EE60012 – Spring Semester 2025-26

Homework 1: Optimal Control

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Q 1.1: Convex Set

Let $x_0, v \in \mathbb{R}^n$. Let $C = \{x \in \mathbb{R}^n | x = x_0 + \alpha v, \alpha \geq 0\}$ be the set of points that lie on the ray originating from x_0 along the direction v. Is C a convex set?

Q 1.2: Convex Sets

Determine if the following sets in $\mathbb{R}^2$ are convex sets together with suitable explanation.

- $S_1 = \{(x_1, x_2) | x_2 \leq x_1(x_1 - 2)(x_1 - 3)\}$.
- $S_2 = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 4, x_1 \geq 0\}$.
- $S_3 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2\}$.
- $S_4 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2, (x_1 - 1)^2 + (x_2 + 1)^2 \leq 5\}$.
- $S_5 = \{x \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 2\} \cup \{x \in \mathbb{R}^2 | (x_1 - 1)^2 + (x_2 + 1)^2 \leq 1\}$.

Q 1.3: Convex Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Show that

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \geq 0, \quad \forall x, y \in \texttt{dom} f.$$

Q 1.4: Gradients

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x) = (x_1 + x_2^2)^2$ where $x = [x_1; x_2] \in \mathbb{R}^2$.

1. Compute the gradient of f.
2. At $x_0 = [0, 1]$, is the direction $d = [1, -1]$ a descent direction, i.e., with directional derivative negative?
3. Find $\alpha > 0$ that minimizes $f(x_0 + \alpha d)$.

Q 1.5: Optimality Conditions

Consider a convex optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad i \in [m]. \end{aligned}$$

Let there exist x^* and λ^* that satisfy the KKT conditions for this problem. Then, show that

$$\nabla f_0(x^*)^\top (x - x^*) \ge 0,$$

for all feasible x.

Q 1.6: LP duality

Determine dual of the following optimization problem and simplify it as much as possible. Determine the optimal solution and the optimal value of the dual problem.

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^3} \quad & x_1 + x_3 \\ \text{subject to} \quad & x_1 + 2x_2 \le 5, \quad x_1 + 2x_3 = 6, \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0. \end{aligned}$$

Q 1.7: Convex duality

Determine the dual of the following optimization problem:

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}} \quad & x^2 + 1 \\ \text{subject to} \quad & (x-2)(x-4) \le 0, \end{aligned}$$

and simplify it as much as possible. Determine the optimal solution and the optimal value of both the primal and the dual problems.

Q 1.8: Convex optimization

Determine if the following problem is a convex optimization problem.

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^2} \quad & |x_1 + 5| + |x_2 - 3| \\ \text{subject to} \quad & 2.5 \le x_1 \le 5, \quad -1 \le x_2 \le 5. \end{aligned}$$

Q 1.9: LP duality

Consider the following primal optimization problem:

$$\begin{aligned} \text{minimize}_{x \in \mathbb{R}^2} \quad & -x_2 \\ \text{subject to} \quad & x_2 \ge 0, x_1 \ge 0, \\ & x_1 - x_2 \le 3. \end{aligned}$$

Find the dual of the above optimization problem. Determine whether the primal is infeasible, unbounded or has an optimal solution. Determine the dual optimization problem and whether the dual is infeasible, unbounded or has an optimal solution. If either one has an optimal solution, then show that strong duality holds.

Q 1.10: Robust Optimization

Consider the following optimization problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & a^\top x \leq b, \qquad \forall a \in \{a \in \mathbb{R}^n | Ca \leq d\}, \end{aligned}$$

where $C \in \mathbb{R}^{m\times n}$ and $d \in \mathbb{R}^m$. Reformulate the above problem as a convex optimization problem with finitely many constraints. Hint: You may need to use linear programming duality in a slightly non trivial manner.

Q 1.11: Stationary Points

Determine the set of all stationary points of the following function:

$$f(x_1, x_2) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2,$$

and for each of those points, determine if it is a local minimum or a local maximum or a saddle point.

Q 1.12: Optimality Conditions

Consider the following optimization problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^2} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 = 1. \end{aligned}$$

State the KKT optimality conditions for the above problem and find all solutions that satisfy the KKT conditions. Determine if any of them is a local minimizer.

Q 1.13: Optimality Conditions

Consider the following optimization problem:

$$\begin{aligned} \text{minimize}_{x\in\mathbb{R}^3} \quad & x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 - 2x_1 - 4x_2 - 6x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 1. \end{aligned}$$

State the KKT optimality conditions for the above problem and find all solutions that satisfy the KKT conditions. Determine if any of them is a local minimizer. Determine if the problem is convex.

Q 1.14: QP

Consider the following optimization problem:

$$\text{minimize}_{x\in\mathbb{R}^2} \quad \frac{1}{2} x^\top \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Determine the optimal solution and optimal value.

Q 1.15: Optimal Value

Consider the problem

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + 2x_2^2 + x_1 \\ \text{s.t.} \quad & x_1 + x_2 \leq a, \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter.

(i) Prove that for any $a \in \mathbb{R}$, the problem has a unique optimal solution (without actually solving it).

(ii) Solve the problem (the solution will be in terms of the parameter a).

(iii) Let $f(a)$ be the optimal value of the problem with parameter a. Write an explicit expression for f and prove that it is a convex function.