EE60012 – Spring Semester 2025-26

Homework 3: Optimal Control

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Q 3.1: Minimum-Energy Control

Consider the scalar system

$$\dot{x}(t) = -x(t) + u(t), \qquad t \in [0,1],$$
$$x(0) = 1, \quad x(1) = 0.$$

Find the optimal control input $u^*(t)$ and corresponding state trajectory $x^*(t)$ that minimizes

$$J = \int_0^1 u(t)^2 \, dt.$$

(a) Form the augmented cost using a co-state $p(t)$ and write down all the first-order necessary conditions (state equation, co-state equation, stationarity condition) together with the boundary conditions.

(b) Solve the resulting two-point boundary value problem for $x^*(t)$, $p^*(t)$, and $u^*(t)$.

Q 3.2: Free-Endpoint Optimal Control

Consider the system

$$\dot{x}(t) = u(t), \qquad t \in [0,2],$$
$$x(0) = 0, \quad x(2) \text{ is free,}$$

with cost functional

$$J = 3\,x(2)^2 + \int_0^2 \left[\, x(t)^2 + u(t)^2 \,\right] dt.$$

(a) Form the Hamiltonian and write down all the necessary conditions. Pay careful attention to the boundary condition at $t = 2$, since $x(2)$ is free and there is a terminal cost $\phi(x(2)) = 3\,x(2)^2$.

(b) Solve the resulting boundary value problem for $x^*(t)$, $p^*(t)$, and $u^*(t)$. Hint: The solution of $\ddot{x} = x$ is $x(t) = c_1 e^t + c_2 e^{-t}$.

Q 3.3: Scalar Finite-Horizon LQR and the Riccati Equation

Consider the scalar linear system

$$\begin{aligned} \dot{x}(t) &= a\,x(t) + b\,u(t), \qquad && t \in [0, t_f], \\ x(0) &= x_0 \quad \text{(given)}, \qquad && x(t_f) \text{ free,} \end{aligned}$$

with the quadratic cost functional

$$J = \frac{1}{2}\, f\, x(t_f)^2 + \frac{1}{2} \int_0^{t_f} \left[\, q\, x(t)^2 + r\, u(t)^2 \,\right] dt,$$

where a, b are given real constants, $f \geq 0$, $q \geq 0$, and $r > 0$.

(a) Form the Hamiltonian $H(x, u, p, t)$, and write down all the first-order necessary conditions (state equation, co-state equation, stationarity condition) and the boundary conditions.

(b) Using the stationarity condition, express $u^*(t)$ in terms of $p(t)$.

(c) Hypothesise that $p(t) = P(t)\, x(t)$ for a scalar function $P(t)$, and derive the scalar differential Riccati equation (DRE) satisfied by $P(t)$, together with the terminal condition.

(d) For the special case $a = 0$, $b = 1$, $q = 1$, $r = 1$, $f = 0$, investigate the limit $t_f \to \infty$. Compute this steady-state value $\bar{P}$ and verify that the closed-loop system is stable.

Q 3.4: Infinite-Horizon LQR for a Two-Dimensional System

Consider the LTI system

$$\dot{x}(t) = A\, x(t) + B\, u(t), \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with $x(0) = x_0$ (given), $x(\infty)$ free, and the infinite-horizon quadratic cost

$$J = \frac{1}{2} \int_0^{\infty} \left[\, x(t)^\top Q\, x(t) + u(t)^2 \,\right] dt, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Verify that (A, B) is stabilisable and $(A, \sqrt{Q})$ is observable.

(b) Write the algebraic Riccati equation (ARE) and solve for the symmetric positive definite matrix $\bar{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ that solves it.

(c) Compute the Kalman gain K and the optimal control law.

(d) Show that the closed-loop matrix is Hurwitz (all eigenvalues have strictly negative real parts).

Q 3.5: Hamilton–Jacobi–Bellman Equation

Consider the scalar system

$$\dot{x}(t) = -x(t) + u(t), \qquad t \in [0, t_f], \quad x(0) = x_0, \quad x(t_f) \text{ free},$$

with cost functional

$$J = \frac{1}{2}\, x(t_f)^2 + \frac{1}{2}\int_0^{t_f} u(t)^2\, dt.$$

(a) Write the Hamilton–Jacobi–Bellman (HJB) equation for this problem. State the boundary condition at $t = t_f$.

(b) Hypothesise that the optimal cost-to-go has the form $J^*(x,t) = \frac{1}{2}P(t)\, x^2$. Substitute into the HJB equation and derive the scalar differential Riccati equation for $P(t)$, together with the terminal condition.

(c) Show that $P(t) = \frac{2}{3e^{2(t_f - t)} - 1}$ is a solution of the differential Riccati equation, and then, find the optimal feedback law $u^*(x,t)$.

Q 3.6: Dynamic Programming for a Nonlinear System

Consider the scalar system

$$\dot{x}(t) = x(t)\, u(t), \qquad t \in [0, t_f], \quad x(0) = x_0 > 0, \quad x(t_f) \text{ free},$$

with cost functional

$$J = \frac{1}{2}\big(\ln x(t_f)\big)^2 + \frac{1}{2}\int_0^{t_f} u(t)^2\, dt.$$

Note that the dynamics are nonlinear, and the terminal cost $S(x) = \frac{1}{2}(\ln x)^2$ achieves its minimum at $x = 1$.

(a) Write the Hamilton–Jacobi–Bellman (HJB) equation for the optimal cost-to-go $J^*(x,t)$, and state the boundary condition at $t = t_f$.

(b) Perform the minimisation over u in the HJB equation to find the optimal control u^* in terms of $\frac{\partial J^*}{\partial x}$, and substitute back to obtain a PDE for $J^*(x,t)$.

(c) Hypothesise that $J^*(x,t) = \frac{1}{2}P(t)\,(\ln x)^2$ for a scalar function $P(t)$. Substitute into the PDE from part (b) to derive an ODE for $P(t)$, together with the terminal condition.

(d) Solve the ODE for $P(t)$ and write down the optimal cost-to-go $J^*(x,t)$, the optimal feedback law $u^*(x,t)$, and the optimal cost $J^*(x_0, 0)$.

Q 3.7: Ruling Out Singular Arcs

Consider the following continuous-time optimal control problem on $[0, T]$:

$$\begin{aligned} \min_{u(t),\, t\in[0,T]} \quad & J = \int_0^T \left[\, c_I\, y(t) + c_V\, u(t) \,\right] dt \\ \text{s.t.} \quad & \dot{x}(t) = -\beta\, x(t)\, y(t) - x(t)\, u(t), \\ & \dot{y}(t) = \beta\, x(t)\, y(t) - \gamma\, y(t), \\ & x(0) = x_0, \quad y(0) = y_0 \quad \text{(given)}, \\ & 0 \le u(t) \le u_{\max}. \end{aligned} \tag{1}$$

Here $\beta, \gamma, c_I, c_V > 0$ are given constants and $x_0, y_0 > 0$ with $x_0 + y_0 < 1$. Assume that $x(t), y(t) \in (0,1)$ for all t.

(a) Form the Hamiltonian $H\big(x, y, u, p_x, p_y, t\big)$ using co-state variables $p_x(t)$ and $p_y(t)$. Write down all the first-order necessary conditions for optimality (state equations, co-state equations, and the stationarity/minimum condition on u), together with the appropriate boundary conditions.

(b) Since the Hamiltonian is affine in u, identify the switching function $\phi(t)$ that determines the optimal control structure. Express the optimal control $u^*(t)$ in terms of $\phi(t)$.

(c) Compute $\dot{\phi}(t)$ using the state and co-state dynamics.

(d) Show that if $\beta\, c_V \neq c_I$, no singular arc exists and the optimal vaccination control is *bang-bang*.

Q 3.8: Computing a Singular Control

Consider the following continuous-time optimal control problem on $[0, T]$:

$$\begin{aligned} \min_{u(t),\, t\in[0,T]} \quad & J = \int_0^T \left[a x(t)^2 + b u(t) \right] dt \\ \text{s.t.} \quad & \dot{x}(t) = \beta\, x(t)\big(1 - x(t)\big) - \gamma\, x(t) - u(t)\, x(t), \\ & x(0) = x_0 \quad \text{(given)}, \\ & 0 \le u(t) \le u_{\max}. \end{aligned} \tag{2}$$

Here $\beta > \gamma > 0$, $a > 0$, $b > 0$, and $x_0 \in (0, 1)$. Assume the parameters satisfy $0 < \frac{b\beta}{2a} < 1$ and $0 < \beta\big(1 - \frac{b\beta}{2a}\big) - \gamma < u_{\max}$.

(a) Form the Hamiltonian $H(x, u, p, t)$ using co-state $p(t)$. Write down all the first-order necessary conditions for optimality (state equation, co-state equation, and the stationarity/minimum condition on u), together with the appropriate boundary conditions.

(b) Identify the switching function $\phi(t)$ and express $u^*(t)$ in terms of $\phi(t)$.

(c) Compute $\dot{\phi}(t)$.

(d) On a singular arc where $\phi(t) = \dot{\phi}(t) = 0$ on an open interval I, compute the singular control u_s. Verify that $u_s \in (0, u_{\max})$.