EE60012 – Spring Semester 2025-26

Homework 2: Optimal Control

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Q 2.1: Optimal Control of Bilinear Systems

Consider the discrete-time system $x_{k+1} = x_k u_k + 1$ with performance index $J = \frac{1}{2}\sum_{k=0}^{N-1} u_k^2$. Let $N = 2$. Given x_0, goal is to ensure $x_2 = 0$. Answer the following questions.

1. Write state and co-state equation eliminating optimal input u.
2. Assume λ_2 is known. Solve for λ_1 and λ_0 in terms of λ_2.
3. Express x_2 in terms of λ_2 and x_0.
4. If $x_0 = 1$, find optimal state and costate trajectories, optimal control and optimal value of the performance index.

Q 2.2: Optimal Control with Linear Performance Index

Consider the discrete-time system $x_{k+1} = Ax_k + Bu_k$ with performance index $J = S_N x_N + \sum_{k=0}^{N-1} Qx_k + Ru_k$. From the first order necessary condition of optimality, write the state and costate equation that the optimal trajectory needs to satisfy. What problem do you notice?

Q 2.3: (Matlab problem) Minimum Energy and Minimum Fuel Control

Consider the following discrete-time LTI system given by

$$x(k+1) = Ax(k)+Bu(k), \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_T = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}.$$

1. Does there exist a sequence of control inputs to drive the states from $x(0)$ to x_T for $T = 1$?
2. What is the smallest value of T for which you can find a control input that drive the states from $x(0)$ to x_T stated above?
3. For $T = 10$, find a sequence of control inputs that minimizes the total energy $\sum_{k=0}^{T-1} u(k)^2$.
4. For $T = 10$, find a sequence of control inputs that minimizes the "fuel" $\sum_{k=0}^{T-1} |u(k)|$.
5. Compare both the inputs on the same plot.
6. Find the state trajectory under both inputs and compare each component of the state in a separate plot.

Q 2.4: Dynamic Programming with Finite States and Actions

Consider the discrete-time system $x_{k+1} = x_k - 0.4x_k^2 + u_k$ with the state and control values are constrained by $0.0 \le x_k \le 1.0$, and $-0.4 \le u_k \le 0.4$. Suppose the state and input are quantized as $x_k \in \{0.0,\ 0.5,\ 1.0\}$, $u_k \in \{-0.4,\ -0.2,\ 0.0,\ 0.2,\ 0.4\}$. The performance measure to be minimized is $J = \sum_{k=0}^{1} \left(x_k^2 + |u_k|\right)$.

(a) Use dynamic programming with linear interpolation to find the optimal control law and the associated cost for all possible values of x_0 and x_1.

(b) From the above results, find the optimal control sequence $\{u_0^*, u_1^*\}$ and the minimum cost if the initial state is $x_0 = 1.0$.

Q 2.5: Discrete-time LQR for Scalar System

For the first-order discrete system $x_{k+1} = 1.05\,x_k + u_k$, the performance measure to be minimized is

$$J = x_N^2 + \sum_{k=0}^{N-1} \left(q\,x_k^2 + r\,u_k^2\right).$$

(a) For the case $N \to \infty$, solve the discrete Riccati equation for the steady-state solution S_{inf} and the feedback gain K_{inf} as a function of q and r.

(b) For $q = r = 1$, confirm that the backwards Riccati recursion agrees with this steady-state prediction.

(c) Compare the performance of the optimal controller (with $q = r = 1$) with the ad-hoc design $u_k = -0.4\,x_k$.

Q 2.6: Discrete-time LQR

For the first-order discrete system with $x_k = [x_{1,k} \quad x_{2,k}]^\top$ and dynamics $x_{1,k+1} = 0.8x_{1,k} + x_{2,k} + u_k$ and $x_{2,k+1} = 0.6x_{2,k} + 0.5u_k$, the performance measure to be minimized is

$$J = (x_{1,N}^2 + 2x_{2,N}^2) + \sum_{k=0}^{N-1} \left(0.5x_{1,k}^2 + 0.5x_{2,k}^2 + 0.5u_k^2\right).$$

Let the initial state be $x_{1,0} = 5, x_{2,0} = 3$ and $N = 10$. Answer the following questions by programming in MATLAB.

(a) Determine the optimal controller gains for the entire horizon.

(b) Determine whether there exists a steady-state state feedback controller that is optimal when $N \to \infty$.

(c) Modify the above problem so that $x_{1,k}$ tracks the reference $r_k = 2$ for $k \le 5$ and $r_k = 4$ for $k > 5$.