LECTURE 25 & 26 : 9th march

Last week, we discussed CT optimal control problem

$$\min_{x(t),u(t)} \quad S(x(t_f),t_f) + \int_{t_0}^{t_f} V(x(t),u(t),t)\,dt$$

s.t. $\quad \dot{x}(t) = f(x(t),u(t),t)$

t_0, $x(t_0)$ are fixed/known

t_f, $x(t_f)$ are free.

<u>optimality conditions</u> $\quad H(x,u,p,t) = V(x,u,t) + p^T f(x,u,t)$

$$\left.\frac{\partial H}{\partial u}\right|_{*} = 0, \quad \dot{p}(t) = -\left(\frac{\partial H}{\partial x}\right)^T, \quad \dot{x}(t) = f(x(t),u(t),t)$$

boundary condition: $\left.\left[\frac{\partial S}{\partial x} - p(t)^T\right]\delta x_f + \left(H + \frac{\partial S}{\partial t}\right)\delta t_f\right|_{t_f} = 0$

$$\left[c^T F C x - c^T F z - p(t_f)\right]\delta x_f + H(t_f)\,\delta t_f = 0$$

CT optimal control: Linear Quadratic case

<u>plant</u>: $\quad \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$

$\quad y(t) = C(t)\,x(t)$

<u>cost f^n</u>: $$J = \frac{1}{2}\left(y(t_f) - z(t_f)\right)^T F(t_f)\left(y(t_f) - z(t_f)\right) \qquad — (1)$$

$$+ \frac{1}{2}\int_{t_0}^{t_f}\left[\left(y(t) - z(t)\right)^T Q(t)\left(y(t) - z(t)\right) + u(t)^T R(t)\,u(t)\right]dt$$

$z(t)$, $t \in [t_0, t_f]$: reference signal

cost function is quadratic in the error $e(t) = y(t) - z(t)$

Special cases

(1) State regulation: Goal is to drive state to origin

this is achieved when $C(t) = I$, $z(t)=0$ $\forall t$

(2) Output regulation: goal is to drive output to origin.

achived by setting $z(t)=0$ $\forall t$

(3) Tracking problem: $z(t) \neq 0$, and specified by user.

Assumptions: $F(t_f) \succeq 0$, $R(t) \succ 0$ (pd $\Rightarrow$ invertible), $Q(t) \succeq 0$.

↳ inputs and states are not subject to any constraints.

Let us now derive the optimality conditions.
We define the Hamiltonian

$$H(x,u,p,t) = \underbrace{\frac{1}{2}(Cx-z)^T Q (Cx-z) + \frac{1}{2}u^T R u}$$
$$+ \underbrace{p^T (Ax + Bu)}$$

optimality conditions

$$\frac{\partial H}{\partial u} = 0 \Rightarrow Ru + B^T p = 0$$

$$\Rightarrow u(t) = -R(t)^{-1} B(t)^T p(t) \quad \forall t \in [t_0, t_f]$$

$$\dot{p}(t) = -\left(\frac{\partial H}{\partial x}\right)^T \Rightarrow \dot{p} = -\left(A^T p + C^T Q C x - C^T Q z\right),$$
$$\forall t \in [t_0, t_f]$$

(A) State-Regulation, t_f is fixed, $z(t)=0$, $C(t)=I$

boundary conditions:- $p(t_f) = C(t_f)^T F(t_f)\, C(t_f)\, x(t_f)$

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R(t)^{-1}B(t)^T \\ -C(t)^T Q(t) C(t) & -A(t)^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \quad x(t_0) = x_f$$

$$= \begin{bmatrix} A(t) & -B(t)R(t)^{-1}B(t)^T \\ -Q(t) & -A(t)^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \quad p(t_f) = F(t_f)\,x(t_f)$$

we hypothesize $p(t) = P(t)\,x(t)$ $\forall\, t \in [t_0, t_f]$,

$P(t_f) = F(t_f)$

$$\dot{p}(t) = \dot{P}(t)\,x(t) + P(t)\,\dot{x}(t)$$

$$\Rightarrow -Q(t)\,x(t) - A(t)^T P(t)\,x(t) = \dot{P}(t)\,x(t) + P(t)\left[A(t)x(t) + B(t)\left(-R(t)^{-1}B(t)^T\right)\cancel{p(t)}\; P(t)\,x(t)\right]$$

$$\Rightarrow -Q(t) - A(t)^T P(t) = \dot{P}(t) + P(t)A(t) - P(t)B(t)R(t)^{-1}B(t)^T P(t)$$

$$\Rightarrow \dot{P}(t) = -Q(t) - A(t)^T P(t) - P(t)A(t) + P(t)B(t)R(t)^{-1}B(t)^T P(t)$$

: differential Riccati equation.

eg: $\begin{bmatrix} \dot{p}_{11}(t) & \dot{p}_{12}(t) \\ \dot{p}_{21}(t) & \dot{p}_{22}(t) \end{bmatrix} = \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}$, $P(t_f) = F(t_f)$

→ solved backwards in time offline

during online execution: we can define $\tau = t_f - t$, $\Rightarrow$ $\underline{t = t_f - \tau}$

when $t = t_f \Rightarrow \underline{\tau = 0}$, $t = 0, \tau = t_f$

determine $\underline{u(t) = -R(t)^{-1} B(t)^T P(t)\, x(t)} = -K(t)\, x(t)$

$K(t)$ = called Kalman gain

Second order optimality conditions

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \left(\frac{\partial^2 H}{\partial x \partial u}\right)^T & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} = \begin{bmatrix} Q(t) & 0 \\ 0 & R(t) \end{bmatrix} \succeq 0,$$

more importantly, $\frac{\partial^2 H}{\partial u^2} = R(t) \succ 0$

$\Rightarrow$ $u(t)$ is the optimal input.

What is the optimal value of $-(1)$?

(HW) If $u(t) = -K(t) x(t)$, the cost incurred from time t_0 to t_f can be written as $\frac{1}{2} x(t_0)^T \underline{S(t_0)} x(t_0)$.

Proposition: $J^* = \frac{1}{2} \underline{x(t_0)^T} \, \underline{P(t_0)} \, \underline{x(t_0)}$

Proof: Note that

$$\frac{1}{2} \int_{t_0}^{t_f} \frac{d}{dt}\left(x(t)^T P(t) x(t)\right) dt = \frac{1}{2} x(t_f)^T P(t_f) x(t_f) - \frac{1}{2} x(t_0)^T P(t_0) x(t_0)$$

Let us now look at the cost function.

$$J = \frac{1}{2} x(t_f)^T P(t_f) x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[x(t)^T Q(t) x(t) + u(t)^T R(t) u(t) \right] dt$$

$$= \frac{1}{2} x(t_0)^T P(t_0) x(t_0) + \frac{1}{2} \int_{t_0}^{t_f} \left[x(t)^T Q(t) x(t) + u(t)^T R(t) u(t) + \frac{d}{dt}\left(x(t)^T P(t) x(t)\right) \right] dt$$

Let us analyze the quantity inside integral and omit the dependence on t. substituting optimal input $u^* = -R^{-1}B^T P x$, yields

$$x^T Q x + \underline{u}^T R u + \dot{x}^T P x + x^T \dot{P} x + x^T P \dot{x}$$

$$= x^T Q x + x^T P B (R^{-1})^T R^{-1} R R^{-1} B^T P x + (Ax - BR^{-1}B^T P x)^T P x$$
$$+ x^T \dot{P} x + x^T P (Ax - BR^{-1}B^T P x)$$

$$= x^T \Big[Q + PBR^{-1}B^T P + A^T P - PBR^{-1}B^T P + \dot{P} + PA$$
$$- PBR^{-1}B^T P \Big] x$$

$$= x^T \left[Q + A^T P + PA + \underline{\dot{P}} - PBR^{-1} B^T P \right] x$$

$= 0$ when $P(t)$ is solution of differential riccati equation.

This concludes the proof.

Note: 1) The above analysis is carried out at the optimal solution where

$$u^*(t) = -R(t)^{-1} B(t)^T P(t) x(t)$$

and $P(t)$ satisfies Diff. Riccati eq^n.

2) $P(t)$ is a symmetric matrix, since $P(t_f) = F(t_f)$ assumed to be symmetric,

and the R.H.S. of $\dot{P}(t)$ equation is symmetric.

3) $P(t)$ is positive semi-definite for $t \in [t_0, t_f)$.

note that at any t, optimal cost of the subproblem from t to t_f

is given by $J^*(x(t)) = \frac{1}{2} x(t)^T P(t) x(t)$.

If $P(t)$ is not positive semi definite, $\exists\ \bar{x}$ s.t. if $x(t) = \bar{x}$,

then optimal cost $J^* = \frac{1}{2} \bar{x}^T P \bar{x} < 0$.

which is not possible since

$$J(x(t)) = \frac{1}{2} x(t_f)^T \underbrace{F(t_f)}_{\geq 0} x(t_f) + \frac{1}{2}\int_0^\infty \left(x(t)^T \underbrace{Q(t)}_{\geq 0} x(t) + u(t)^T \underbrace{R(t)}_{> 0} u(t) \right) dt$$

≥ 0.

(4) <u>optimal value</u> is always finite, even when the system is unstable, not controllable, etc.

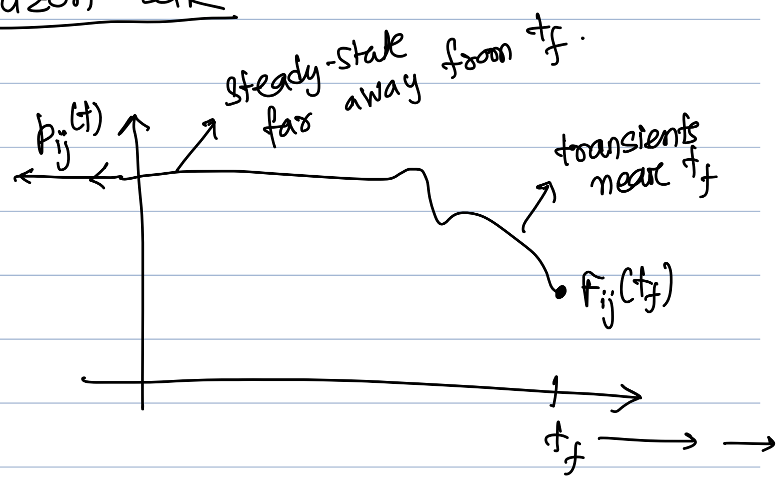
<u>Infinite-horizon LQR</u>

$P(0)$ = backward integration of $\dot{P}(t)$ from ∞ to 0.

$\tau = t_f - t$, $\tau \to \infty$

$\Leftrightarrow t_f \to \infty$

$t \to 0$



we do not consider the terminal cost, i.e., $F(t_f) = 0$

cost function $J = \frac{1}{2} \int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t) \right] dt$.

we mostly consider time-invariant dynamics and cost function,

i.e., $A(t)$, $B(t)$, $Q(t)$, $R(t)$ are constants, do not vary with time.

To solve for optimal input, consider the differential riccati eqn:

$$\dot{P}(t) = -Q - A^T P(t) - P(t) A + P(t) B R^{-1} B^T P(t).$$

$P(t)$ reaches steady-state when $\dot{P}(t) = 0$, which implies

$$\boxed{Q + A^T \bar{P} + \bar{P} A - \bar{P} B R^{-1} B^T \bar{P} = 0} \rightarrow \text{Algebraic Riccati eqn (ARE).}$$

Theorem:

(1) Let (A, B) be stabilizable. Then, for any $P(T)$, $\exists$ exists a limiting solution $\bar{P}$ of the differential Riccati eqn as $T \to \infty$, and $\bar{P}$ is also a soln of ARE with $\bar{P} \succeq 0$.

Note: If Differential Riccati eqn converges to some $\bar{P}$,

$$u^* = -\underbrace{R^{-1} B^T \bar{P}}\, x(t)$$

$\hookrightarrow$ does not depend on time t

closed-loop system $\dot{x} = \underline{(A - BR^{-1}B^T \bar{P}) x}$ $\hookrightarrow$ autonomous system

if (A, B) not stabilizable, even if P converges to $\bar{P}$, $(A - BR^{-1}B^T\bar{P})$ is not Hurwitz.

optimal cost: $J^* = \frac{1}{2} x(0)^T \bar{P} x(0)$ is finite.

however $\int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt \to \infty$ as $x(t) \to \infty$.

which is a contradiction.

(2) Let $(A, \sqrt{Q})$ be detectable. Then

(A, B) is stabilizable $\Leftrightarrow$ $\exists$ unique positive semi-definite solution $\bar{P}$ of the ARE, and $(A - BR^{-1}B^T\bar{P})$ is Hurwitz.

i.e., closed-loop system is asymptotically stable.

Let $y(t) = \sqrt{Q}\, x(t)$

then $x(t)^T Q x(t) = y(t)^T y(t)$

cost function $\int_0^{\infty} \left(y(t)^T y(t) + u(t)^T R\, u(t) \right) dt.$

$(A, \sqrt{Q})$ detectable ensures that all unstable modes are reflected in the output and penalized.

(3) If $(A, \sqrt{Q})$ is observable, then

(A, B) stabilizable $\Leftrightarrow$ $\exists$ unique positive definite sol^n $\bar{P}$ of the ARE, and $(A - BR^{-1}B^T P)$ is Hurwitz.

Lectures 27 & 28: 10th March

Linear Quadratic Tracking Problem

$$\min_{x(t),u(t)} \frac{1}{2}\left(C(t_f)x(t_f) - z(t_f)\right)^T F(t_f)\left(C(t_f)x(t_f) - z(t_f)\right)$$

$$+ \frac{1}{2}\int_0^{\infty} \left[\left(C(t)x(t) - z(t)\right)^T Q(t)\left(C(t)x(t) - z(t)\right) + u(t)^T R(t) u(t)\right] dt$$

s.t. $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

$t_f, t_0, x(t_0)$: given

$x(t_f)$: free.

$$H(x,u,\lambda,t) = \frac{1}{2}\left(C(t)x(t) - z(t)\right)^T Q(t)\left(C(t)x(t) - z(t)\right) + \underbrace{\frac{1}{2}u(t)^T R(t) u(t)}$$

$$+ \underbrace{\lambda(t)^T \left[A(t)x(t) + B(t)u(t)\right]}.$$

$$\frac{1}{2}x^T C^T Q C x - \frac{1}{2}\underline{z^T Q C x} - \frac{1}{2}\underline{x^T C^T Q z} = \frac{1}{2}x^T C^T Q C x - z^T Q C x$$

scalars

optimality conditions

$$\frac{\partial H}{\partial u} = 0 \Rightarrow R(t)u(t) = B(t)^T \lambda(t)$$

$$\Rightarrow u^*(t) = -R(t)^{-1} B(t)^T \lambda^*(t)$$

$$\dot{\lambda}(t) = -\left(\frac{\partial H}{\partial x}\right)^T = -A(t)^T \lambda(t) - C(t)^T Q(t) C(t) x(t) + C(t)^T Q(t) z(t)$$

$$\dot{x}(t) = A(t)x(t) - B(t)R(t)^{-1}B(t)^T \lambda(t)$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^T Q C & -A^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ C^T Q \end{bmatrix} z$$

↳ all terms above are functions of time.

Boundary condition: $$\lambda(t_f) = \frac{\partial}{\partial x}\left[\frac{1}{2}(Cx - z)^T F(t_f)(Cx - z)\right]\Bigg|_{t_f}$$

$$= C(t_f)^T F(t_f) C(t_f)\, x(t_f) - C(t_f)^T F(t_f)\, z(t_f)$$

We hypothesize at the optimal solution,

$$\lambda(t) = P(t)x(t) - g(t),$$

$$P(t_f) = C(t_f)^T F(t_f) C(t_f)$$
$$g(t_f) = -C(t_f)^T F(t_f)\, z(t_f).$$

As before, $$\dot\lambda(t) = \dot P(t)x(t) + P(t)\dot x(t) - \dot g(t)$$
$$= -C(t)^T Q(t) C(t) x(t) - A(t)^T \lambda(t) + C(t)^T Q(t) z(t)$$

Let us drop (t) in the following analysis for better readability.

$$\dot P x + P\left[Ax - BR^{-1}B^T\lambda\right] - \dot g = -C^TQCx - A^T\lambda + C^TQz$$

$$\Rightarrow \dot P x + PAx - PBR^{-1}B^T(Px - g) - \dot g = -C^TQCx - A^T[Px - g] + C^TQz$$

$$\Rightarrow \left[\dot P + PA - PBR^{-1}B^TP + C^TQC + A^TP\right]x + \left[PBR^{-1}B^Tg - \dot g - A^Tg - C^TQz\right] = 0$$

$\to 0$ in infinite horizon LTI case

$$\Rightarrow \left.\begin{aligned} &\cancel{\dot P} + PA + A^TP + C^TQC - PBR^{-1}B^TP = 0 \\ &\dot g = PBR^{-1}B^Tg - A^Tg - C^TQz \end{aligned}\right\}$$

$\rightarrow \frac{n(n+1)}{2}$ number of ODEs

$\rightarrow n$ number of ODEs.

Optimal input: $u^*(t) = -R(t)^{-1}B(t)^T \left[P(t)x(t) - g(t)\right]$

$$u^*(t) = -\underbrace{K(t)}_{\text{same as that of the regulation problem.}}x(t) + R(t)^{-1}B(t)^T g(t)$$

Optimal value of cost function

$$J^*(x(t_0)) = \frac{1}{2}x(t_0)^T P(t_0)x(t_0) - x(t_0)^T g(t_0) + h(t_0)$$

where
$$\dot{h}(t) = -\frac{1}{2}g(t)^T B(t)R(t)^{-1}B(t)^T g(t) - \frac{1}{2}z(t)^T Q(t)z(t),$$

$$h(t_f) = -z(t_f)^T P(t_f)z(t_f).$$

please try to show this.

Heuristic to choose Q and R matrices

Let x_i^{max} and u_i^{max} be the largest acceptable value of x_i and u_i, respectively.

$$Q = \begin{bmatrix} \frac{\alpha_1^2}{(x_1^{max})^2} & & 0 \\ & \frac{\alpha_2^2}{(x_2^{max})^2} & \\ & & \ddots \\ 0 & & & \frac{\alpha_n^2}{(x_n^{max})^2} \end{bmatrix}$$

$\sum_i \alpha_i^2 = 1$

$\sum_i \beta_i^2 = 1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \frac{1}{2}x_1^2 + \frac{100}{2}x_2^2$

penalizes x_2 more than x_1

diagonal matrix

$\Rightarrow (A, \sqrt{Q})$ is observable.

$$R = \rho \begin{bmatrix} \frac{\beta_1^2}{(u_1^{max})^2} & & 0 \\ & \ddots & \\ 0 & & \frac{\beta_m^2}{(u_m^{max})^2} \end{bmatrix},$$

ρ large $\Rightarrow$ priority is to save control energy.

Continuous-time optimal control using Dynamic Programming

Consider the CT-OPT problem

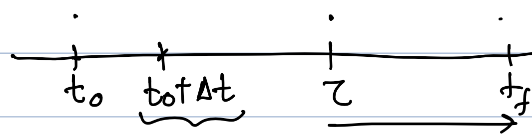
$$\min_{u(t),x(t)} J(x(t_f),t_0) = \underline{S(x(t_f),t_f)} + \int_{t_0}^{t_f} V(x(t),u(t),t)\,dt$$

s.t. $\dot{x}(t) = f(x(t),u(t),t)$

$t_f, t_0, x(t_0)$: given

$x(t_f)$: free

Suppose optimal value of this problem is $\underline{J^*(x(t_f),t_0)}$.



using dynamic programming principle, we can say

$$J^*(x(t_0),t_0) = \min_{\substack{u(\tau)\\ t_0\le\tau\le t_0+\Delta t}} \left[\int_{t_0}^{t_0+\Delta t} V(x(t),u(t),t)\,dt + \underbrace{J^*(x(t_0+\Delta t),t_0+\Delta t)} \right] \quad \text{— (DP)}$$

<u>Assumption</u>: $J^*(x(t),t)$ has bounded second derivatives in both arguments.

Then using Taylor series expansion, we obtain

$$\underset{\downarrow\,\text{scalar}}{J^*(x(t_0+\Delta t),t_0+\Delta t)} = \underset{\downarrow\,\text{scalar}}{J^*(x(t_0),t_0)} + \overset{\text{scalar}}{\frac{\partial J^*}{\partial t}(x(t_0),t_0)}\cdot\Delta t$$

$$+ \underbrace{\frac{\partial J^*}{\partial x}(x(t_0),t_0)}_{\text{row vector}} \cdot \underbrace{\frac{dx}{dt}(x(t_0),t_0)}_{\text{column vector}} \cdot \Delta t$$

$$\Rightarrow J^*(x(t_0+\Delta t), t_0+\Delta t) \cong J^*(x(t_0), t_0) + \frac{\partial J^*}{\partial t}(x(t_0), t_0)\,\Delta t + \frac{\partial J^*}{\partial x}(x(t_0), t_0) . f(x(t_0), u(t_0), t) . \Delta t$$

Let us substitute the above in DP eq^n. when Δt sufficiently small.

$$J^*(x(t_0), t_0) \cong \min_{\substack{u(\tau)\\ t_0 \le \tau \le t_0+\Delta t}} \left[\int_{t_0}^{t_0+\Delta t} V(x(t), u(t), t)\, dt + J^*(x(t_0), t_0) + \frac{\partial J^*}{\partial t}\Big|_{t_0} \Delta t + \frac{\partial J^*}{\partial x}\Big|_{t_0} . f(x(t_0), u(t_0), t) . \Delta t \right]$$

$$\Rightarrow 0 = \min_{\substack{u(\tau)\\ t_0 \le \tau \le t_0+\Delta t}} \left[\underline{V(x(t_0), u(t_0), t_0)} . \Delta t + \frac{\partial J^*}{\partial t}\Big|_{t_0} \Delta t + \frac{\partial J^*}{\partial x}\Big|_{t_0} \underline{f(x(t_0), u(t_0), t_0)} \Delta t \right]$$

$$0 = \min_{u(t_0)} \left[V(x(t_0), u(t_0), t_0) + \frac{\partial J^*}{\partial t}(x(t_0), t_0) + \frac{\partial J^*}{\partial x}(x(t_0), t_0) . f(x(t_0), u(t_0), t_0) \right]$$

HJB eq^n (Hamiltonian-Jacobi-Bellman)

$-J_t^* = \min_u \left[V + J_x^* f \right]$ → short-form.

Note: If we find J^* satisfying the above equation at all times, then we have solved the problem. optimal input at any time t_0 is the solution of the above problem.

→ decision variable $u(t_0) \in \mathbb{R}^m$, not a signal or function of time.

→ solved pointwise.

Boundary condition: $J^*(x(t_f), t_f) = S(x(t_f), t_f)$.

Note: (1) HJB equation is both necessary & sufficient for optimality.

(2) In general, HJB eqn is a PDE.

Let us now specialize the above theory to the linear quadratic regulator setting.

$$f(x(t), u(t), t) = A(t)x(t) + B(t)u(t)$$

$$V(x(t), u(t), t) = \frac{1}{2} x(t)^T Q(t) x(t) + \frac{1}{2} u(t)^T R(t) u(t)$$

$$S(x(t_f), t_f) = \frac{1}{2} x(t_f)^T \underline{F(t_f)} x(t_f).$$

can we make some hypothesis about J^* ?

Suppose $J^*(x(t), t) = \frac{1}{2} x(t)^T P(t) x(t)$ has this form.

for some symmetric matrix $P(t)$

$$\frac{\partial J^*}{\partial t}(x(t), t) = \frac{1}{2} x(t)^T \dot{P}(t) x(t)$$

$$\frac{\partial J^*}{\partial x}(x(t), t) = x(t)^T P(t)$$

HJB eqn

$$0 = \min_{u(t)} \Big[\frac{1}{2} x(t)^T Q(t) x(t) + \frac{1}{2} u(t)^T R(t) u(t) + \frac{1}{2} x(t)^T \dot{P}(t) x(t) + x(t)^T P(t) \big(A(t) x(t) + B(t) u(t)\big) \Big]$$

optimal input: $R(t) u(t) + B(t)^T P(t) x(t) = 0$

$$\Rightarrow u^*(t) = -R(t)^{-1} B(t)^T P(t) x(t)$$

minimizer when $R(t)$ is positive definite.

substituting $u^*(t)$, we obtain

$$0 = \frac{1}{2} x(t)^T Q(t) x(t) + \frac{1}{2} x(t)^T \left[P(t) B(t) \cancel{R(t)^{-1}} \right] \cancel{R(t)} \left[R(t)^{-1} B(t)^T P(t) \right] x(t)$$

$$+ \frac{1}{2} x(t)^T \dot{P}(t) x(t) + x(t)^T P(t) A(t) x(t)$$

$$+ x(t)^T P(t) B(t) \left(-R(t)^{-1} B(t)^T P(t) \right) x(t)$$

$$x^T P A x = \frac{1}{2} x^T \left(PA + A^T P \right) x$$

For any matrix M, we have

$$x^T M x = \frac{1}{2} \left(x^T M x + x^T M^T x \right)$$

$$\Rightarrow 0 = \frac{1}{2} x(t)^T \Big[\underbrace{Q(t) + P(t) A(t) + A(t)^T P(t) + \dot{P}(t) - P(t) B(t) R(t)^{-1} B(t)^T P(t)}_{\text{differential Riccati eq}^n \text{ we derived earlier}} \Big] x(t)$$

$$\forall \, x(t)$$

$\Rightarrow$ if $P(t)$ is sol^n of Riccati eq^n, then HJB equation is satisfied.

Lecture 29 : 16th March

Minimum-time optimal control with input constraints

$$\min_{u(t),\, t_f} \quad t_f = t_f - t_0 = \int_{t_0}^{t_f} 1 \cdot dt$$

s.t.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$t_0,\ x(t_0) = x_0 \ : \text{Known}$$

$$x(t_f) = 0 \ : \text{Known.}$$

$$u^{min} \leq u(t) \leq u^{max} \quad \forall t$$

you could also have $x(t_f) \in \mathcal{X}$

: if we don't specify final value of state, t_f will end up being equal to t_0

optimality conditions

$$H(x, u, p) = 1 + p^T[Ax + Bu]$$

optimality conditions: It can be shown that

$$u^*(t) = \underset{u^{min} \leq u(t) \leq u^{max}}{\operatorname{argmin}} \ H(x^*(t), u(t), p^*(t), t)$$

$$= \underset{u^{min} \leq u(t) \leq u^{max}}{\operatorname{argmin}} \left[1 + p^*(t)^T A x^*(t) + p^*(t)^T B u(t)\right]$$

$$= 1 + p^*(t)^T A x^*(t) + \underset{u^{min} \leq u(t) \leq u^{max}}{\operatorname{argmin}} \ \underbrace{p^*(t)^T B\, u(t)}_{q(t)^T u(t)}$$

$$u_i^*(t) = \begin{cases} u_i^{min}, & [p^*(t)^T B]_i > 0 \\ u_i^{max}, & [p^*(t) B]_i < 0 \\ \text{indeterminate}, & [p^*(t) B]_i = 0 \end{cases}$$

→ Bang-bang control

→ singularity.

define <u>switching function</u> $\underline{q(t) = B^T p^*(t)}$

<u>other optimality conditions</u>: $\dot{x}^*(t) = Ax^*(t) + Bu^*(t)$

$$\underline{\dot{p}^*(t) = -A^T p^*(t)} \Rightarrow \underline{p^*(t) = e^{-A^T t} p^*(0)}$$

+ some boundary conditions. autonomous linear system.

<u>Note</u>: Singularity arises whenever Hamiltonian is an affine function of $u(t)$, not just in the specific problem given here.

Basically, if stage cost is affine in $u(t)$, and dynamics is control-affine, i.e.,

$$\dot{x}(t) = f(x(t)) + g(x(t))\, u(t), \text{ with}$$

f and g being possibly nonlinear.

We will deal with singular inputs in more generality later.

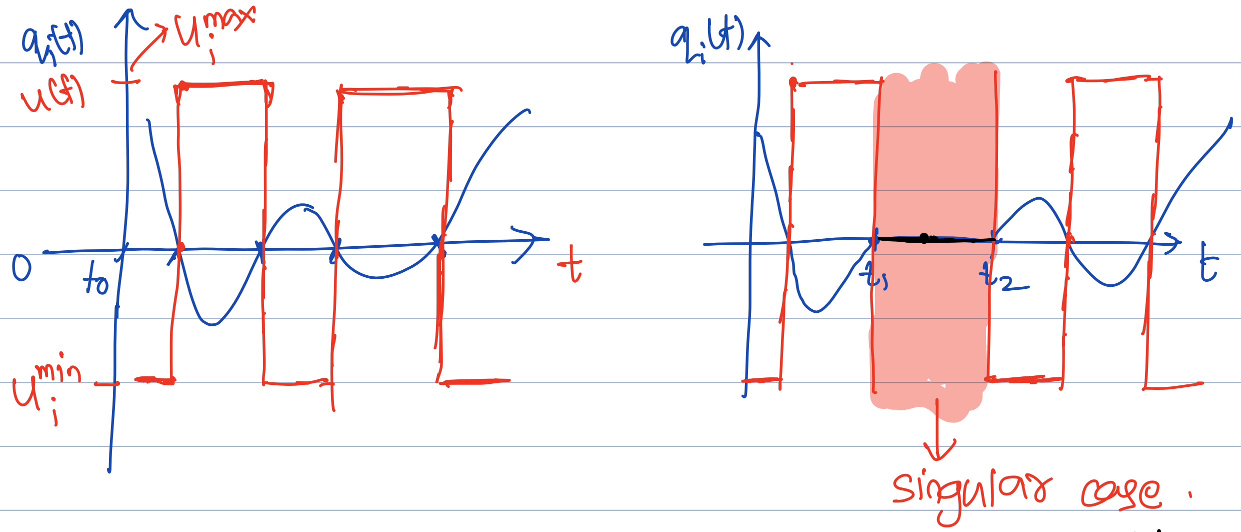
For now, let us focus on the minimum-time optimal control problem.

There are two cases that arise.

(i) <u>Normal case</u>: components of switching function take value zero at finitely many time points.

(ii) <u>Singular case</u>:

$\exists\ [t_1, t_2]$ s.t. $q_i(t) = 0 \quad \forall\, t \in [t_1, t_2]$.



we will now derive conditions that rule out possibility of singularity.

observe that when $q_i(t) = 0 \quad \forall t \in [t_1, t_2]$,

$$\left.\begin{aligned} \Rightarrow \dot{q}_i(t) &= 0 \\ \ddot{q}_i(t) &= 0 \\ &\vdots \\ q_i^{n-1}(t) &= 0 \end{aligned}\right\} \quad \forall t \in (t_1, t_2)$$

Assume $p^*(0) \neq 0 \Rightarrow p^*(t) = e^{-A^T t} p^*(0)$

else $p(t) = 0 \ \forall t$ which means $q(t) = 0 \ \forall t$ $\Rightarrow$ problem is not well-posed.

$$q(t) = B^T p^*(t), \qquad B \in \mathbb{R}^{n \times m}, \quad q(t) \in \mathbb{R}^m$$

$$q_i(t) = b_i^T p^*(t), \qquad b_i : \text{i-th column of } B.$$

$$\dot{q}_i(t) = b_i^T \dot{p}^*(t)$$

$$= b_i^T \frac{d}{dt}\left(e^{-A^T t} p^*(0)\right)$$

$$= b_i^T (-A^T) \, p^*(t)$$

$$\ddot{q}_i(t) = b_i^T (-A^T)^2 p^*(t)$$

Singularity requires:

$$\begin{cases} q_i(t) = \underline{b_i^T\, p^+(t)} = 0 = (p^+(t))^T b_i \\ \dot{q}_i(t) = b_i^T A^T\, p^+(t) = 0 = (p^+(t))^T A b_i \\ \ddot{q}_i(t) = b_i^T (A^T)^2\, p^+(t) = 0 \quad , \\ \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad , \\ q_i^{n-1}(t) = b_i^T (A^T)^{n-1} p^+(t) = 0 \quad , \end{cases}$$

Stacking the above togethers, we get.

$$\left[b_i \;\vdots\; Ab_i \;\vdots\; A^2 b_i \;\; - - \;\vdots\; A^{n-1} b_i \right] \in \mathbb{R}^{n \times n}$$

Stacking all columns of B, we get

$$\mathcal{C} = \left[B \;\vdots\; AB \;\vdots\; A^2 B \;\vdots\; --- \;\vdots\; A^{n-1} B \right] \in \mathbb{R}^{n \times mn}$$

which is nothing but controllability matrix.

<u>Proposition</u>: If (A,B) is controllable, i.e., $\operatorname{rank}(\mathcal{C}) = n$, then singularity is not possible and vice versa.

<u>proof</u>: If $\mathcal{C}$ has full row-rank, there does not exist any $p(t) \neq 0$ s.t. $p(t)^T \mathcal{C} = 0$

$\Rightarrow$ $q_i^{(j)}(t) = 0$ cannot simultaneously hold for $j = 0, 1, 2 \ldots n-1$ for any i.

$\Rightarrow$ we donot have singularity.

Lecture 30: 16th March

A typical approach to solve the above problem is the following.

(i) guess $p(t_0)$, let $t_0 = 0$

(ii) solve for $p(t) = e^{-A^T t} p(0)$

(iii) find $u^*(t)$ using above analysis

(iv) solve for $x(t)$ starting from $x(0)$ & applying $u^*(t)$

(v) Check if $x(t_f) = 0$

else repeat.

another difficulty is with $p(t) = e^{-A^T t} p(0)$

if A is stable, i.e., $Re(\lambda_i(A)) < 0$,

then $-A^T$ has all eigenvalues on ORHP.

+ve real parts.

So, instead of following the above recipe, some feedback scheme is often desired, but it is difficult to do so except for special cases.

Idea is to express $q(t)$ as a function of $x(t)$.

usually a complex nonlinear function in contrast with the LQR case where $p(t) = P(t)x(t)$

Minimum fuel optimal control

In systems where motion is involved, amount of fuel spent is proportional to magnitude of input.

First let us consider a simpler setting with the dynamics being a double integrator system.

$\ddot{x}(t) = \frac{f(t)}{m}$: input

$= u(t)$

$x_1(t)$: position

$\dot{x}_1(t) = x_2(t)$: velocity

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \end{aligned}\right\} \Rightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$\mathcal{C} = [B \,\vdots\, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\text{rank}[\mathcal{C}] = 2$

$\Rightarrow (A, B)$ is controllable.

Minimum fuel optimal control problem is

$$\min_{u(t)} \int_{t_0}^{t_f} |u(t)| \, dt$$

s.t. $\dot{x}_1(t) = x_2(t)$.

$\dot{x}_2(t) = u(t)$.

$-1 \leq u(t) \leq 1$ (or $|u(t)| \leq 1$)

$x(t_0), t_0, t_f$: given

$x(t_f)$: typically free

To derive the optimal input, define the Hamiltonian as

$$H(x,u,p,t) = |u(t)| + \underbrace{p_1(t)\,x_2(t)} + p_2(t)\,u(t)$$

$$u^*(t) = \underset{|u(t)|\le 1}{\text{argmin}} \left[\underline{\underline{|u(t)|}} + \underline{p_2(t)\,u(t)} \right]$$

(Rough)

$$\min_{|x|\le 1} |x| - x = \min_{|x|\le 1} (|x| + x)$$

$\Rightarrow$ optimal value $= 0$

optimal sol^n : $x \in [-1, 0]$

$$\min_{|x|\le 1} |x| + \underline{\underline{10}}x \Rightarrow$$

$x = -0.5$, $0.5 + 10(-0.5) < 0$

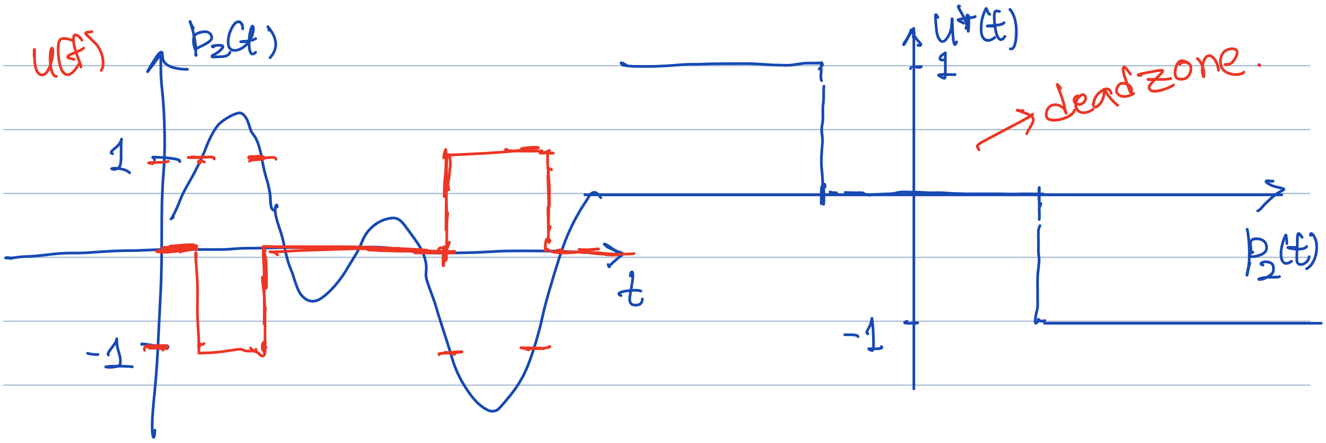
$$\min_{|x|\le 1} \left[\underline{|x|} - \underline{0.3x} \right] = \begin{cases} 0.7x & x \geq 0 \\ -1.3x & x < 0 \\ \quad = 1.3|x| & \end{cases}$$

we obtain the following:

$$u^*(t) = \begin{cases} -1 & \text{if } p_2(t) > 1 \\ [-1, 0] & \text{if } p_2(t) = 1 \quad : \text{not unique} \\ 0 & \text{if } -1 < p_2(t) < 1 \\ [0, 1] & \text{if } p_2(t) = -1 \\ 1 & \text{if } p_2(t) < -1 \end{cases}$$

if $\underline{p_2(t) = 1}$ for $t \in [t_1, t_2]$, we have singularity.

we can not determine $u^*(t)$ uniquely.



$$\text{dez}(x) = \begin{cases} 1 & \text{if} \quad x > 1 \\ 0 & \text{if} \quad x \in [-1, 1] \\ -1 & \text{if} \quad x < -1 \end{cases}$$

then $\underline{u^*(t) = -\text{dez}(p_2^*(t))}$.

Once again, we can try to find conditions that rule out singularity.

please do the derivation for $\dot{x} = Ax + Bu$, $B \in \mathbb{R}^{n \times 1}$, <u>single input</u>.

singularity means all derivatives of switching function are simultaneously zero for $t \in [t_1, t_2)$.

$$H(x, u, p, t) = |u(t)| + p(t)^T [Ax + Bu]$$

$$u^*(t) = \underset{|u(t)| \le 1}{\text{argmin}} \left[|u(t)| + p(t)^T B \, u(t) \right]$$

$$= \underset{|u(t)| \le 1}{\text{argmin}} \left[|u(t)| + q(t)^T u(t) \right]$$

Switching function $q(t) = B^T p(t)$ scalar in this case.

$$\boxed{u^*(t) = -\mathrm{dez}(q(t))}$$, with singularity when $q(t) = 1$ or $q(t) = -1$

From the Hamiltonian, $\dot{p}(t) = -A^T p(t)$

$$\Rightarrow p(t) = e^{-A^T t} p(0)$$

$$q(t) = B^T e^{-A^T t} p(0) \in \mathbb{R} \quad \text{(scalar)}$$

$$\left.\begin{aligned} \dot{q}(t) &= (-)B^T A^T e^{-A^T t} p(0) = 0 \\ \ddot{q}(t) &= (+)B^T (A^T)^2 e^{-A^T t} p(0) = 0 \\ &\vdots \\ q^{(n)}(t) &= (+)B^T (A^T)^n e^{-A^T t} p(0) = 0 \end{aligned}\right\} \text{simultaneously.}$$

$\left[B^T A^T \quad B^T (A^T)^2 \quad \dots \quad B^T (A^T)^n\right]$ should have rank $< n$

$\Rightarrow \left[B^T \quad B^T A^T \quad \dots \quad B^T (A^T)^{n-1}\right] . A^T$ "

$\Rightarrow \; \mathcal{C}^T . A^T$ should have rank $< n$.

Theorem : Singularity is not possible if & only if (A, B) is controllable and A is not invertible.