Improved Lower Bounds on Multicolor Diagonal Ramsey Numbers

S. Venkitesh

IIT Bombay

April 23, 2021

E-Seminar on Graphs and Matrices

Ramsey numbers

Ramsey number $r(t_1, t_2, \ldots, t_\ell; \ell)$

The least $n \in \mathbb{Z}^+$ such that every ℓ-(edge) labeling of K_n contains a monochromatic K_{t_i}, for some $i \in [\ell]$.

Diagonal Ramsey number $r(t; \ell)$
$(t_1 = \cdots = t_\ell = t)$

The least $n \in \mathbb{Z}^+$ such that every ℓ-labeling of K_n contains a monochromatic K_t.

For $r(t) := r(t; 2)$,

[Erdös] $(1 + o(1))\dfrac{t}{\sqrt{2}e}2^{t/2} < r(t) < 2^{2t}$ [Erdös-Szekeres]

Ramsey numbers

Further bounds:

- $r(t;\ell) < \ell^{\ell t}, \quad r(t;3) > 3^{t/2}$
- $r(t;\ell_1+\ell_2) \geq (r(t;\ell_1)-1)(r(t;\ell_2)-1)$ [Lefmann]
- For $\ell \geq 2$,
$$r(t;\ell) \geq \left(2^{\frac{79\ell}{300}+C}\right)^{t+o(t)},$$
where C is a constant dependent on $\ell \pmod 3$.

Theorem (Conlon, Ferber, 2020)

For $\ell \geq 3$,
$$r(t;\ell) \geq \left(2^{\frac{7\ell}{24}+C}\right)^{t-o(t)},$$
where C is a constant dependent on $\ell \pmod 3$.

Conlon-Ferber Theorem

Theorem (Conlon, Ferber, 2020)

For $\ell \geq 3$, $\quad r(t;\ell) \geq \left(2^{\frac{7\ell}{24}+C}\right)^{t-o(t)}$, where C is a constant dependent on $\ell \pmod 3$.

Follows from [Lefmann] and the following main theorem.

Theorem (Main Theorem, Conlon, Ferber, 2020)

For any prime p, $\quad r(t;p+1) > 2^{t/2}p^{3t/8+o(t)}$.

Improvement: $\ell = 3$: from 1.732^t to 1.834^t
$\ell = 4$: from 2^t to 2.135^t

Main Theorem

Theorem (Conlon, Ferber, 2020)

For any prime p, $\quad r(t; p+1) > 2^{t/2} p^{3t/8+o(t)}$.

We need to prove that there is a $(p+1)$-labeling of K_n, $n = 2^{t/2} p^{3t/8+o(t)}$ that does not contain a monochromatic K_t. We will show that for a random $(p+1)$-labeling,

$$\mathbb{P}\big(\exists \text{ monochromatic } K_t\big) < 1.$$

This proves the theorem.

Proof of Main Theorem

Let p be a prime and

$$V = \{v = (v_1, \ldots, v_t) \in \mathbb{F}_p^t : v \cdot v = v_1^2 + \cdots + v_t^2 = 0\} \subseteq \mathbb{F}_p^t.$$

Fact. $\forall\, a \in \mathbb{F}_p$, $\exists\, b, c \in \mathbb{F}_p : a = b^2 + c^2$. *Proof.*

Assume $p > 2$. Let $S_p = \{a^2 : a \in \mathbb{F}_p\}$. Then $|S_p| = (p+1)/2$.

Lemma. (*Cauchy-Davenport Theorem*) For any $A, B \subseteq \mathbb{F}_p$, $|A + B| \geq \min\{p, |A| + |B| - 1\}$.

Thus $|S_p + S_p| \geq p$. $\square$ So for any $v_1, \ldots, v_{t-2} \in \mathbb{F}_p$, there

exist $v_{t-1}, v_t \in \mathbb{F}_p$ such that $-\left(v_1^2 + \cdots + v_{t-2}^2\right) = v_{t-1}^2 + v_t^2$. Thus $p^{t-2} \leq |V| \leq p^t$.

Proof of Main Theorem

We have $V = \{v =\in \mathbb{F}_p^t : v \cdot v = 0\}$ and $p^{t-2} \leq |V| \leq p^t$. We now label $E(K_V)$.

- If $u \cdot v = i \neq 0$, then set $\chi(uv) = i$.
- If $u \cdot v = 0$, then set $\chi(uv) \in \{p, p+1\}$ independently and uniformly at random.

Labels in $[p-1]$; easy. There is no monochromatic K_s with label $i \in [p-1]$, for any $s > t$. This follows by taking the vertex set $\{y_1, \ldots, y_s\}$ of K_s and observing that it is linearly independent.

Proof of Main Theorem

Labels in $\{p, p+1\}$: Define $X \subseteq V$ to be a *potential clique* if $|X| = t$ and $u \cdot v = 0$ for all $u, v \in X$. Let M_X be the matrix formed by taking vectors in X as rows of M_X. Then $M_X M_X^T = 0$. Let X be a potential clique with rank r

and suppose the first r rows of M_X are linearly independent. The number of such X is at most the number of $t \times t$ matrices M_X of rank r having first r rows linearly independent. The number of such matrices is

$$\left(\prod_{i=0}^{r-1} p^{t-i}\right) \cdot p^{(t-r)r} = p^{tr - \binom{r}{2} + tr - r^2} = p^{2tr - \frac{3r^2}{2} + \frac{r}{2}}.$$

Proof of Main Theorem

So the number of potential cliques N_t is at most

$$p^{2tr-\frac{3r^2}{2}+\frac{r}{2}} \leq p^{\frac{5t^2}{8}+o(t^2)}, \quad \text{maximizing at } r = t/2.$$

Now for $n = 2^{t/2}p^{3t/8+o(t)}$, let $\alpha = n/2|V| = np^{-t+O(1)}$. Choose a random subset of V by picking each element of V independently with probability α. The expected number of monochromatic potential cliques is

$$\begin{aligned} \alpha^t 2^{1-\binom{t}{2}} N_t &\leq p^{-t^2+o(t^2)} n^t 2^{-t^2/2+o(t^2)} p^{5t^2/8+o(t^2)} \\ &= \left(2^{-t/2}p^{-3t/8+o(t)}n\right)^t < 1/2. \end{aligned}$$

So there is a choice of subset of size n such that there is no monochromatic potential clique. $\square$

References

- David Conlon, Asaf Ferber. Lower bounds for multicolor Ramsey numbers. https://arxiv.org/abs/2009.10458.
- Yuval Wigderson. An improved lower bound on multicolor Ramsey numbers. https://arxiv.org/abs/2009.12020.

Thank you!