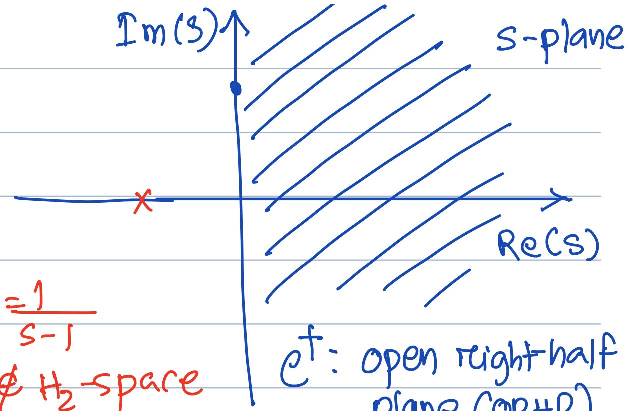
$\Lambda u$ : function of $s$

$\Lambda^{-1}\hat{y}$ : function of time $t$

<u>Goal</u>: Relate $\|G\|_{\mathcal{L}_2}$ with $\underline{\hat{G}(s)}$.

In order to that, we will define some suitable spaces defined on the complex plane.

$\mathbb{C}^+ = \{ s \mid Re(s) > 0 \}$




$\mathbb{C}^+$: open right-half plane (ORHP)

$H_2$-space : Let $F : \mathbb{C}^+ \to \mathbb{C}^{n \times m}$.

e.g. $F(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{(s+1)^2} \end{bmatrix}$ $\qquad F(s) = \frac{1}{s-1} \notin H_2$-space

F belongs to the $H_2$-space if

(i) $F(s)$ is analytic in ORHP $\Rightarrow$ $F(s)$ is differentiable at all points in ORHP and some neighborhood of all those points

(ii) $\|F\|^2_{H_2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} Tr\big( \underbrace{F(j\omega)^* F(j\omega)}_{\in \mathbb{C}^{m \times m} \text{ matrix}} \big) d\omega$

$< \infty$.

Proposition: If $u(t) \in \mathcal{L}_2[0,\infty)$, then $\Lambda u \in H_2$.
If $\hat{u}(s) \in H_2$, then $\Lambda^{-1}\hat{u} \in \mathcal{L}_2[0,\infty)$.

Relationship in terms of inner products.

Inner product on $\mathcal{L}_2$ space: If $u(t) \in \mathcal{L}_2[0,\infty)$, $y(t) \in \mathcal{L}_2[0,\infty)$,

$$\langle u(t), y(t) \rangle_{\mathcal{L}_2} = \int_0^{\infty} u(t)^T y(t)\, dt$$

Inner product on $H_2$-space

$$\langle \hat{u}(s), \hat{y}(s) \rangle_{H_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega)^* \hat{y}(j\omega)\, d\omega$$

Parseval relation: $\langle u(t), y(t) \rangle_{\mathcal{L}_2} = \langle \hat{u}(s), \hat{y}(s) \rangle_{H_2}$ //

Special case: for scalar function.

$$\langle u(t), u(t)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(j\omega)^{*}\, u(j\omega)\, d\omega$$
$$= \|\hat{u}(s)\|_{\mathcal{H}_2}^{2}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |u(j\omega)|^{2}\, d\omega .$$

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The transfer function $\hat{G}$ is essentially an operator from $\mathcal{H}_2$-space to $\mathcal{H}_2$-space.

$$\hat{u}(s) \longrightarrow \boxed{\hat{G}(s)} \longrightarrow \hat{y}(s)$$

$\mathcal{H}_\infty$-space: $\hat{G} : \mathbb{C}^{+} \to \mathbb{C}^{m\times n}$ is in $\mathcal{H}_\infty$-space if

(i) $\hat{G}$ is analytic in ORHP ($\Rightarrow$ no poles on RHP)

(ii) $$\|\hat{G}\|_{\mathcal{H}_\infty} = \sup_{Re(s)>0} \bar{\sigma}\left[\hat{G}(s)\right]$$
$$= \sup_{\omega\in\mathbb{R}} \underbrace{\bar{\sigma}\left[\hat{G}(j\omega)\right]}_{\text{largest singular value.}} < \infty .$$

In the special case of SISO systems, $\hat{G}(j\omega)$ is a scalar.

$$\bar{\sigma}\left[\hat{G}(j\omega)\right] = |\hat{G}(j\omega)|$$

$$\|\hat{G}\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} |\hat{G}(j\omega)|$$

Proposition: $\|G\|_{\mathcal{L}_2} = \|\hat{G}\|_{\mathcal{H}_\infty}$ (when $G$ is a LTI system)

|| if $x(t) = \sin(\omega t) \notin \mathcal{L}_2$ naturally $x(j\omega) \notin \mathcal{H}_2$ space as it has poles on imaginary axis.

Any matrix $A \in \mathbb{R}^{m \times n}$ can be written as

$$A = U \Sigma V^T$$

$\sqrt{\lambda_i(A^T A)}$ singular values.

$A^T A$ is psd matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_m \end{bmatrix}$$

$\geq 0$.

## Lecture 35 & 36: 6th April

Consider a LTI system $\dot{x}(t) = Ax(t) + Bu(t)$ ---- (1a)

$y(t) = Cx(t) + Du(t)$, --- (1b)

Let $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$,

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$

Applying Laplace transform to (1a) with $x(0) = 0$, we obtain

$$s\hat{X}(s) = A\hat{X}(s) + B\hat{U}(s) \Rightarrow (sI - A)\hat{X}(s) = B\hat{U}(s)$$

$$\Rightarrow \hat{X}(s) = (sI - A)^{-1} B \hat{U}(s)$$

likewise, applying Laplace transform to (1b), we obtain

$$\hat{Y}(s) = C\hat{X}(s) + D\hat{U}(s)$$

$$= \left[C(sI - A)^{-1}B + D\right]\hat{U}(s)$$

$$\Rightarrow \hat{G}(s) = \left[C(sI - A)^{-1}B + D\right] \in \mathbb{C}^{p \times m} \rightarrow \text{transfer function matrix.}$$

$(i,j)$-th element of $\hat{G}(s)$ shows how $j$-th input affects $i$-th output.

**Notation:** $\hat{G}(s) = \underbrace{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]} = C(sI-A)^{-1}B + D.$

**Theorem:** Let $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI-A)^{-1}B+D$. Then

$\|\hat{G}(s)\|_{H_\infty} < \gamma$ iff $\Leftrightarrow$ $\exists\ X \succ 0$ s.t.

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} \prec 0. \quad \text{---}(1)$$

basically a LMI in $X \in S^n$.

(proof) ($\Leftarrow$ direction). We have $X \succ 0$ s.t. (1) holds.

Since the L.H.S. of (1) is negative definite, multiplying $\begin{bmatrix} x \\ u \end{bmatrix}^T$ from the left and $\begin{bmatrix} x \\ u \end{bmatrix}$ from the right will yield

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \left( \begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} \right)\begin{bmatrix} x \\ u \end{bmatrix} < 0$$

$$\Rightarrow \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \underbrace{\begin{bmatrix} C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}} < 0$$

$$\Rightarrow \begin{bmatrix} x^T & u^T \end{bmatrix}\begin{bmatrix} A^T X x + XAx + XBu \\ B^T X x - \gamma u \end{bmatrix} + \begin{bmatrix} \frac{1}{\gamma} \end{bmatrix}(Cx+Du)^T(Cx+Du) < 0$$

$$\Rightarrow x^T A^T X x + x^T X A x + x^T X B u + u^T B^T X x - \gamma u^T u + \frac{1}{\gamma} y^T y < 0$$

$$\Rightarrow (x^T A^T + u^T B^T) X x + x^T X (Ax + Bu) - \gamma u^T u + \frac{1}{\gamma} y^T y < 0$$

$$\Rightarrow (\dot{x})^T X x + x^T X \dot{x} - \gamma u^T u + \frac{1}{\gamma} y^T y < 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(x^T X x\right) - \gamma u^T u + \frac{1}{\gamma} y^T y < 0 \qquad \forall\, t,\ x(t), u(t)$$

$$\Rightarrow \quad \int_0^T \left[ \frac{d}{dt}\big(\underbrace{x(t)^T X x(t)}_{V(x(t))}\big) - \gamma\, u(t)^T u(t) + \frac{1}{\gamma}\, y(t)^T y(t) \right] dt < 0$$

$$\Rightarrow \quad V(x(T)) - V(x(0)) - \gamma \int_0^T u(t)^T u(t)\, dt + \frac{1}{\gamma} \int_0^T y(t)^T y(t)\, dt < 0$$

let $x(0) = 0$, $\Rightarrow$ $V(x(0)) = x(0)^T X x(0) = 0$

$$\frac{1}{\gamma} \int_0^T y(t)^T y(t)\, dt < \gamma \int_0^T u(t)^T u(t)\, dt - \underbrace{V(x(T))}_{\geq 0 \text{ as } X \succ 0}$$

$$\Rightarrow \quad \frac{1}{\gamma} \int_0^T y(t)^T y(t)\, dt < \gamma \int_0^T u(t)^T u(t)\, dt \qquad \forall\, T$$

setting $T \to \infty$, we obtain

$$\| y(t) \|_{\mathcal{L}_2}^2 < \gamma^2 \, \| u(t) \|_{\mathcal{L}_2}^2$$

$$\Rightarrow \quad \underline{\| y(t) \|_{\mathcal{L}_2} < \gamma \, \| u(t) \|_{\mathcal{L}_2}} \qquad \forall \text{ input-output pair } (u(t), y(t)) \text{ of the system.}$$

$$\Rightarrow \quad \| G \|_{\mathcal{L}_2} < \gamma$$

$$\Updownarrow$$

$$\Rightarrow \quad \underline{\| \hat{G} \|_{\mathcal{H}_\infty} < \gamma}$$

Our closed-loop dynamics is
$$\dot{x} = \bar{A} x + \bar{B} w$$
$$y = \bar{C} x + \bar{D} w,$$

$\bar{A} = A + B_1 K$, $\quad \bar{B} = B_2$, $\quad \bar{C} = C + D_1 K$, $\quad \bar{D} = D_2$

## Equivalent LMI formulations of (1)

**Proposition:** The condition $X \succ 0$, and (1) holds is equivalent to:

(a) $X \succ 0$ s.t. $\begin{bmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \prec 0$

does not remain a LMI when K & X are both unknowns.

(b) $X \succ 0$ s.t. $\begin{bmatrix} XA^T+AX & B & XC^T \\ B^T & -\gamma I & D^T \\ CX & D & -\gamma I \end{bmatrix} \prec 0$

**proof:** To show (a) is equivalent to (1), we apply Schur Complement Lemma to (a), we obtain

$$\begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} - \begin{bmatrix} C^T \\ D^T \end{bmatrix} (-\gamma I)^{-1} \begin{bmatrix} C & D \end{bmatrix} < 0$$

$$\Leftrightarrow \begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0,$$

which is nothing but (1).

Next, we show that (b) is equivalent to (a). We multiply the matrix $T = \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ from both left & right to (a) and obtain

symmetric pd matrix

$$\begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} X^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^T + XAX^{-1} & XB & C^T \\ B^T & -\gamma I & D^T \\ CX^{-1} & D & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} X^{-1}A^T + AX^{-1} & B & X^{-1}C^T \\ B^T & -\gamma I & D^T \\ CX^{-1} & D & -\gamma I \end{bmatrix}$$

let $X^{-1} = P$,
$X^{-1} \succ 0 \Leftrightarrow P \succ 0$

$$= \begin{bmatrix} PA^T + AP & B & PC^T \\ B^T & -\gamma I & D^T \\ CP & D & -\gamma I \end{bmatrix}$$

which is nothing but the matrix in (b) with $X$ replaced by $P \succ 0$.

Since congruence transformation preserves positive definiteness, then (b) has a solution if and only if (a) has a solution.

We will use the above derivations to compute a state-feedback controller that minimizes $H_\infty$ norm of the transfer function between a disturbance input and the output.

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 w(t)$$
$$y(t) = Cx(t) + D_1 u(t) + D_2 w(t),$$

where $u(t)$ : input applied by the controller
$w(t)$ : disturbance affecting the system

The goal is to find $K \in \mathbb{R}^{m \times n}$ s.t. when $u(t) = Kx(t)$, then $\| \hat{G}_{wy}(s) \|_{H_\infty}$ is minimized.

when $u(t) = Kx(t)$, then the dynamics of closed-loop system is given by:

$$\dot{x}(t) = Ax(t) + B_1 K x(t) + B_2 w(t)$$
$$y(t) = Cx(t) + D_1 K x(t) + D_2 w(t)$$

$$\Rightarrow \quad \begin{aligned} \dot{x} &= (A + B_1 K)x + B_2 w \\ y &= (C + D_1 K)x + D_2 w \end{aligned}$$

$$\hat{G}_{wy}(s) = \left[\begin{array}{c|c} A + B_1 K & B_2 \\ \hline C + D_1 K & D_2 \end{array}\right] = (C + D_1 K)(sI - A - B_1 K)^{-1} B_2 + D_2$$

Goal: $\min_{K \in \mathbb{R}^{m \times n}} \| \hat{G}_{wy}(s) \|_{H_\infty}$ — (P)

Using LMI formulation (b) in the proposition, we formulate the following convex optimization problem to solve (P).

$$\min_{\gamma, X, K} \quad \gamma$$

$$\text{s.t.} \quad X \succ 0,$$

not a LMI due to product terms $KX$ $\leftarrow$

$$\begin{bmatrix} X(A+B_1K)^T + (A+B_1K)X & B_2 & X(C+D_1K)^T \\ B_2^T & -\gamma I & D_2^T \\ (C+D_1K)X & D_2 & -\gamma I \end{bmatrix} \prec 0$$

$\downarrow$ can be reformulated as a LMI constrained problem by $W := KX$

$$\min_{\gamma, X, W} \quad \gamma$$

$$\text{s.t} \quad X \succ 0, \quad \begin{bmatrix} W^T B_1^T + XA^T + AX + B_1 W & B_2 & XC^T + W^T D_1^T \\ B_2^T & -\gamma I & D_2^T \\ CX + D_1 W & D_2 & -\gamma I \end{bmatrix} \prec 0$$

$\hookrightarrow$ LMI in variables $(\gamma, X, W)$

If the above problem has optimal solutions $\underline{(X^*, W^*)}$, then

$$W^* = K^{opt} X^* \Rightarrow \underline{K^{opt} = W^* (X^*)^{-1}}.$$

apply input $\underline{u(t) = K^{opt} x(t)}$.

# Lecture 37 & 38 : 7th April

For a transfer function $\hat{G}(s)$, its $H_2$-norm is given by

$$\|\hat{G}\|_{H_2}^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Tr}\left[\hat{G}(j\omega)^* \hat{G}(j\omega)\right] d\omega .$$

Note: If $D \neq 0$, then $\|\hat{G}\|_{H_2}^2$ is not finite.

$$\hat{G}(j\omega) = C(j\omega I - A)^{-1}B + D$$

$$\int_{-\infty}^{\infty} \operatorname{Tr}\left[\hat{G}(j\omega)^* \hat{G}(j\omega)\right] d\omega = \int_{-\infty}^{\infty} \operatorname{Tr}\left[\left(C(j\omega I - A)^{-1}B\right)^*\left(C(j\omega I - A)^{-1}B\right) + (\cdot)\right] d\omega$$

$$+ \int_{-\infty}^{\infty} \operatorname{Tr}[\underbrace{D^T D}_{\text{does not depend on } \omega}]\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \operatorname{Tr}[D^T D]\, d\omega = \operatorname{Tr}[D^T D] \cdot \int_{-\infty}^{\infty} d\omega = \infty .$$

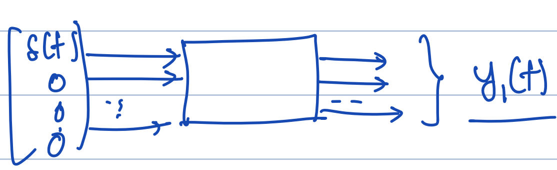
Thus, for a transfer function to have finite $H_2$-norm, the feedforward term must be absent in the output equation.

## Interpretation of $H_2$-norm

Suppose $u(t) \in \mathbb{R}^m$. Let $u_i(t) = \delta(t)\, e_i$, $e_i$ : unit vector in $\mathbb{R}^m$ with 1 at the $i$-th component & 0 everywhere else.

$\delta(t)$ : unit impulse

Let $y_i(t) \in \mathbb{R}^p$ be the output of the system when input is $u_i(t)$.



Claim: $\underline{\|\hat{G}(s)\|^2_{H_2}} = \underline{\sum_{i=1}^{m} \|y_i(t)\|^2_{L_2}}$ $\rightarrow$ total energy of the output signals produced by exciting the system by applying unit impulse to each input channel separately.

proof:

Note that the system is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t).$$

$$y(t) = C\int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$y_i(t) = C\int_0^t e^{A(t-\tau)} Be_i\, \delta(\tau)\, d\tau = Ce^{At}Be_i$$

$$\|y_i(t)\|_2^2 = \int_0^\infty (Ce^{At}Be_i)^T (Ce^{At}B)e_i\, dt$$

$$= \int_0^\infty e_i^T (Ce^{At}B)^T (Ce^{At}B)\, e_i\, dt$$

$$\Rightarrow \sum_{i=1}^{m} \|y_i(t)\|^2_{L_2} = \int_0^\infty \Big[\sum_{i=1}^{m} \underbrace{e_i^T (Ce^{At}B)^T (Ce^{At}B)\, e_i}_{}\Big] dt$$

ith diagonal element of $(Ce^{At}B)^T(Ce^{At}B)$

$$\Rightarrow \sum_{i=1}^{m} \|y_i(t)\|^2_{L_2} = \underline{\int_0^\infty \mathrm{Tr}\left[(Ce^{At}B)^T(Ce^{At}B)\right] dt} = \underbrace{\mathrm{Tr}\left(\int_0^\infty (Ce^{At}B)^T (Ce^{At}B)\, dt\right)}_{}$$

Recall that

$\mathcal{L}(e^{At}) = (sI - A)^{-1}$

$\mathcal{L}(Ce^{At}B) = C(sI-A)^{-1}B$.

$$\Rightarrow \sum_{i=1}^{m} \|y_i(t)\|_{L_2}^2$$

$$= \mathrm{Tr}\left[\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(C(j\omega I - A)^{-1}B\right)^{*} \left(C(j\omega I - A)^{-1}B\right) d\omega}\right]$$

→ follows from Parseval's identity.

$$= \|\hat{G}(s)\|_{H_2}^2$$

## Computing $H_2$-norm of a LTI system

First observe that

$$\|\hat{G}\|_{H_2}^2 = \mathrm{Tr}\left[\int_0^{\infty} \left(Ce^{At}B\right)^T \left(Ce^{At}B\right) dt\right]$$

$(Ce^{At}B)^T = B^T e^{A^T t} C^T$

$$= \mathrm{Tr}\left[B^T \underbrace{\left(\int_0^{\infty} e^{A^T t} C^T C e^{At}\, dt\right)}_{\text{observability Gramian of the system, denoted by } X_O} B\right] = \mathrm{Tr}\left[B^T X_O B\right]$$

- If A is Hurwitz, $X_O$ is a solution of

$$A^T X_O + X_O A + C^T C = 0. \quad \text{--- (1)}$$

If $(C, A)$ is observable, it is the unique sol<sup></sup>n of (1) with $X_O \succ 0$.

$\|\hat{G}\|_{H_2}^2$ is also related to the controllability gramian.

Note that $\mathrm{Tr}[A^T A] = \mathrm{Tr}[A A^T]$.

$$\|\hat{G}\|_{H_2}^2 = \mathrm{Tr}\left[\int_0^{\infty} \left(Ce^{At}BB^T e^{A^T t} C^T\right) dt\right] = \mathrm{Tr}\left[C X_c C^T\right],$$

where $X_c = \int_0^{\infty} \left(e^{At}BB^T e^{A^T t}\right) dt$ is the

# Controllability Gramian.

**Recall:**

If $A$ is Hurwitz, $X_c$ is a solution of

$$AX_c + X_cA^T + BB^T = 0 \quad \cdots (2)$$

If $(A,B)$ controllable, $X_c$ is unique positive definite solution of (2).

**Theorem:** Let $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = C(sI-A)^{-1}B$. Then, the following are equivalent.

(i) $A$ is Hurwitz and $\|\hat{G}(s)\|_{H_2}^2 < \gamma^2$

(ii) $\exists\, X \succ 0$ s.t. $A^TX + XA + C^TC \prec 0$, $\operatorname{tr}(B^TXB) < \gamma^2$

(iii) $\exists\, X \succ 0$ s.t. $AX + XA^T + BB^T \prec 0$, $\operatorname{tr}(CXC^T) < \gamma^2$.

**proof:** (i) $\Rightarrow$ (iii).

$$X_c \succeq 0,\ P \succ 0 \Rightarrow X_c + \varepsilon P \succ 0$$

$A$ is Hurwitz $\Rightarrow \exists\, P$ s.t. $A^TP + PA \prec 0$, $P \succ 0$ (*)

$A$ is Hurwitz $\Rightarrow$ the controllability Grammian $X_c \succeq 0$, is a solution of

$$AX_c + X_cA^T + BB^T = 0$$

It is also given

$$\|\hat{G}(s)\|_{H_2}^2 = \operatorname{Tr}[CX_cC^T] < \gamma^2.$$

Let $\varepsilon > 0$ be sufficiently small s.t. $\operatorname{Tr}[C(X_c+\varepsilon P)C^T] < \gamma^2$

Claim: $X_c + \varepsilon P$ satisfies all the conditions stated in (iii).

$$A(X_c+\varepsilon P) + (X_c+\varepsilon P)A^T + BB^T$$

$$= \underbrace{AX_c + X_cA^T + BB^T}_{=0} + \underbrace{\varepsilon(AP + PA^T)}_{\prec 0} \prec 0.$$

Note: $A$ is Hurwitz $\Leftrightarrow$ $A^T$ is also Hurwitz

$\Updownarrow$

$\exists P \succ 0$ s.t. $A^TP + PA \prec 0$ $\Leftrightarrow$ $\exists P \succ 0$ s.t. $AP + PA^T \prec 0$.

Thus, to compute $H_2$-norm of $\hat{G}(s)$, we can equivalently solve a LMI.

$$\boxed{\begin{aligned} &\min_{\rho, X} \quad \rho \\ &\text{s.t.} \quad X \succ 0, \quad \mathrm{Tr}[CXC^T] < \rho, \\ &\qquad AX + XA^T + BB^T \prec 0, \\ &\qquad \rho \geq 0. \end{aligned}}$$

Then, $\gamma = +\sqrt{\rho}$

## $H_2$-optimal State Feedback Controller

Consider the system:
$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2w(t)$$
$$y(t) = Cx(t) + D_1u(t)$$

We want to find $K$ s.t. if $u(t) = Kx(t)$, then

$$\|\hat{G}_{w \to y}\|_{H_2}^2 \text{ is minimized.}$$

Closed-loop system.
$$\dot{x}(t) = (A + B_1K)x(t) + B_2w(t)$$
$$y(t) = (C + D_1K)x(t).$$

$$\hat{G}_{wy} = \left[\begin{array}{c|c} A+B_1K & B_2 \\ \hline C+D_1K & 0 \end{array}\right].$$

Applying the previous theorem, our goal is to find $X, K$ s.t.

$X \succ 0$, $\quad (A+B_1K)X + X(A+B_1K)^T + B_2B_2^T \prec 0$ --- (a)

$$\mathrm{Tr}\left[(C+D_1K)X(C+D_1K)^T\right] < \rho \quad \text{--- (b)}.$$

observe that (a) can be written as:

$$AX + B_1\underbrace{KX}_{=:Z} + XA^T + \underbrace{XK^T}_{=:Z^T}B_1^T + B_2B_2^T \prec 0$$

$\iff$ $AX + B_1Z + XA^T + Z^TB_1^T + B_2B_2^T \prec 0$ -- (a')

LMI in $(X, Z)$

Now for (b) observe that it is equivalent to

$$\mathrm{Tr}\left[(C+D_1K)XX^{-1}X(C+D_1K)^T\right] < \rho$$

$$\iff \mathrm{Tr}\left[(CX+D_1Z)X^{-1}(CX+D_1Z)^T\right] < \rho$$

$\iff$ $\mathrm{Tr}[W] < \rho$ and $(CX+D_1Z)X^{-1}(CX+D_1Z)^T \prec W$

for some $W$.

$$W - (CX+D_1Z)X^{-1}(CX+D_1Z)^T \succ 0$$

$$\iff \begin{bmatrix} W & CX+D_1Z \\ (CX+D_1Z)^T & X \end{bmatrix} \succ 0$$

applying Schur complement lemma.

Finally, we solve the following problem:

$$\min_{X, Z, W} \quad Tr(W)$$

$$\text{s.t.} \quad \left.\begin{array}{l} \begin{bmatrix} W & CX + D_1 Z \\ (CX + D_1 Z)^T & X \end{bmatrix} \succ 0 \\ AX + B_1 Z + XA^T + Z^T B_1^T + B_2 B_2^T \prec 0 . \end{array}\right\} \text{LMIs.}$$

once we solve the above problem, and find $(X^*, Z^*, W^*)$,

we use $KX^* = Z^* \Rightarrow K_{H_2}^{opt} := Z^* (X^*)^{-1}$ - - - ✓

$$Tr[W^*] = \|\hat{G}_{wy}\|_{H_2}^2 .$$