THE MEDIAN PROBLEM IN GRAPHS

by

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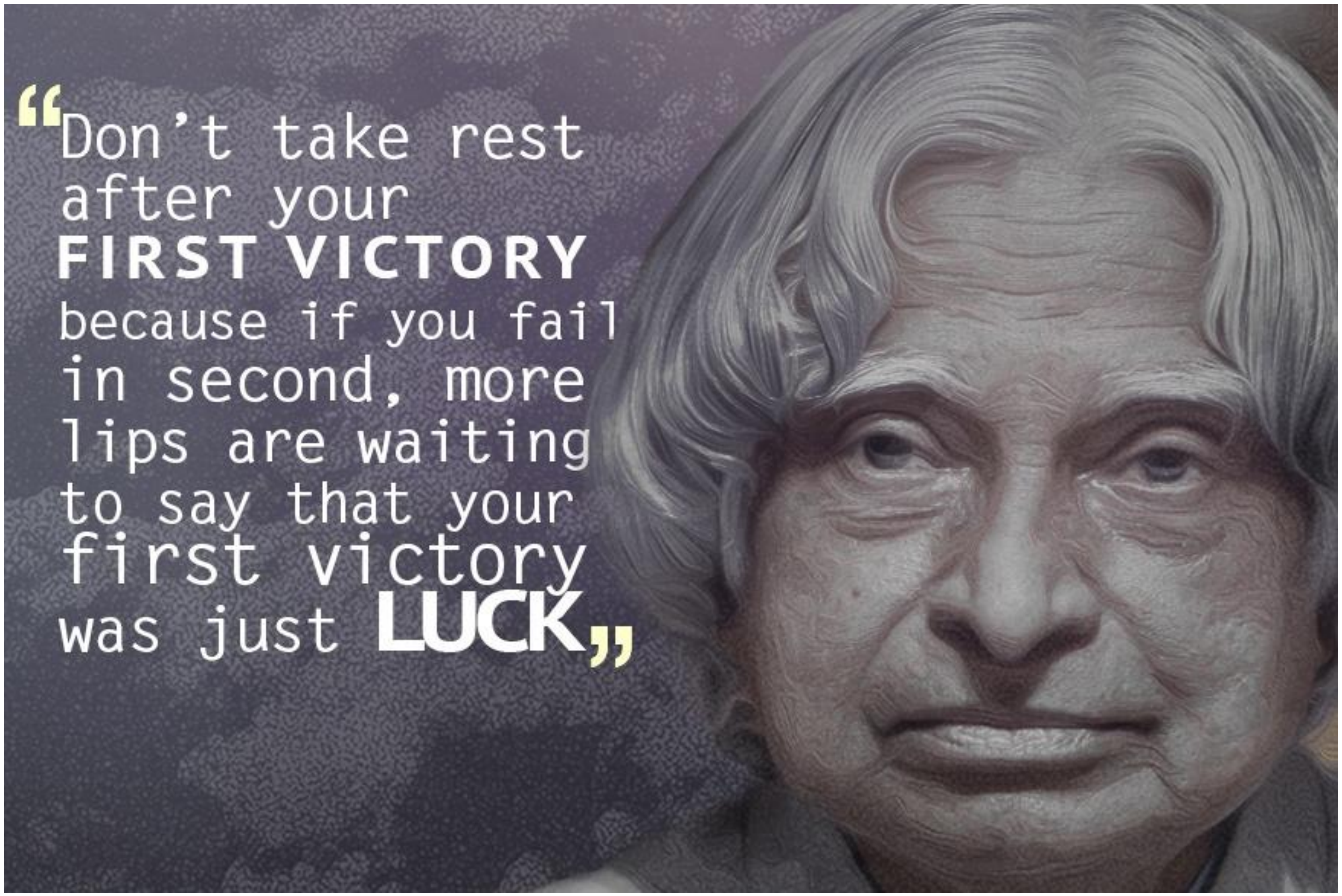
IIT, Kharagpur

09 April 2021

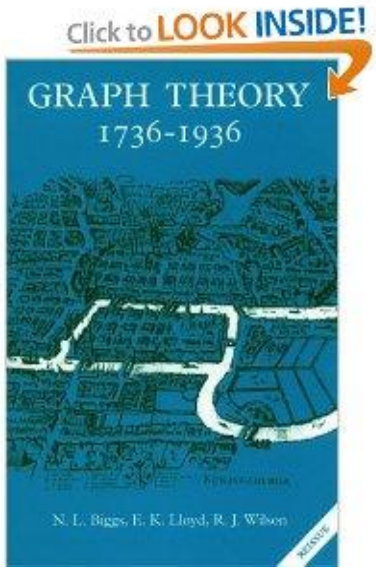
Homage to an Inspiring Teacher……
(15th Oct 1931- 27th July2015)

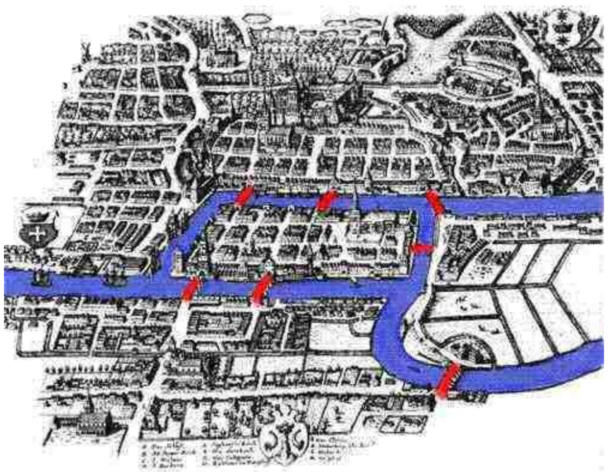


Outline

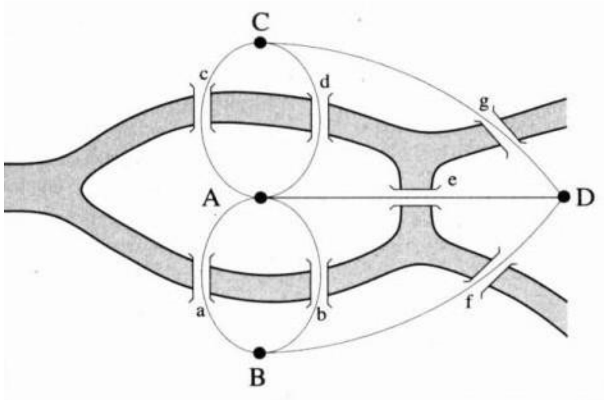
Leonhard Euler (1707-1783)
The Analysis Incarnate

The solution of a problem relating to the geometry of position – 1736 - GRAPH THEORY originates…..

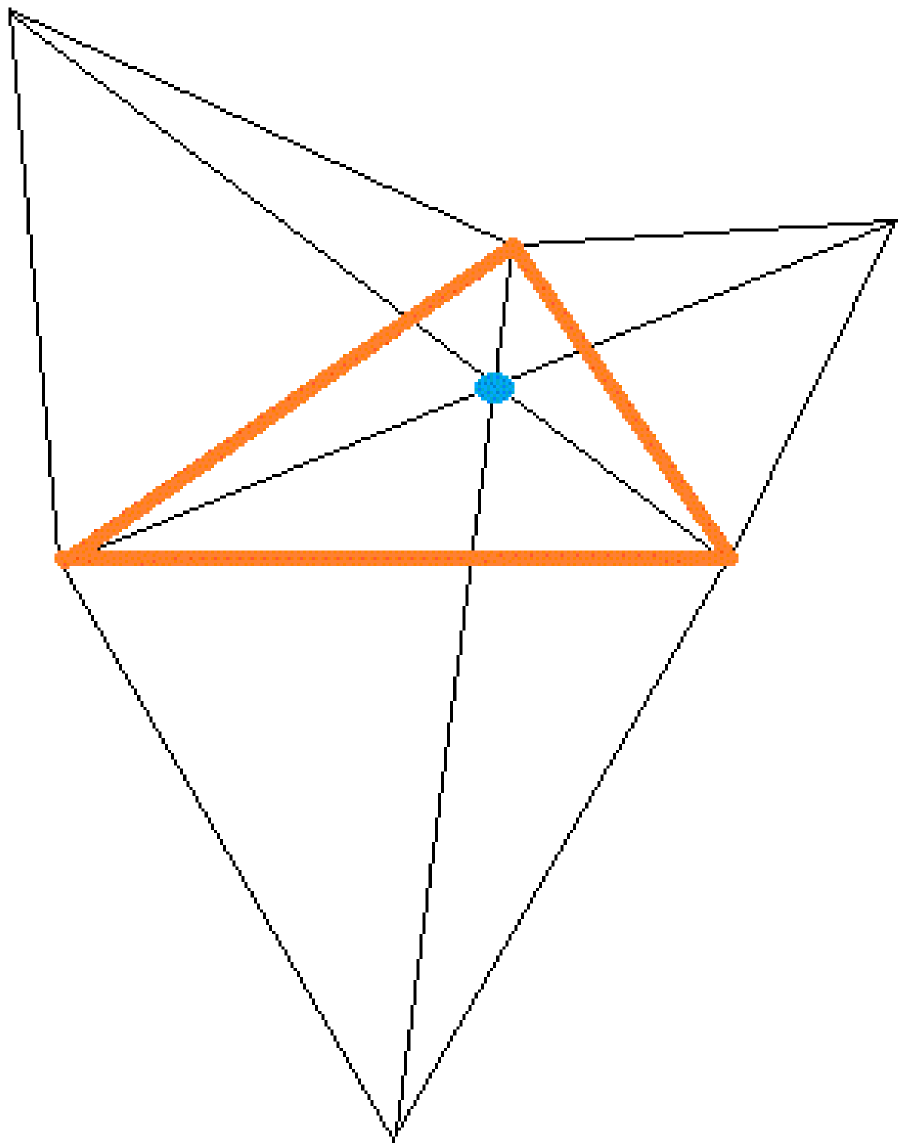




The (Eulerian) path should cross over each of the seven bridges exactly once



A PROBLEM DUE TO FERMAT



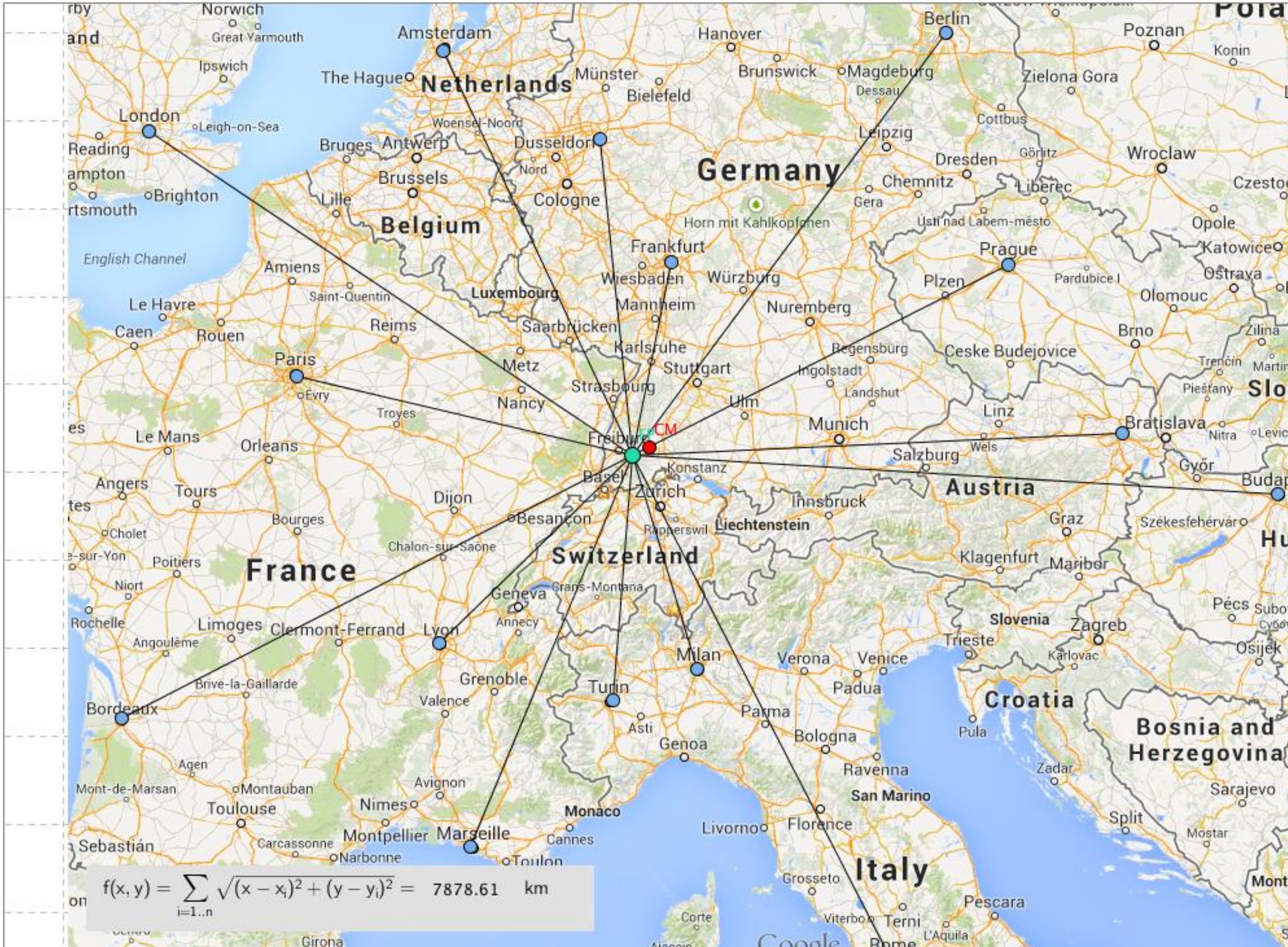
Fermat posed the problem:

'Given a triangle find the point where the sum of the distances to the vertices is a minimum.'

Weber problem

- It requires finding a point in the plane that minimizes the sum of the transportation costs from this point to **n** destination points

Fermat Weber Point of European Cyties

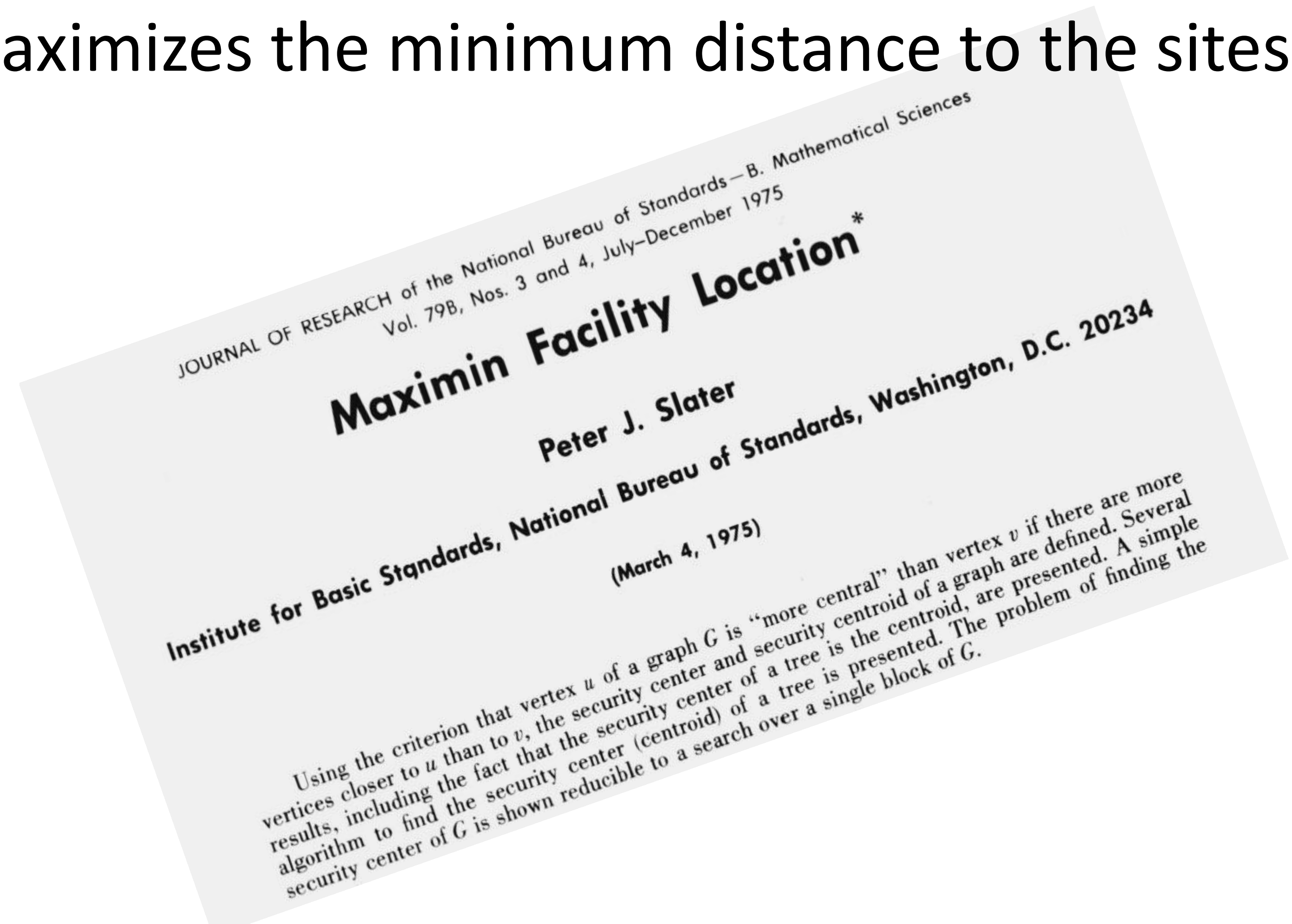


Minimax facility location

The **minimax facility location** problem seeks a location which minimizes the maximum distance to the sites, where the distance from one point to the sites is the distance from the point to its nearest site.

Maxmin facility location

The maxmin facility location or obnoxious facility location problem seeks a location which maximizes the minimum distance to the sites.


JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 79B, Nos. 3 and 4, July–December 1975

Maximin Facility Location*

Peter J. Slater

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(March 4, 1975)

Using the criterion that vertex u of a graph G is "more central" than vertex v if there are more vertices closer to u than to v, the security center and security centroid of a graph are defined. Several results, including the fact that the security center of a tree is the centroid, are presented. A simple algorithm to find the security center (centroid) of a tree is presented. The problem of finding the security center of G is shown reducible to a search over a single block of G.


STATIC AND DYNAMIC
NEURAL NETWORKS
From Fundamentals to Advanced Theory
UNDERSTANDING
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Theories, Concepts,
and Findings
CHARLES KADUSHIN
Graph Theory and
Interconnection Networks
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Stephen P Borgatti
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Jeffrey C Johnson
NETWORK THEORY
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Distance in
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Centrality concepts

- How important a room is in a building?
- How influential a person is in a social network?
- How well used a road is in a transportation network?
- How important a web page is?

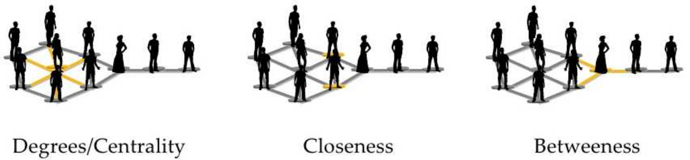
Degree Centrality

- Node with the highest degree is most important.
- Gossip network: central actor more likely to hear a gossip.

Degrees/Centrality   Closeness   Betweeneness

Closeness Centrality

- Closeness is a measure of the degree to which an individual is near all other individuals in a network.
- In a biological network, a protein with high closeness will be easily central to the regulation of other proteins.



Betweenness centrality

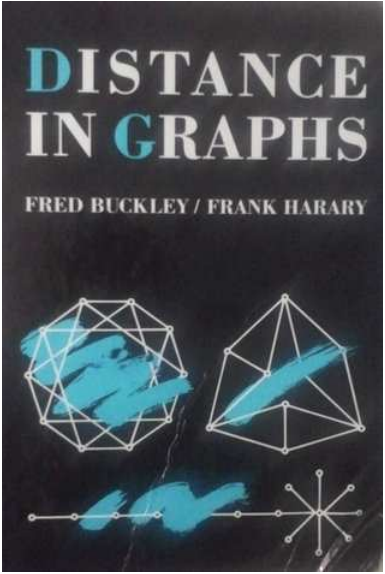
- Betweenness centrality quantifies the number of times a node acts as a bridge along the shortest path between two other nodes.
- In telecommunications network, a node with higher betweenness centrality would have more control over the network, because more information will pass through that node.

Degrees/Centrality Closeness Betweeneness

Centrality Problems continued...

Facility Location problems

1. What is the optimal location of a hospital such that the worst case response time of an ambulance is minimal?
2. What is the optimal location of a shopping mall so that the average driving time to the mall is minimal?
3. What is the optimal location of a shop if customers buy at the nearest shop and there will be a competitor placing its shop after we have placed ours?


DISTANCE
IN GRAPHS
FRED BUCKLEY / FRANK HARARY

Distance - based graph invariants

- Wiener index (or average distance)
- Balaban distance connectivity index
- Average (or total) eccentricity
- Degree distance
- Wiener polarity index
- Distance spectral radius
- Minimum status (or its normalized version called proximity)
- Maximum status (or its normalized version called remoteness)
- Status difference (or the normalized version called the difference between remoteness and proximity)

The Center of a graph

A vertex v of G is a **central vertex** if $e(v) = rad(G)$. The graph induced by the set of central vertices of G is the **center** $C(G)$ of G. Graphs G where $rad(G) = diam(G)$ are called **self-centered**. A vertex v of G is a **peripheral vertex** if $e(v) = diam(G)$. The graph induced by the set of peripheral vertices of G is the **periphery** $P(G)$ of G.

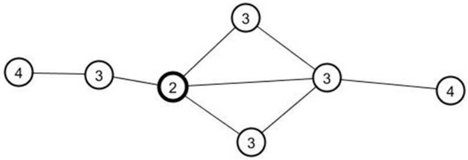


Figure : A graph with different eccentricities and marked center.

The centroid of a graph

For a tree T, the **branch weight** of vertex u, denoted by $bw(u)$, is the maximum number of edges in a subtree with u as an endpoint. The **branch weight centroid** of T is $\{u \in V(T) : bw(u) \leq bw(v), \text{for all} v \in V(T)\}$ and the **centroid** is the subgraph induced by the set of branch weight centroid vertices.

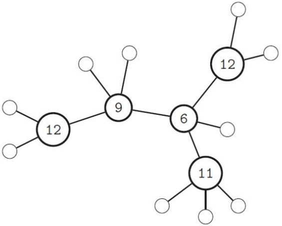


Figure : Here maximum branch weight of vertices are given and the centroid is the vertex with minimum branch weight

Median and anti-median of a graph

The **status** of a vertex $v \in V(G)$, denoted by $S_G(v)$ or $S(v)$, is the sum of the distances from v to all other vertices in G. The subgraph induced by the vertices of minimum (maximum) status in G is called the median (anti-median) of G, denoted by $M(G)(AM(G))$.

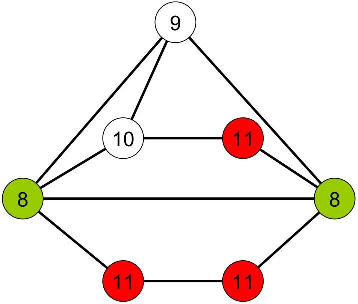
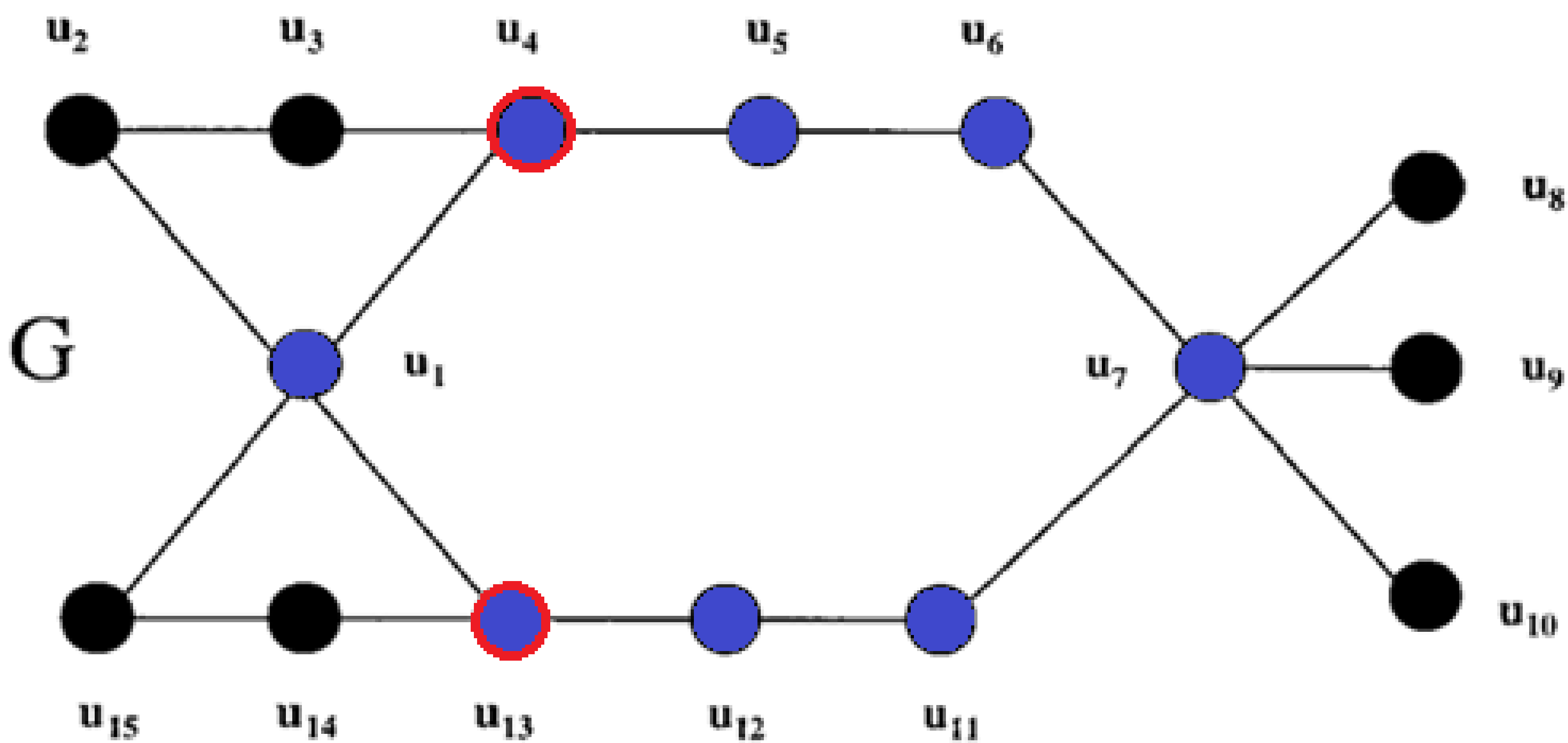


Figure : A graph with connected median and disconnected anti-median.



A graph with median $\{u_4, u_{13}\}$ and centroid $\{u_1, u_4, u_5, u_6, u_7, u_{11}, u_{12}, u_{13}\}$.

The extremal properties for minimum status of trees

C. Liang, B. Zhou, H. Guo, Minimum status, matching and domination of graphs, Comp. J. (2020), doi: 10.1093/comjnl/bxaa057.

Maximum status are also studied

R. Rissner , R.E. Burkard , Bounds on the radius and status of graphs, Networks 64 (2014) 76 - 83.

M. Aouchiche , P. Hansen , Proximity, remoteness and girth in graphs, Discrete Appl. Math. 222 (2017) 31 - 39

The status difference

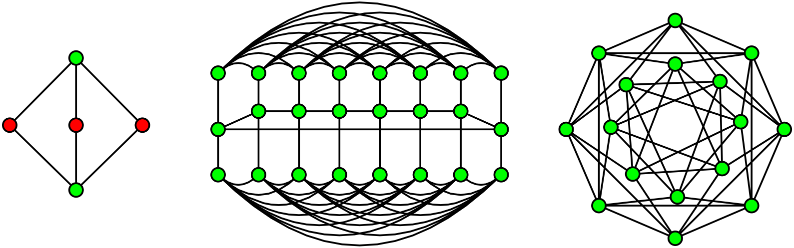
A connected graph may be used to represent a complex network. In the context of a communication network, the status of a vertex is interpreted as the contribution of the vertex to the communicational cost of the network and is used to measure its closeness centrality in the network, while the status difference may be used as a network descriptor to measure the communicational cost of the whole netork.

D. Vukičević , G. Caporossi , Network descriptors based on betweenness centrality and transmission and their extremal values, Discrete Appl. Math. 161 (2013) 2678 - 2686 .

Lin, Hongying, and Bo Zhou. "Which numbers are status differences?." Applied Mathematics and Computation 399 (2021): 126004.

Self centered and self median graphs

- A graph is self-centered if $C(G) \cong G$ [F.Buckley & F.Harary(1990)].
- A graph is self-median if $M(G) \cong G$ [G. Sabidussi(1966)].



1. A graph with both median and anti-median disconnected.
2. A non-regular self-median graph.
3. Identify this (self-median) graph. (Here all graphs are self-centered.)

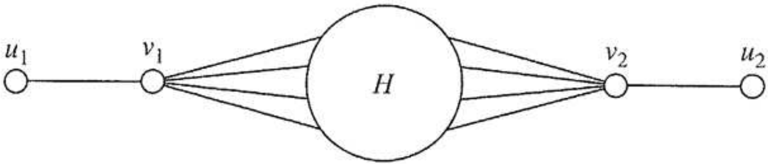
[G. Sabidussi(1966)] G.Sabidussi, The centrality index of a graph, Psychometrica 31(1966)581-603.

The median of graph was introduced in [C.Jordan(1869)],where it has been proved that the median of trees is K_1 or K_2 . In [M. Truszczyǹski(1985)], it was shown that the median of any graph G must lie in some block of G. The anti-median of a graph is discussed in [H.Bielak & M.M.Syslo(1983)].

[C.Jordan(1869)] C.Jordan, Sur les assemblages de lignes, J. Reine Angew. Math, 70(1869), 185-190.
[M. Truszczyǹski(1985)] M. Truszczyǹski, Centers and centroids of unicyclic graphs, Math.Slovaca, 35 (1985), 223-228.
[H.Bielak & M.M.Syslo(1983)] H.Bielak, M.M.Syslo; Peripheral vertices in graphs, Studia Sci. Math. Hungar, 18(1983), 269-275.

Existence of Graphs with prescribed facility structure?

- Hedetniemi (See [F.Buckley, et.al.(1981)]) showed that every graph is the center of some connected graph; that is, for every graph H, there exists a connected graph G such that $C(G) \cong H$.
- Not every graph is the periphery of some graph, however, [H.Bielak & M.M.Syslo(1983)] proved that a graph G is the periphery of some connected graph if and only if no vertex of G has eccentricity 1 or all vertices of G have eccentricity 1.



[F.Buckley, et.al.(1981)]F. Buckley, Z. Miller and P.J. Slater, On graphs containing a given graph as a center. J. Graph Theory 5 (1981)427-434.

[H.Bielak & M.M.Syslo(1983)] H.Bielak, M.M.Syslo; Peripheral vertices in graphs, Studia Sci. Math. Hungar, 18(1983), 269-275.

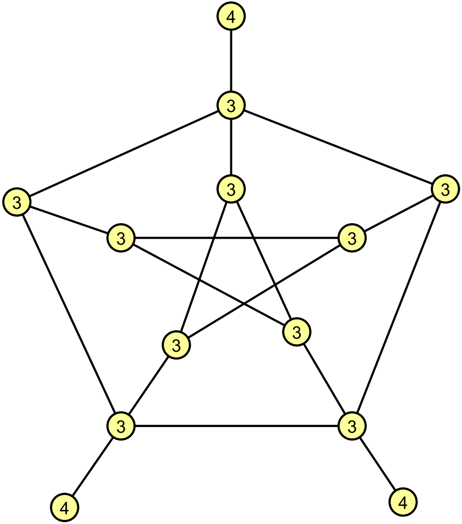


Figure : A graph with Peterson graph as center

The median problem

Given a graph H, the problem of finding a graph G such that $M(G) \cong H$ is referred to as the median problem.

Significance of the problem

The median vertices represent facility locations with minimum average distance. In network theory the median problem is significant as it is related to the optimization problems involving the placement of network servers, the core of the entire networks, specially in very large interconnection networks.

Solutions on the median problem

- [Slater(1980)] Constructed a graph H with $M(H) \cong G$ using $|V_G|^3$ vertices.
- [Z.Miller(1983)] Improved the number of vertices required to $2|V_G|$.
- [G.R.T. Hendry(1985)] Improved this to $2|V_G| - \delta(G) + 1$.

- [H.Bielak & M.M.Syslo(1983)] Proved that any graph G is the anti-median of some connected graph.

[Slater(1980)] P.J.Slater; Medians of arbitrary graphs, J. Graph Theory, 4 (1980), 389-392.
[H.Bielak & M.M.Syslo(1983)] H.Bielak, M.M.Syslo; Peripheral vertices in graphs, Studia Sci. Math. Hungar, 18(1983), 269-275.
[G.R.T. Hendry(1985)] G.R.T. Hendry; On graphs with prescribed median I, J. Graph Theory, 9(1985), 477-481.
[Z.Miller(1983)] Z.Miller; Medians and distance sequences in graphs,Ars Combin, 15(1983), 169-177.

Limitation of general solutions

The median constructions for general graphs cannot be directly applied to many networks as their underlying graph belong to different classes of graphs.

Median problem in different graph classes

Solutions for median problems can be found for

- Ptolemaic graphs in [J. Nieminen(1988), S.V. Yushmanov(1988)]
- Distance-hereditary graphs in [H.G.Yeh & G.J.Chang(2003)]
- Cographs in [S.B.Rao & A.Vijayakumar (2008)]
- k-partite graphs in [K. Pravas & A. Vijayakumar(2015)]
- Symmetric bipartite graphs in [K. Pravas & A. Vijayakumar(2017)]

[J. Nieminen(1988)] J. Nieminen, The center and the distance center ofa Ptolemaic graph, Oper. Res. Lett. 7 (1988) 91-94.
[S.V. Yushmanov(1988)] S.V. Yushmanov; On the median of Ptolemaic graphs, Issled Oper ASU 32 (1988), 67-70.
[H.G.Yeh & G.J.Chang(2003)] H.G.Yeh, G.J.Chang; Centers and medians of distance-hereditary graphs, Discrete Math., 265 (2003), 297-310.
[S.B.Rao & A.Vijayakumar (2008)] S.B.Rao, A.Vijayakumar; On the median and the anti-median of a cograph, Int. J. Pure Appl. Math., 46, 5 (2008), 703 - 710.

Distance-hereditary

A graph G is distance-hereditary if for any induced subgraph H, $d_H(u,v) = d_G(u,v)$, for any $u, v \in V(H)$ [H. Bandelt & H.M. Mulder (1986)].

Chordal

A graph G is chordal if every cycle of length at least four in G has an edge(chord) joining two non-adjacent vertices of the cycle [R.Balakrishnan & K.Ranganathan(2012)].

Ptolemaic

A graph is Ptolemaic if it is both distance-hereditary and chordal [E. Howorka(1977)].

[H. Bandelt & H.M. Mulder (1986)] H. Bandelt and H.M. Mulder; Distance-hereditary graphs, J. Combin.Theory Ser. B 41(1986)182-208.
[R.Balakrishnan & K.Ranganathan(2012)] R.Balakrishnan, K.Ranganathan; A Textbook of Graph Theory, second edition, Heidelberg: Springer (2012).
[E. Howorka(1977)] E. Howorka; A characterization of distance-hereditary graphs, Quart.J.Math.Oxford.Ser. 28 (1977)417-420.

Median problem in different graph classes

Ptolemaic graphs:[S.V. Yushmanov(1988)]

- The median of a Ptolemaic graph is complete. That is, it is not possible to find a ptolemaic graph H such that $M(H) \cong G$, when G is not complete.
- Every connected component of the subgraph induced by the antimedian of a Ptolemaic graph is complete.

2-trees

A 2-tree is a graph formed by starting with a K_3 and then repeatedly adding vertices in such a way that each added vertex u has exactly two neighbors v and w such that uvw forms a K_3.

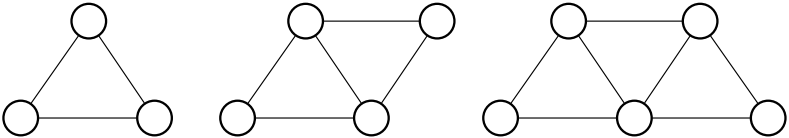


Figure : Examples of 2-trees

Theorem [Slater(1980)]

The median of a 2-tree is isomorphic to K_1, K_2 or K_3.

Cographs

Complement-reducible graphs, or cographs, are the graphs belonging to the following recursively defined family:

1. K_1 is a cograph,
2. If G is a cograph, then so is its complement $\bar{G}$, and
3. If G and H are cographs, then so is their disjoint union $G \cup H$.

A graph is a cograph if and only if it does not have the path P_4 as an induced subgraph.

[T.A. Mckee(2000)]T.A. Mckee, Dimensions for cographs, Ars.Combin.56(2000), 85-95.

Median problem in different graph classes

Cographs:[S.B.Rao & A.Vijayakumar (2008)]

- The median graph of a connected cograph is the subgraph induced by the vertices of maximum degree in G.
- If M_1 and M_2 are the median graphs of connected cographs G_1 and G_2 respectively, then $M_1 \cup M_2$ is also the median graph of a connected cograph.
- If there is an Eulerian cograph G of order p such that $M(G) = H$ then there exists an Eulerian cograph G of even order such that $M(G) = H$.
- Every cograph G is the median graph of some connected, Eulerian cograph.

Median problem in different graph classes

Cographs:[S.B.Rao & A.Vijayakumar (2008)]

- Any cograph H is the antimedian graph of a connected Eulerian cograph G of even order.
- Let G_1 and G_2 be two cographs. Then there is a Hamiltonian, Eulerian cograph G such that $M(G) = G_1$ and $AM(G) = G_2$.

- The median graph of a planar, connected cograph is one of the following graphs $K_1, K_2, K_3, K_4, \bar{K}_2, K_4 - e, C_4, K_{1,2}, C_4 + \bar{K}_2$.
- The antimedian graph of a planar, connected cograph is one of the following graphs $K_1, K_2, K_3, K_4, \bar{K}_n, nK_2, C_4, C_4 + \bar{K}_2$.

$G + H$ is the join of the graphs G and H.

k-partite graphs

A graph G is k-partite if $V(G)$ can be partitioned into k nonempty subsets such that no edge in G has its both ends in the same subset. G is bipartite when $k = 2$.

k-partite graphs

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Bi-partite graphs as underlying graphs

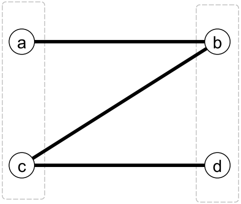
It can be seen that most of the Very Large Scale Interconnection networks are bipartite. In particular, citation networks, recommendation systems in online purchasing, protein interaction networks and movie-actor networks in social networks are all bipartite. Also, most of the analysis in network communities are done using preference networks [H.Kautz, et.al.(1997)] and they are modeled using bipartite graphs.

[H.Kautz, et.al.(1997)] H.Kautz, B. Selman and M. Shah; Referral Web: combining social networks and collaborative filtering, Communications of the ACM 40(3)(1997)63 - 65.
[J. Xu(2002)] J. Xu; Topological structure and analysis of interconnection networks, Kluwer Academic,2002.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.

Example: Construct H', when $G \cong P_4$



[K. Pravas & A. Vijayakumar(2015)]

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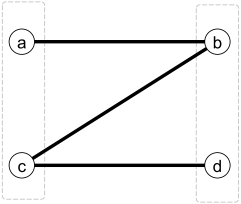
Example



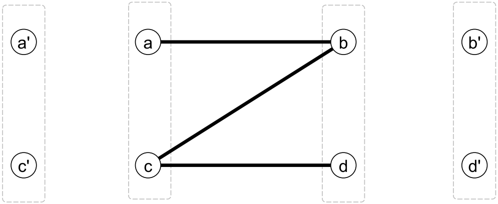
Construction

Let X, Y be a bipartition of $V(G)$.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.

Example



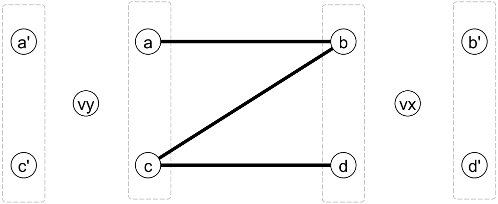
Construction

Let X', Y' be the copy of X, Y such that v' denote the copy of a vertex $v \in V(G)$.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.

Example



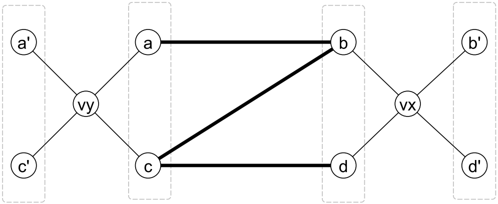
Construction

Consider two new vertices v_x and v_y.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.

Example



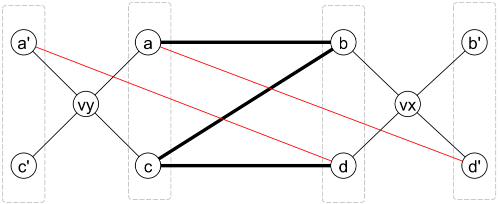
Construction

Make v_y adjacent to $X \cup X'$ and v_x adjacent to $Y \cup Y'$.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.

Example



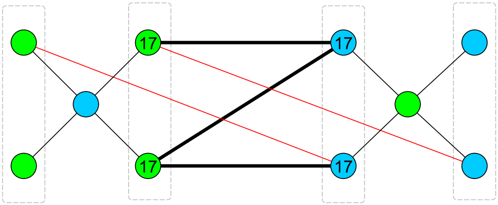
Construction

Also, for each $v \in X$ (Y) make v' adjacent to $Y \backslash N(v)$ $(X \backslash N(v))$. Call this graph H'.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a connected bipartite graph H' such that G is an induced subgraph of H' and all the vertices of G in H' have equal status in H'.

Example



Construction

It follows that H' is bipartite and $S_{H'}(v) = 4n + 1, \forall\ v \in V(G)$.
The graph H' is called the bipartite gadget graph of G.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a bipartite graph H such that $M(H) \cong G$.

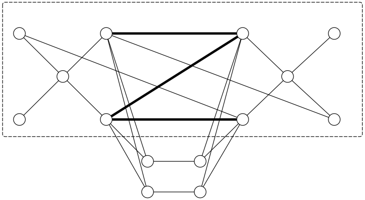


Figure : A graph with P_4 as the median. Here, the subgraph in the dotted box is the bipartite gadget graph of P_4.

[K. Pravas & A. Vijayakumar(2015)]

Given a bipartite graph G, there exists a bipartite graph H such that $AM(H) \cong G$

[K. Pravas & A. Vijayakumar(2015)]

Given a k-partite graph G, there exists a k-partite graph H such that $M(H) \cong G$.

[K. Pravas & A. Vijayakumar(2015)]

Given G, a k-partite graph, there exists a k-partite graph H' such that $AM(H') \cong G$

Equivalently...

Any k-colorable graph is the median of a k-colorable graph G and anti-median of a k-colorable graph H.

Inserting a new vertex in constructions

The constructions of a graph with prescribed median naturally faces the following problem. The addition of a vertex in any part of the graph changes the status of each vertex in that graph, thus changing the median preferences in that graph.

Embedding another graph in our constructions

In our constructions, it is possible to embed another k-partite graph as the center of the newly constructed graph keeping the median same in the graphs, which are obtained using previous constructions.

[K. Pravas & A. Vijayakumar(2015)]

Given two bipartite graphs G and J, there are bipartite graphs H and H' such that $M(H) \cong G$, $C(H) \cong J$, $AM(H') \cong G$ and $C(H') \cong J$.

Median and Center embedding in k-partite graphs [K. Pravas & A. Vijayakumar(2015)]

Given two k-partite graphs G and J, there is a k-partite graph W such that $M(W) \cong G$ and $C(W) \cong J$.

Anti-median and Center embedding in k-partite graphs [K. Pravas & A. Vijayakumar(2015)]

Given two k-partite graphs G and J, there is a k-partite graph W' such that $AM(W') \cong G$ and $C(W') \cong J$.

Median problem in symmetric bipartite graphs

Symmetric bi-partite graphs

A bipartite graph G is symmetric if for a bi-partition (X, Y) of G, there is a map f from X onto Y such that for every edge $(u, f(v))$ in G, there is an edge $(v, f(u))$ in G, where $u, v \in X$.

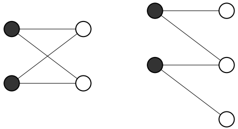


Figure : Here, the graph on left is a symmetric bipartite.

Symmetric bipartite graphs with prescribed center and median[K. Pravas & A. Vijayakumar(2017)]

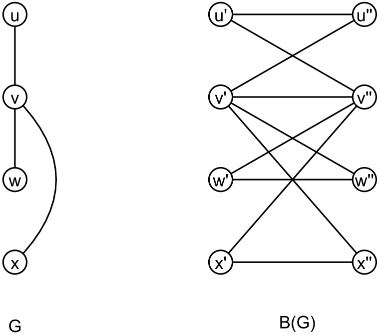
Given two symmetric bipartite graphs G and J there exists a symmetric bipartite graph H with $\mathrm{M}(H) \cong G$ and $C(H) \cong J$.

Symmetric bipartite graphs with prescribed center and ant-median[K. Pravas & A. Vijayakumar(2017)]

Given two symmetric bipartite graphs G and J there exists a symmetric bipartite graph H with $\mathrm{AM}(H) \cong G$ and $C(H) \cong J$.

Bipartite graph $B(G)$ of a graph [R.Balakrishnan & K.Ranganathan(2012)]

The bipartite graph $B(G)$ of a graph G can be constructed as follows[R.Balakrishnan & K.Ranganathan(2012)]. For each vertex $v \in V$, form $v' \in X$ and $v'' \in Y$ and let $N(v') = \{u'' \in Y : u \in N[v]\}$ and $N(v'') = \{u' \in X : u \in N[v]\}$.



[R.Balakrishnan & K.Ranganathan(2012)] R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Second edition, Heidelberg: Springer (2012).

[K. Pravas & A. Vijayakumar(2017)]

The operator $B(\cdot)$ commute with both median operators. That is, $B(M(G)) \cong M(B(G))$ and $B(AM(G)) \cong AM(B(G))$.

Median,anti-median and center embedding with B(.) [K. Pravas & A. Vijayakumar(2017)]

Let $G' \cong B(G)$ and $J' \cong B(J)$ be two connected graphs. Then the following results hold.

1. There exist graphs H_1 and H_1' such that $M(H_1') = G'$ and $C(H_1') = J'$ and $H_1' \cong B(H_1)$.
2. There exist graphs H_2 and H_2' such that $AM(H_2') = G'$ and $C(H_2') = J'$ and $H_2' \cong B(H_2)$.

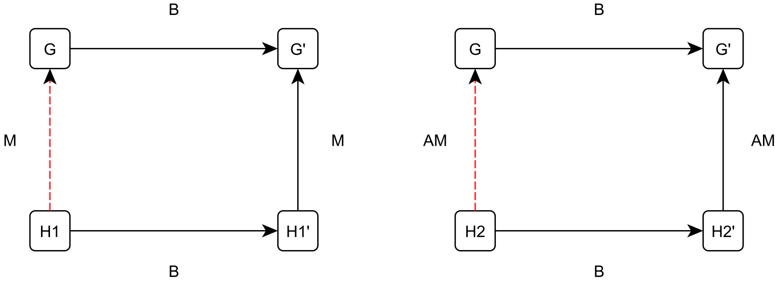
A generalised result

[K. Pravas & A. Vijayakumar(2017)][a]

[a] This result is independently obtained using B(.) operator

Let G and J be two connected graphs. Then,

1. There exist a graph H_1 such that $M(H_1) \cong G$ and $C(H_1) \cong J$.
2. There exist a graph H_2 such that $AM(H_2) \cong G$ and $C(H_2) \cong J$.



Square of a graph

The square G^2 of a graph G has the same vertex set as G and two vertices $u, v \in V(G^2)$ are adjacent if $d_G(u, v) \leq 2$.

Square-subgraph of a graph

A subgraph H of G is a square-subgraph of G if $H^2 \cong G^2[V(H)]$.

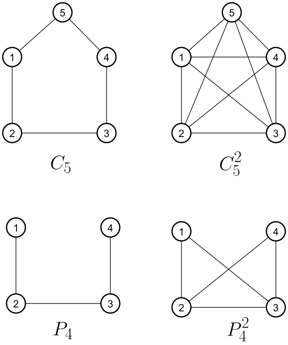


Figure : P_4 is not a square-subgraph of C_5

When do square operator commute with both median operators?

[K. Pravas & A. Vijayakumar(2017)]

Let G be a graph such that the number of vertices at odd distance from u is a constant for all $u \in V(G)$. If $M(G)$ and $AM(G)$ are square-subgraphs of G, then $M(G^2) = (M(G))^2$ and $AM(G^2) = (AM(G))^2$.

Median problem on square of graphs[K. Pravas & A. Vijayakumar(2017)]

Let G be a bipartite graph with bi-partition (X, Y) and $|X| = |Y|$, then there are bipartite graphs H_1 and H_2 such that Median set of H_1^2 is G^2 and Anti-median set of H_2^2 is G^2.

Corollary

Let G be a bipartite graph with bi-partition (X, Y) and $|X| = |Y|$, then there are bipartite graphs H_1 and H_2 such that Median set of H_1^2 is G^2 and Anti-median set of H_2^2 is G^2.

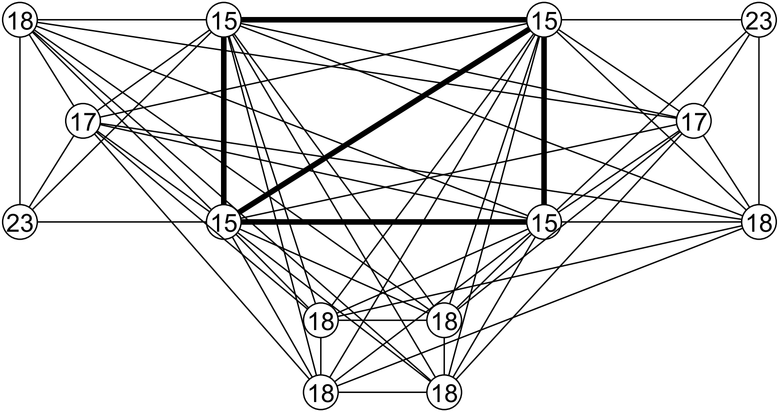


Figure : $M(G^2) \cong (M(G))^2 \cong P_4^2 \cong K_4 - e$.

Convex median and anti-median at prescribed distance

Motivation

In network terms, the concept of median of a graph generalises the locations of desired facilities such as servers or data centers and for anti-median, the obnoxious or disposal/ dumping centers of different forms of data. A convex structure ensures the existence of a safe path (a path within the structure) for any data exchange between the nodes.

So we ask the following question: How close these facility locations can be made?

Notations

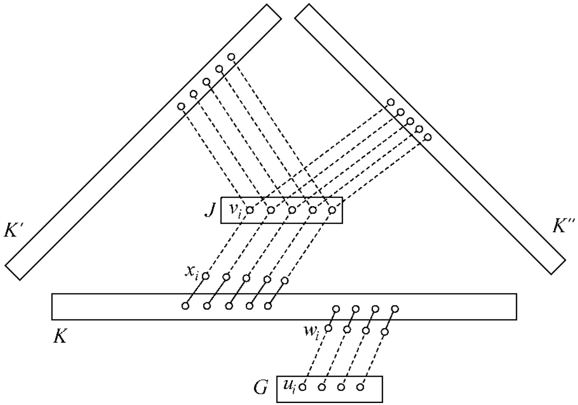
- $d_G(u, v)$ is the length of a shortest path between u and v.
- The distance between two subgraphs G_1 and G_2 of G is $d(G_1, G_2) = \underset{\substack{x \in G_1 \\ y \in G_2}}{\text{Min}}\, d_G(x, y)$.
- The status difference of G, $SD(G)$, is the maximum value of $|S_G(u) - S_G(v)|$ for all u and v in G.

[K. Pravas & A. Vijayakumar(2017)]

The maximum SD of a graph on n vertices is $\frac{n^2-2n+1}{4}$, when n is odd, and ,$\frac{n^2-2n}{4}$ when n is even.

[K.Balakrishnan, et.al.(2010)]

Given two graphs J and G, there exist a graph H with $M(H) = J, AM(H) = G$ and $d(J, G) = r$, for every $r \geq 2$.



Convex embedding

[P.Dankelmann & G.Sabidussi(2008)]

- Given a graph G, there exists a graph H such that $M(H) = G$ and G is an isometric subgraph of H.
- Then number of vertices used in this construction is $O(2\text{Diam}(G)^n)$, where n is the number of vertices in G.

The number of vertices used in this construction is improved in [K.Balakrishnan, et.al.(2010)]

For a positive integer r, let $H = (G_1, G_2, r)$ denote a graph with $d_H(G_1, G_2) = r$, $M(H) = G_1$, $AM(H) = G_2$ and both G_1 and G_2 are convex subgraphs of H.

The construction of a (G_1, G_2, r) graph, specially when $r = 1$, shows the existence of two such facilities which can transfer data through a safe path.

[K.Balakrishnan, et.al.(2010)]

(G_1, G_2, r) exists for graphs satisfying $r \geq \lfloor Diam(G_1)/2 \rfloor + \lfloor Diam(G_2)/2 \rfloor + 2$.

[K.Balakrishnan, et.al.(2010)] K.Balakrishnan, B.Brešar, M.Changat, S.Klavžar, M.Kovše and A.R.Subhamathi, Simultaneous embeddings of graphs as median and antimedian subgraphs, Networks, 56(2010), 90 - 94.

[K. Pravas & A. Vijayakumar(2017)]

Let G_1 and G_2 be any two graphs, $r \geq 1$. Then, there exists a graph H_0 with the property that both G_1 and G_2 are convex subgraphs of H_0 and $d_{H_0}(G_1, G_2) = r$.

[K. Pravas & A. Vijayakumar(2017)]

(G_1, G_2, r) exists for all graphs G_1 and G_2, and for every integer $r \geq 1$.

Problems

- See [J. Xu(2002)] and [L. H. Hsu & C. K. Lin(2009)] for Interconnection networks.
- What about the median problem on these networks?

- See [E. Prisner(1995)] for graph operators.
- Which operators commute with $M(.)$ and $AM(.)$?

Questions?

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Break the chain

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Thank you