EE60012 – Spring Semester 2025-26

Homework 4: Optimal Control

Prof. Ashish Ranjan Hota
Department of Electrical Engineering, IIT Kharagpur

Q 4.1: Matrix Inequality

If P is a symmetric positive definite matrix, then show that

$$\begin{bmatrix} q^\top x + r & x^\top \\ x & -2P^{-1} \end{bmatrix} \prec 0$$

is equivalent to $q^\top x + r + \frac{1}{2}x^\top P x \prec 0$.

Q 4.2: Optimization with LMI Constraints

Determine the optimal value of the following optimization problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^2, t\in\mathbb{R}} \quad & t \\ \text{subject to} \quad & -1 \leq x_1 + x_2 \leq 1, \\ & t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} - x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - x_2 \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} \succeq 0. \end{aligned}$$

You may solve the problem using YALMIP.

Q 4.3: LMI for Stability

Consider the matrix

$$A = \begin{bmatrix} -0.018 & -0.2077 & -0.7150 \\ -0.5814 & -4.29 & 0 \\ 1.067 & 4.273 & -6.654 \end{bmatrix}.$$

By solving a suitable LMI, determine the stability of the system $\dot{x} = Ax$.

Q 4.4: H_∞ Norm

Let $\widehat{G}(s) = \frac{1}{s+1}$. Determine $||\widehat{G}(s)||_{H_\infty}$.

Q 4.5: H_∞ Norm

Consider the following CT-LTI system

$$\dot{x} = \begin{bmatrix} -5 & 1 & 2 \\ 1 & -9 & 1 \\ -1 & -10 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

Determine the smallest γ such that $||\widehat{G}(s)||_{H_\infty} < \gamma$. Implement the LMI constraints as

`LMI <= -1e-6*eye(n) or LMI >= 1e-6*eye(n).`

Q 4.6: True/False Questions

Determine if the following statements are true or false, alongwith suitable explanation.

(a) The unit step signal belongs to the $L_2[0, \infty)$ space.

(b) The signal $x(t) = \sin t$ belongs to the $L_2[0, \infty)$ space.

(c) The transfer function $\widehat{G}(s) = \frac{2s^2+1}{s^2+1}$ belongs to the H_∞ space, but not the H_2 space.

Q 4.7: Transfer Function

Consider the following CT-LTI system

$$\dot{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} u.$$

Determine its transfer function matrix and compute the H_∞ norm of the system.

Q 4.8: H_2 Norm

Consider the following CT-LTI system

$$\dot{x} = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0.0729 & 0.0001 \\ -4.7500 & 1.2300 \\ 1.5300 & 10.6300 \\ 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.$$

Determine the smallest γ such that $||\widehat{G}(s)||^2_{H_2} < \gamma^2$.

Implement the LMI constraints as

`LMI <= -1e-6*eye(n) or LMI >= 1e-6*eye(n).`

Q 4.9: H_2-Optimal State Feedback Control

Consider the LTI system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w,$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + u,$$

where w is an exogenous disturbance. Determine the smallest $||\widehat{G}_{wy}(s)||_{H_2}$ that can be achieved via a suitably designed state feedback controller.

Implement the LMI constraints as

```
LMI <= -1e-6*eye(n) or LMI >= 1e-6*eye(n).
```

Q 4.10: H_∞-Optimal State Feedback Control

Consider the LTI system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w,$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + u + 0.05w,$$

where w is an exogenous disturbance. Determine the smallest $||\widehat{G}_{wy}(s)||_{H_\infty}$ that can be achieved via a suitably designed state feedback controller.

Implement the LMI constraints as

```
LMI <= -1e-6*eye(n) or LMI >= 1e-6*eye(n).
```