Lectures 20: 18th Feb.

Module 3: Optimal control of continuous-time systems.

Given a CT dynamical system $\dot{x}(t) = f(x(t), u(t), t)$, the goal is to find an input signal $u(t)$ for $t \in [t_0, T]$ such that the resulting state trajectory $x(t)$, $t \in [t_0, T]$ together with the input signal $u(t)$ minimize the following functional:

$$\underline{\underline{J}}\left(x(t), u(t); t \in [t_0, T]\right) = \phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)\, dt .$$

Note: i) for a given time t, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

ii) J is a functional, i.e., it is a function of functions.
input to J are functions $x(t)$, $u(t)$, $t \in [t_0, T]$
output of J is a real-number.

The above problem can be written as

$$\min_{\substack{x(t), u(t), \\ t \in [t_0, T]}} \quad \phi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)\, dt$$

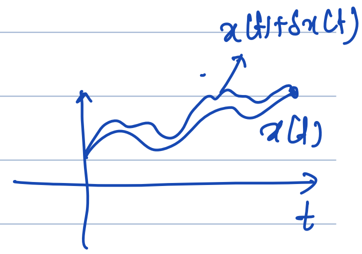
$$\text{s.t.} \quad \begin{aligned} &\dot{x}(t) = f(x(t), u(t), t) \\ &\psi(x(T), t) = 0 \\ &x(t_0) = x_0 \text{ is known.} \end{aligned} \qquad \text{(CT-OPT)}$$

decision variables: $x(t)$, $t \in [t_0, T]$
$u(t)$, $t \in [t_0, T]$

In order to analyze optimal solutions of the above problem, we need to define quantities for <u>functionals</u> that are analogous to notions of derivative/gradient, Hessian, etc for <u>functions</u>.

$$f(x(t)) = 2x^2(t) + 1$$
$$\partial f / \partial x = \underline{4x(t)}$$

<u>Ex</u>: $\quad J_1(x(t)) = \int_{t_0}^{T} [2x^2(t) + 1]\, dt$



consider a perturbed signal $x(t) + \delta x(t)$

$$\boxed{\underline{\text{Increment}} \quad \Delta J(x(t), \delta x(t)) := J[x(t) + \delta x(t)] - J(x(t))}$$

$$\simeq \underline{\delta J(x(t), \delta x(t))} \quad \text{for } \underline{\delta x(t)} \text{ sufficiently small.}$$

$(\delta J)(\delta x(t))$

Recall: (taylor series)

$$\underline{f(x+\Delta x)} \simeq \underline{f(x)} + \underline{\nabla f(x)^T \Delta x} + \frac{1}{2} \Delta x^T H(x) \Delta x + \text{h.o.t.}$$

$\underline{\Delta J_1(x(t), \delta x(t))}$

$$= \int_{t_0}^{T} [2(x(t) + \delta x(t))^2 + 1] - [2x(t)^2 + 1]\, dt$$

$$= \int_{t_0}^{T} [\underbrace{4x(t)\,\delta x(t)}_{\text{linear in } \delta x(t)} + \underbrace{2\,\delta x(t)^2}_{\text{quadratic.}}]\, dt$$

if $x(t) \in \mathbb{R}^n$

$\delta x(t) \in \mathbb{R}^n$

$\underline{(\ )(\delta x(t))_1} + \underline{(\cdot)(\delta x(t))_2}$

(First) variation of J at $x(t)$ is $\quad \dfrac{\partial f}{\partial x}$

$$\delta J(x(t), \delta x(t)) = \int_{t_0}^{T} \underline{4x(t)\,\delta x(t)}\, dt$$

$$\delta J = \int \underline{D(x)^T \delta x(t)\, dt}$$

$$\underline{Dx(t)^T \delta x(t)\, dt}$$

<u>second variation</u> $\quad \delta^2 J(x(t), \delta x(t)) = \int_{t_0}^{T} 2\,\delta x(t)^2\, dt$

Norm: Given a vector space $\mathcal{V}$, norm $\|\cdot\|: \mathcal{V} \to \mathbb{R}$ s.t.

i) $\|x\| \geq 0 \quad \forall x \in \mathcal{V}$, and $\|x\| = 0 \Leftrightarrow x = 0$

ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for any $\alpha \in \mathbb{R}$, $x \in \mathcal{V}$

iii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$, $\forall x_1, x_2 \in \mathcal{V}$.

If $\mathcal{V}$ is the space of functions defined over $[0,T]$,

$$\|x(t)\|_2 = \left[\int_0^T x(t)^T x(t) \, dt \right]^{\frac{1}{2}}.$$

$$\|x(t)\|_1 = \int_0^T \|x(t)\|_1 \, dt \quad \text{, and so on.}$$

zero element $x(t) = 0 \quad \forall t \in [0,T]$

Local minimum: $x^*(t)$ is a local minimizer of the functional $J(x(t))$ if $J(x(t)) \geq J(x^*(t)) \quad \forall x(t)$ s.t. $\|x(t) - x^*(t)\|_2 \leq \varepsilon$ for some $\varepsilon > 0$.

or there does not exist any signal $x(t)$ within distance ε from $x^*(t)$ which has a strictly smaller functional value.

Fundamental Theorem of Calculus of Variations (analogous FONC)

Let $J(x(t))$ be a differentiable functional of $x(t)$. If $x^*(t)$ is an extremal function, then

$\delta J(x^*(t), \delta x(t)) = 0$ for all admissible $\delta x(t)$.

Sufficient condition for x^* to be a minimizer is $\delta^2 J(x^*) > 0$

sufficient condition for x^* to be a maximizer is $\delta^2 J(x^*) < 0$.

Leibniz Rule: Consider a functional $J(x(t)) = \int_{t_0}^{T} f(x(t))\,dt$.

The first variation of J is given by:

$$\delta J(\bar{x}(t), \delta \bar{x}(t)) = \int_{t_0}^{T} \left[\frac{\partial f}{\partial x}(\bar{x}(t)) \cdot \delta \bar{x}(t) \right] dt + f(\bar{x}(T))\,dT - f(\bar{x}(t_0))\,dt_0$$

Recall differentiation under integral sign.

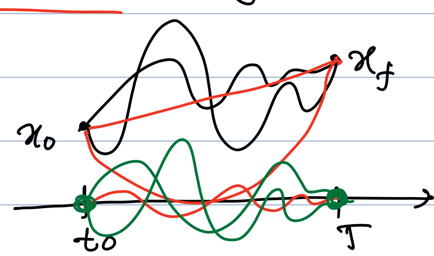
Basic Variational Problem

(analogous to unconstrained optimization, precursor to solving CT-OPT)

Find continuously differentiable $x(t)$, $t \in [t_0, T]$ to minimize

$$J(x(t)) = \int_{t_0}^{T} V(x(t), \dot{x}(t), t)\,dt,$$

with $t_0, T, x(t_0) = x_0$ and $x(T) = x_f$ are given/fixed.



Let $x^*(t)$ be a candidate minimizer.

Then

$\delta J(x^*(t), \delta x(t)) = 0$

$\forall\ \delta x(t)$ s.t. $x^*(t) + \delta x(t)$ must satisfy the boundary conditions.

$\Rightarrow \delta x(t) = 0$ at $t = t_0$ and $t = T$.

Lecture 21 & 22: 2nd March

Let us compute first variation using Leibniz rule.

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{T} \left[\frac{\partial V}{\partial x}(x^*(t), \dot{x}^*(t), t)\, \delta x(t) + \frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\, \delta \dot{x}(t)\right] dt$$

$$+ \underline{V(x^*(T), \dot{x}^*(T), T)}\, dT - V(x^*(t_0), \dot{x}^*(t_0), t_0)\, dt_0$$

Since T and t_0 are fixed, $dT = dt_0 = 0$ for this problem.

$$\Rightarrow \delta J(x^*(t), \delta x(t)) = \int_{t_0}^{T} \left[\frac{\partial V}{\partial x}(x^*(t), \dot{x}^*(t), t)\, \underline{\delta x(t)} + \underbrace{\frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\, \delta \dot{x}(t)}_{\text{apply integration by parts.}}\right] dt$$

Note that $\delta \dot{x}(t)$ is not independent of $\delta x(t)$.

$$\int_{t_0}^{T} \frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\, \underline{\delta \dot{x}(t)}\, dt = \frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\, \delta x(t) \Big|_{t_0}^{T} - \int_{t_0}^{T} \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right) \underline{\delta x(t)}\, dt$$

Consequently, we obtain

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{T} \left[\frac{\partial V}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right)\right] \delta x(t)\, dt$$

$$+ \frac{\partial V}{\partial \dot{x}}(x^*(T), \dot{x}^*(T), T)\, \underline{\underline{\delta x(T)}}$$

$$- \frac{\partial V}{\partial \dot{x}}(x^*(t_0), \dot{x}^*(t_0), t_0)\, \underline{\underline{\delta x(t_0)}}$$

When $x(T)$ and $x(t_0)$ are fixed, $\delta x(T) = \delta x(t_0) = 0$, and the last two terms are equal to 0.

Thus, if $x^*(t)$ is an extremal signal,

$$\delta J(x^*(t), \delta x(t)) = \int_{t_0}^{T} \left[\frac{\partial V}{\partial x}\Big|_* - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\Big|_* \right) \right] \delta x(t)\, dt = 0$$

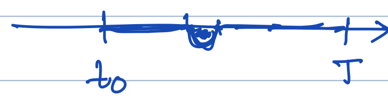
$\forall$ admissible $\delta x(t)$.

In practice, we do not need to check for infinitely many $\delta x(t)$.

Fundamental Lemma: If $g(t)$ is a continuous function and $\int_{t_0}^{T} g(t)\,\delta x(t) = 0 \quad \forall\ \delta x(t)$ that are continuous, then

$$g(t) = 0 \quad \forall\, t \in [t_0, T].$$

proof can be done by contradiction.



Applying the fundamental lemma, we have the following necessary condition for $x^*(t)$ to be an extremal signal.

$$\frac{\partial V}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t) \right) = 0$$

$\forall\, t \in [t_0, T]$.

(Euler-Lagrange equation)

$$\frac{\partial^2 V}{\partial x \partial \dot{x}} \frac{dx}{dt} + \frac{\partial^2 V}{\partial \dot{x}^2} \underbrace{\frac{d\dot{x}}{dt}}_{\ddot{x}(t)} + \frac{\partial V}{\partial t \partial \dot{x}}$$

2nd order diff. eqn

Example: $J(x(t)) = \int_0^2 \left[\dot{x}(t)^2 - 2t\,x(t) \right] dt$

$x(0) = 1, \quad x(2) = 5$

here $V(x(t), \dot{x}(t), t) = \dot{x}(t)^2 - 2t\,x(t)$

$\frac{\partial V}{\partial x} = -2t, \quad \frac{\partial V}{\partial \dot{x}} = 2\dot{x}(t), \quad \frac{d}{dt}\frac{\partial V}{\partial \dot{x}} = 2\ddot{x}(t)$

EL eqn: $-2t - 2\ddot{x}(t) = 0$

$$\Rightarrow \quad \ddot{x}(t) = -t$$

$$\Rightarrow \quad \dot{x}(t) = -\frac{t^2}{2} + C_1$$

$$\Rightarrow \quad x(t) = -\frac{t^3}{6} + C_1 t + C_2$$

$$x(0) = C_2 = 1$$

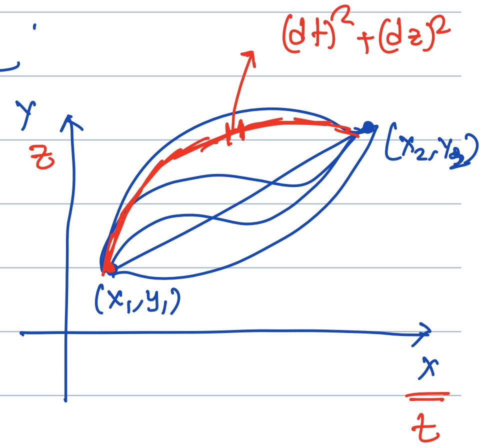
$$x(2) = \frac{-8}{6} + 2C_1 + 1 = 5$$

$$\Rightarrow \quad 2C_1 = 4 + \frac{4}{3} = \frac{16}{3} \Rightarrow C_1 = 8/3$$

$$x(t) = \frac{-t^3}{6} + \frac{8}{3}t + 1$$

Example

We can express the length of a curve as



$$J = \int_{x_1}^{x_2} \sqrt{(dt)^2 + (dz)^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$J(z(t)) = \int_{x_1}^{x_2} \underbrace{\left(1 + \dot{z}(t)^2\right)^{\frac{1}{2}}} dt$$

$$V(z(t), \dot{z}(t), t) = \left(1 + \dot{z}(t)^2\right)^{\frac{1}{2}}$$

Let $z^*(t)$ satisfy the EL eqn.

$$\frac{\partial V}{\partial z} = 0$$

$$\frac{\partial V}{\partial \dot{z}} = \frac{1}{2}\left(1 + \dot{z}(t)^2\right)^{-\frac{1}{2}} \times 2\dot{z}(t)$$

EL eqn: $0 - \frac{d}{dt}\left(\frac{\dot{z}(t)}{(1 + \dot{z}(t)^2)^{1/2}}\right) = 0 \quad \forall t$

$$\Rightarrow \quad \frac{\dot{z}(t)}{\sqrt{1 + \dot{z}(t)^2}} = c \quad \forall t \in [x_1, x_2]$$

$$\Rightarrow \quad \dot{z}(t) = C\left(1+\dot{z}(t)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \dot{z}(t)^2\left[1-C^2\right] = C$$

$$\Rightarrow \quad \dot{z}(t)^2 = \bar{C}$$

$$\Rightarrow \quad \dot{z}(t) = \sqrt{\bar{C}}$$

$$\Rightarrow \quad \boxed{z(t) = c_1 t + c_2}$$

$\Rightarrow$ extremal signal is a straight line, with c_1, c_2 determined by boundary conditions.

Let us now generalize the above setting.

Let $x(t) \in \mathbb{R}^n$, t_0, t_f, $x(t_0)$: fixed

$x(t_f)$: free variable.

$$J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt \in \mathbb{R}$$

$$\delta J = \int_{t_0}^{t_f} \Big[\underbrace{g_x}_{\text{row vector}}(x(t), \dot{x}(t), t)\, \delta x(t) + g_{\dot{x}}(x(t), \dot{x}(t), t)\, \delta \dot{x}(t)\Big] dt$$

$$\underbrace{\left[\frac{\partial g}{\partial x_1}(\cdot) \quad \frac{\partial g}{\partial x_2}(\cdot) \quad \cdots \quad \frac{\partial g}{\partial x_n}(\cdot)\right]}_{\nabla_x g(\cdot)^T} \cdot \begin{bmatrix} \delta x_1(t) \\ \delta x_2(t) \\ \vdots \\ \delta x_n(t) \end{bmatrix}$$

$$g_{\dot{x}}\, \delta x(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt} g_{\dot{x}}\, \delta x(t)\, dt$$

Once again, we apply integration by parts to the second term to obtain

$$\delta J = \int_{t_0}^{t_f} \left[g_x - \frac{d}{dt}(g_{\dot{x}})\right] \delta x(t)\, dt + g_{\dot{x}}\left(x(t_f), \dot{x}(t_f), t_f\right) \delta x(t_f)$$

Boundary condition: If $x(t_f) = x_f$ fixed, then $\delta x(t_f) = 0$

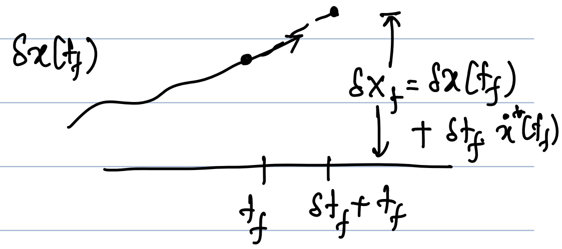
If $x(t_f)$ is free, then $\delta x(t_f) \neq 0$

If $x^*(t)$ is an optimal solution, $\delta J(x^*(t), \delta x(t)) = 0$
$\forall$ admissible $\delta x(t)$.

$$\Rightarrow \quad g_x(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt} g_{\dot{x}}(x^*(t), \dot{x}^*(t), t) = 0 \quad \forall t \in [t_0, t_f]$$

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0$$

Let us now consider the case where both t_f and $x(t_f)$ are free.



$$J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

applying Leibniz rule, we obtain:

$$\delta J = \int_{t_0}^{t_f} \left(g_x \delta x(t) + g_{\dot{x}} \delta \dot{x}(t) \right) dt + g(x(t_f), \dot{x}(t_f), t_f)\, \delta t_f$$

$$= \int_{t_0}^{t_f} \left(g_x - \frac{d}{dt} g_{\dot{x}} \right) \delta x(t)\, dt + g(x(t_f), \dot{x}(t_f), t_f)\, \underline{\delta t_f}$$

$$+ g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f)\, \underline{\delta x(t_f)},$$

where $\underline{\delta x_f = \delta x(t_f) + \dot{x}(t_f) \delta t_f}$: assuming δx_f and δt_f are free and independent.

$$= \int_{t_0}^{t_f} \left(g_x - \frac{d}{dt} g_{\dot{x}} \right) \delta x(t)\, dt + g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f)\, \delta x_f$$

$$+ \left[g(x(t_f), \dot{x}(t_f), t_f) - g_{\dot{x}}(x(t_f), \dot{x}(t_f), t_f)\, \dot{x}(t_f) \right] \delta t_f$$

If $x^*(t)$ is an extremal signal, $\left. g_x - \frac{d}{dt} g_{\dot{x}} \right|_{*} = 0$ (EL eqn)

$$\left. g_{\dot{x}} \right|_{x^*(t_f)} = 0, \quad \left. \left(g - g_{\dot{x}} \dot{x} \right) \right|_{x^*(t_f)} = 0$$

Let us now add constraints.

$$\min_{x(t)} \quad J(x(t),t) = \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)\,dt$$

s.t. $\quad \underline{f(x(t),\dot{x}(t),t) = 0}\,,$

$x(t_0),\ x(t_f),\ t_0, t_f$ are fixed.

sometimes, there is a terminal constraint $\boxed{m(x(t_f),t_f)=0}$: $\in \mathbb{R}^{k_2}$

v is not a f^n of time $\leftarrow$ $v \in \mathbb{R}^{k_2}$

Recall: for conventional optimization problem: $\min_x \ g(x)$ s.t. $\underline{f(x)=0}$,

$$L(x,\lambda) = g(x) + \lambda^T f(x).$$

optimality condition: $\begin{Vmatrix} \nabla_x L(x,\lambda) = 0 \ \Rightarrow \ g_x(x) + \lambda^T f_x(x) = 0 \\ \nabla_\lambda L(x,\lambda) = 0 \ \Rightarrow \ \underline{f(x) = 0} \end{Vmatrix}$

earlier when $f(x) \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^m$.

Now, $f(x(t),\dot{x}(t),t) \in \mathbb{R}^k$, we need a multiplier $p(t) \in \mathbb{R}^k$, and $p(t)$ should be a function of time t.

Let us now define augmented cost / Lagrangian for the optimization problem stated above.

$$J'(x(t),\dot{x}(t),p(t)) = \int_{t_0}^{t_f} \Big(g(x(t),\dot{x}(t),t) + p(t)^T \underline{f(x(t),\dot{x}(t),t)} \Big)\,dt$$

$$\delta J' = \int_{t_0}^{t_f} \Big[\big(\cdot \big)\, \delta x(t) + \big(f^T \big)\, \delta p(t) \Big]\,dt$$

$$\big(g_x + p^T f_x\big) - \frac{d}{dt}\big(g_{\dot{x}} + p^T f_{\dot{x}}\big)$$

applying Leibniz rule:

$$\int_{t_0}^{t_f} \Big(g_x\,\delta x + g_{\dot{x}}\,\delta\dot{x} + p^T f_x\,\delta x + p^T f_{\dot{x}}\,\delta\dot{x} + f^T \delta p \Big)\,dt$$

$$= \int_{t_0}^{t_f} \Big(g_x - \frac{d}{dt} g_{\dot{x}} + p^T f_x - \frac{d}{dt}\big(p^T f_{\dot{x}}\big) + f^T \delta p \Big)\,dt$$

First order necessary condition

differential eq^n in both x & p.

$$\left(g_x + p^T f_x\right) - \frac{d}{dt}\left(g_{\dot{x}} + p^T f_{\dot{x}}\right) = 0 \qquad \forall t$$

$$f(x, \dot{x}, t) = 0 \qquad \forall t$$

constraint present in the problem.

Other types of boundary conditions can be handled as discussed above.

Example:
$$\min_{x(t),u(t)} \int_0^1 \left(x^2(t) + u^2(t)\right) dt$$

s.t.
$$\dot{x}(t) = -x(t) + u(t)$$
$$x(0) = 1, \quad x(1) = 0$$

Express the FONC in terms of differential eq^n in $x(t)$ & multiplier $p(t)$, and solve for $x(t)$.

Augmented cost:

$$J' = \int_0^1 \left(x^2(t) + u^2(t) + p(t)\left[\dot{x}(t) + x(t) - u(t)\right]\right) dt$$

$$\delta J' = \int_0^1 \Big(\big(\quad \big) \delta x(t) + \big(\quad \big) \delta u(t)$$

$$\left(2x(t) + p(t) - \frac{d}{dt} p(t)\right) \qquad 2u(t) - p(t)$$

$$+ \big(\quad \big) \delta p(t) \Big) dt$$

$$\dot{x}(t) + x(t) - u(t)$$

FONC:

$$2x(t) + p(t) - \dot{p}(t) = 0 \Rightarrow \boxed{\dot{p}(t) = 2x(t) + p(t)} \text{ costate equation.}$$

$$2u(t) - p(t) = 0 \Rightarrow u(t) = p(t)/2$$

$$\dot{x}(t) + x(t) - u(t) = 0 \Rightarrow \boxed{\dot{x}(t) = -x(t) + \frac{1}{2} p(t)}$$

In this case, since $x(t_f)$ is given, we have no boundary condition on costate $p(t)$.

So, to solve for $x(t)$, we need to eliminate $p(t)$.

$$\ddot{x}(t) = -\dot{x}(t) + \frac{1}{2}\dot{p}(t)$$

$$= -\dot{x}(t) + \frac{1}{2}\left(2x(t) + p(t)\right)$$

$$= -\dot{x}(t) + x(t) + \frac{p(t)}{2}$$

$$= -\dot{x}(t) + x(t) + \dot{x}(t) + x(t) = 2x(t)$$

So, extremal $x(t)$ will satisfy $\ddot{x}(t) = 2x(t)$

with boundary conditions $x(0) = 1, x(1) = 0.$

$$u(t) = \dot{x}(t) + x(t)$$

Lecture 23 & 24: 3rd March

- optimality conditions for continuous-time optimal control (taught on board)

- Notes by Dantu Phani Surya appended below.

$$\rightarrow \quad 2x + p - \frac{dp}{dt} = 0 \Rightarrow \dot{p} = p + 2x \longrightarrow \text{co state}$$

$$2u - p = 0 \Rightarrow u = p/2$$

$$x + \dot{x} - u = 0 \Rightarrow \dot{x} = -x + \frac{p}{2} \longrightarrow \text{state}$$

In this case, since $x(t_f)$ is given, we have no boundary condition on p.

$$\rightarrow \quad \ddot{x} = -\dot{x} + \frac{\dot{p}}{2} = -\dot{x} + \frac{p+2x}{2} = -\dot{x} + x + \frac{p}{2} = -\dot{x} + x + x + \dot{x}$$

$$= 2x$$

$$\Rightarrow \ddot{x} = 2x \Rightarrow x = Ae^{-\sqrt{2}t} + Be^{\sqrt{2}t}$$

Any extremal $x(t)$ will satisfy $\ddot{x} = 2x$, with boundary conditions.

$$\Rightarrow u(t) = \dot{x}(t) + x(t)$$

Continuous time optimal control

$$\min_{x(t),u(t)} J(x(t),u(t)) = s(x(t_f),t_f) + \int_{t_0}^{t_f} V(x(t),u(t),t)\,dt$$

s.t. $\dot{x}(t) = f(x(t),u(t),t)$

t_0, $x(t_0)$: given

t_f, $x(t_f)$: free

$p(t)$: multipliers for Lagrangian

minimising J is equivalent to minimising J_2.

$$J_2(x(t),u(t)) = \int_{t_0}^{t_f} \Big[V(x(t),u(t),t) + \underbrace{\frac{ds}{dt}(x(t),t)}_{s(x(t_f),t_f) - \underbrace{s(x(t_0),t_0)}_{\text{constant}}} \Big]\,dt$$

$$= \int_{t_0}^{t_f} \Big[V(x(t),u(t),t) + \frac{\partial s}{\partial x}(x(t),t)\,\dot{x}(t) + \frac{\partial s}{\partial t}(x(t),t) \Big]\,dt$$

Augmented cost : $J_2' = \int_{t_0}^{t_f} \Big[\underbrace{V(x(t),u(t),t) + p(t)^T\big(f(x(t),u(t),t)}_{\text{Hamiltonian}} - \dot{x}(t)\big)$

$$+ \frac{\partial S}{\partial x}(x(t),t)\,\dot{x}(t) + \frac{\partial S}{\partial t}(x(t),t)\Big]\, dt$$

$$= \int_{t_0}^{t_f} \left[H - p^T\dot{x} + \frac{\partial S}{\partial x}\dot{x} + \frac{\partial S}{\partial t} \right] dt = \int_{t_0}^{t_f} L(x,\dot{x},u,p,t)\, dt$$

Necessary condition for $x^*(t), u^*(t)$ to be the optimal signal is $\delta J_2' = 0$

$$\Rightarrow \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial u}\delta u \right] dt + L\Big|_{t_f} \delta t_f$$

$$= \int_{t_0}^{t_f} \left[\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)\delta x + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial p}\delta p \right] dt + \overbrace{L\Big|_{t_f}\delta t_f + \frac{\partial L}{\partial \dot{x}}\delta x(t_f)}^{\text{Boundary condition}} = 0$$

$$L = V(x(t),u(t),t) + p(t)^T f(x(t),u(t),t) - p(t)^T \dot{x}(t) + \frac{\partial S}{\partial x}(x(t),t)\,\dot{x}(t) + \frac{\partial S}{\partial t}(x(t),t)$$

At optimal solution, $\frac{\partial L}{\partial u} = 0 \Rightarrow \frac{\partial V}{\partial u} + \frac{\partial f}{\partial u}(x(t),u(t),t)\cdot p(t) = \underbrace{\frac{\partial H}{\partial u}(x(t),u(t),p(t),t) = 0}_{①}$

$$\frac{\partial L}{\partial p} = 0 \Rightarrow f(x(t),u(t),t) - \dot{x}(t) = 0 \quad \text{— ②}$$

Now, $\frac{\partial L}{\partial x} = \frac{\partial V}{\partial x} + \frac{\partial f}{\partial x}\cdot p + \frac{\partial^2 S}{\partial x^2}\dot{x} + \frac{\partial^2 S}{\partial x \partial t}$

$$\frac{\partial L}{\partial \dot{x}} = -p^T + \frac{\partial S}{\partial x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -\dot{p}^T + \frac{\partial^2 S}{\partial x^2}\dot{x} + \frac{\partial^2 S}{\partial x \partial t}$$

$$\Rightarrow \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial V}{\partial x} + \frac{\partial f}{\partial x}\cdot p + \dot{p}^T = 0 \quad \forall\, t$$

$$\Rightarrow \quad \dot{p}(t) = -\left(\frac{\partial H}{\partial x}\right)^T \quad — \ 3$$

Boundary conditions at t_f :

$$\left[H(t_f) - p(t_f)^T \dot{x}(t_f) + \frac{\partial S}{\partial x}\bigg|_{t_f} \dot{x}(t_f) + \frac{\partial S}{\partial t}\bigg|_{t_f}\right] \delta t_f + \frac{\partial S}{\partial x}\bigg|_{t_f} \delta x(t_f) - p(t_f)^T \delta x(t_f)$$

$$\Rightarrow \underbrace{\left(H + \frac{\partial S}{\partial t}\right)\bigg|_{t_f}}_{=0 \ 4} \delta t_f + \underbrace{\left(\frac{\partial S}{\partial x} - p^T\right)\bigg|_{t_f}}_{=0 \ 5} \underbrace{\left(\delta x(t_f) + \dot{x}(t_f)\, \delta t_f\right)}_{\delta x_f = \text{perturbation in final value at } t_f + \delta t_f}$$

Optimality conditions

$$H(x,u,p,t) = V(x,u,t) + p^T f(x,u,t)$$

$$\left.\begin{aligned} \frac{\partial H}{\partial u} &= 0 && — \ 1 \\ \dot{x} &= f(x,u,t) && — \ 2 \\ \dot{p} &= -\left(\frac{\partial H}{\partial x}\right)^T && — \ 3 \end{aligned}\right\} \quad \forall\, t \in [t_0, t_f]$$

$$\left.\begin{aligned} \left(H + \frac{\partial S}{\partial t}\right)\bigg|_{t_f} \delta t_f &= 0 && — \ 4 \\ \left(\frac{\partial S}{\partial x} - p^T\right)\bigg|_{t_f} \cdot \delta x_f &= 0 && — \ 5 \end{aligned}\right\} \quad \text{Boundary conditions}$$

Special cases:

(a) free final state, free final time

$$H + \frac{\partial S}{\partial t} = 0 \quad , \quad \frac{\partial S}{\partial x} = p^T$$

(b) free final state, fixed final time

t_f : known, $\delta t_f = 0$, $\frac{\partial S}{\partial x} = p^T$

(c) fixed final state, free final time

$$H + \frac{\partial S}{\partial t} = 0 \quad , \quad x_f : \text{known} \quad , \quad \delta x_f = 0$$

(d) fixed final state, fixed final time

t_f, x_f : known, $\delta x_f = \delta t_f = 0$

(e) free final time, final state depends on final time $x(t_f) = \theta(t_f) \Rightarrow \delta x_f = \dot{\theta}(t_f)\,\delta t_f$

$$\Rightarrow \left[H + \frac{\partial S}{\partial t} + \left(\frac{\partial S}{\partial x} - p^T \right) \dot{\theta} \right] \Bigg|_{t_f} = 0$$

Sufficient condition for optimal signal

$$\delta^2 J_2' = \int_{t_0}^{t_f} \begin{bmatrix} \delta x^T & \delta u^T \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix}} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt > 0 \quad \forall \text{ permissible perturbations}$$

should be a positive definite matrix $\forall\, t \in [t_0, t_f]$ at the optimal solution.

Example

1. $$\min_{u(t)} \frac{1}{2}\int_0^2 u^2(t)\,dt$$

s.t $\dot{x}_1(t) = x_2(t)$ $\qquad x(0)=\begin{bmatrix}1\\2\end{bmatrix}, \quad x(2)=\begin{bmatrix}1\\0\end{bmatrix}$

$\dot{x}_2(t) = u(t)$

2. $$\min_{u(t)} \frac{1}{2}\int_{t_0}^{t_f} u^2(t)\,dt$$

$x(0)=\begin{bmatrix}1\\2\end{bmatrix}, \quad x_2(2)$: free

$x_1(2)=0$

s.t $\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = u_2(t)$

3. $$\min \frac{1}{2}\left(x_1(2)-4\right)^2 + \frac{1}{2}\left(x_2(2)-2\right)^2 + \frac{1}{2}\int_0^2 u^2(t)\,dt$$

$x(0)=\begin{bmatrix}1\\2\end{bmatrix}, \quad x(2) =$ free

$t_f = 2$ fixed

1.

$$H = V + p^T f = \frac{u^2}{2} + p_1 x_2 + p_2 u$$

$$\frac{\partial H}{\partial u} = u + p_2 = 0$$

$$\frac{\partial H}{\partial x} = \begin{bmatrix}0 & p_1\end{bmatrix} \Rightarrow \dot{p} = \begin{bmatrix}0\\-p_1\end{bmatrix} \Rightarrow \begin{array}{l}\dot{p}_1 = 0 \rightarrow p_1 = c_1\\ \dot{p}_2 = -p_1 \Rightarrow p_2 = -c_1 t + c_2\end{array}$$

$$\Rightarrow u = c_1 t - c_2$$

$$\dot{x}_2 = u \Rightarrow x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3$$

$$\dot{x}_1 = x_2 \Rightarrow x_1 = \frac{c_1}{2}\frac{t^3}{3} - \frac{c_2}{2}t^2 + c_3 t + c_4$$

$x_1(0)=1 \Rightarrow c_4 = 1$, $x_2(0)=2 \Rightarrow c_3 = 2$, $x_1(2)=1$, $x_2(2)=0 \rightarrow$ give c_1, c_2

2. $\delta x_{f_2} \neq 0 \Rightarrow \left.\dfrac{\partial S}{\partial x_2} - P_2\right|_{t=2} = 0 \Rightarrow P_2(2) = 0$

$\Rightarrow 2c_1 = c_2$ and $x_1(2) = 0 \longrightarrow$ give c_1, c_2

3. $H = \dfrac{u^2}{2} + P_1 x_2 + P_2 u$

$\Rightarrow \dfrac{\partial H}{\partial u} = 0 \Rightarrow u + P_2 = 0$

$\dot{P}_1 = -\dfrac{\partial H}{\partial x_1} = 0 \Rightarrow P_1 = c_1$

$\dot{P}_2 = -\dfrac{\partial H}{\partial x_2} = -P_1 \Rightarrow P_2 = -c_1 t + c_2$

$\delta x_f \neq 0 \Rightarrow \left.\dfrac{\partial S}{\partial x_1} - P_1\right|_2 = 0, \quad \left.\dfrac{\partial S}{\partial x_2} - P_2\right|_2 = 0$

$$\left.\begin{aligned} & x_1(2) - 4 - P_1 = 0 \Rightarrow x_1(2) = 4 + c_1 \\ & x_2(2) - 2 - P_2 = 0 \Rightarrow x_2(2) = 2 - 2c_1 + c_2 \\ & x_1(0) = 1 \\ & x_2(0) = 2 \end{aligned}\right\} \quad \text{4 eq}^{n}\text{s 4 vars can be solved}$$