Distance Matrix of a Multi-block Graph: Determinant and Inverse

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Notations and Definitions

Let $G = (V(G), E(G))$ be a finite, simple, connected graph with $V(G)$ as the set of vertices and $E(G) \subset V(G) \times V(G)$ as the set of edges in G.

- We simply write $G = (V, E)$ if there is no scope of confusion.
- We write $i \sim j$ to indicate that the vertices $i, j \in V$ are adjacent in G.
- The degree of the vertex i, denoted by δ_i, equals the number of vertices in V that are adjacent to i.

Notations and Definitions

Definition

Let G be a graph with n vertices. The adjacency matrix of G is an $n \times n$ matrix, denoted as $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j, i \sim j \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

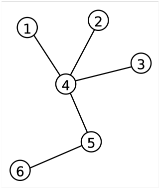

Figure: G

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Notations and Definitions

Definition

Let G be a graph with n vertices. The Laplacian matrix of G is an $n \times n$ matrix, denoted as $L(G) = [l_{ij}]$, where

$$L(G) = \delta(G) - A(G),$$

where $\delta(G) = \text{diag}(\delta_1, \delta_2, \cdots, \delta_n)$.

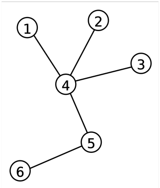

Figure: G

$$L(G) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

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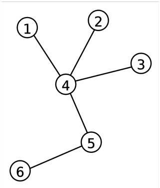

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Note that, $L(G)$ is a symmetric, positive semi-definite matrix. The constant vector $\mathbf{1}$ is the eigenvector of $L(G)$ corresponding to the smallest eigenvalue 0 and hence satisfies $L(G)\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^t L(G) = \mathbf{0}$

Notations and Definitions

A connected graph G is a metric space with respect to the metric d, where $d(i,j)$ equals the length of the shortest path between vertices i and j.

Definition

Let G be a graph with n vertices. The distance matrix of graph G is an $n \times n$ matrix, denoted by $D(G) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} d(i,j) & \text{if } i \neq j,\ i,j \in V, \\ 0 & \text{if } i = j,\ i,j \in V. \end{cases}$$

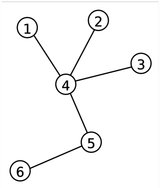

Figure: G

$$D(G) = \begin{pmatrix} 0 & 2 & 2 & 1 & 2 & 3 \\ 2 & 0 & 2 & 1 & 2 & 3 \\ 2 & 2 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 0 & 1 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Results on Distance Matrix for Tree

Theorem[Graham et. al., 1971]

Let T be a tree on n vertices. The determinant of the distance matrix of T is given by

$$\det D(T) = (-1)^{n-1}(n-1)2^{n-2}.$$

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$$D(T)^{-1} = -\frac{1}{2}L(T) + \frac{1}{2(n-1)}\tau\tau^T,$$

where $\tau = (2-\delta_1, 2-\delta_2, ..., 2-\delta_n)^T$ is a column vector.

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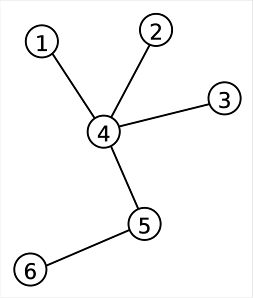
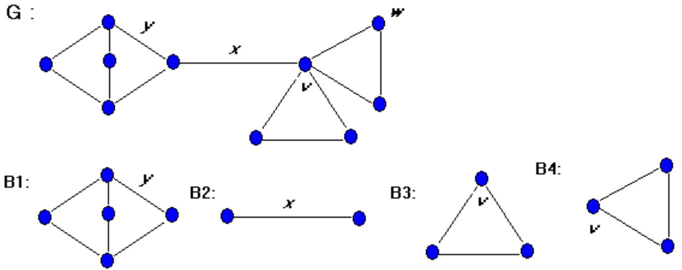


Figure: T

Cut Vertex and Block

Definition

A vertex v of a connected graph G is a cut vertex of G if $G - v$ is disconnected. A block of the graph G is a maximal connected subgraph of G that has no cut-vertex.



Few Graphs of Our Interest

Definition

A graph with n vertices is called complete, if each vertex of the graph is adjacent to every other vertex and is denoted by K_n.

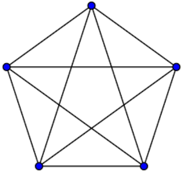


Figure: K_5

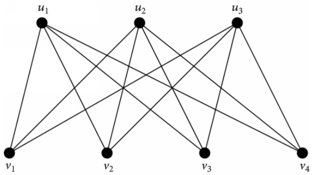


Figure: $K_{3,4}$

Definition

A graph $G = (V, E)$ said to be bipartite if V can be partitioned into two subsets V_1 and V_2 such that $E \subset V_1 \times V_2$. A bipartite graph $G = (V, E)$ with the partition V_1 and V_2 is said to be a complete bipartite graph, if every vertex in V_1 is adjacent to every vertex of V_2. If $|V_1| = n_1$ and $|V_2| = n_2$, the complete bipartite graph is denoted by K_{n_1,n_2}.

Definition

For $m \geq 2$, a graph is said to be m-partite if the vertex set can be partitioned into m subsets $V_i,\ 1 \leq i \leq m$ with $|V_i| = n_i$ and $|V| = \sum_{i=1}^{m} n_i$ such that $E \subset \bigcup_{\substack{i,j \\ i \neq j}} V_i \times V_j$. A m-partite graph is said to be a complete m-partite graph, denoted by $K_{n_1, n_2, \cdots, n_m}$ if every vertex in V_i is adjacent to every vertex of V_j and vice versa for $i \neq j$ and $i, j = 1, 2, \ldots, m$.

Existing Results and Our Aim

In literature the following graphs has been studied.

- Block graph (Bapat et. al., 2011) [each of its blocks is a complete graph].
- Cycle-clique graph (Hou et. al. 2015) [each of its blocks is either a cycle or a complete graph].
- Cactoid graph (Hou et. al. 2015) [each of its blocks is a oriented cycle].
- Bi-block graph(Hou et. al. 2016) [each of its blocks is a complete bipartite graph].
- Weighted cactoid graph (Zhou et. al. 2019) [each of its blocks is a oriented weighted cycle].

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Our Aim: To compute the determinant and inverse of the distance matrix for graphs where each of its block is a complete m-partite graph; $m \geq 2$, we call such graphs multi-block graph.

A Rough Sketch of the Main Result

Given an $n \times n$ matrix B, we define $B(i \mid j)$ to be the matrix obtained from B by deleting the i^{th} row and j^{th} column. For $1 \leq i, j \leq n$, the cofactor c_{ij} is defined as

$$c_{ij} = (-1)^{i+j} \det B(i \mid j).$$

We use the notation cof B to denote the sum of all cofactors of B, *i.e.*,

$$\text{cof } B = \sum_{1 \leq i,j \leq n} c_{ij}.$$

Theorem (Graham et. al., 1977)

Let G be a connected graph with blocks $G_1, G_2, \cdots, G_b$. Then

$$\text{cof } D(G) = \prod_{i=1}^{b} \text{cof } D(G_i),$$

$$\det D(G) = \sum_{i=1}^{b} \det D(G_i) \prod_{j \neq i} \text{cof } D(G_j).$$

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Our Aim:

Let G be a multi-block graph. Then, the inverse of the distnce matrix of G is given by

$$D(G)^{-1} = -\mathcal{L}_G + \frac{1}{\lambda_G}\mu_G\mu_G^t$$

where

- The matrix $\mathcal{L}$ satiesfies $\mathcal{L}\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^t\mathcal{L}_G = \mathbf{0}$ and is a called Laplacian-like matrix.
- μ_G is a column vector
- λ_G a suitable constant.

A Rough Sketch of the Main Result

We need to find $\mathcal{L}_G, \mu_G, \lambda_G$ satisfying the following.

1. $\det D(G) \neq 0$ iff $\lambda_G \neq 0$.
2. $D(G)\mu_G = \lambda_G \mathbf{1}$.
3. $\mathcal{L}_G D(G) + I = \mu_G \mathbf{1}^t$

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By (1) and (2), we have $\mu_G \mathbf{1}^t = \frac{1}{\lambda_G}\mu_G \mu_G^t D(G)$. Next by (3), we have

$$\begin{aligned} & \mathcal{L}_G D(G) + I = \frac{1}{\lambda_G}\mu_G \mu_G^t D(G) \\ \Rightarrow \quad & \mathcal{L}_G + D(G)^{-1} = \frac{1}{\lambda_G}\mu_G \mu_G^t \\ \Rightarrow \quad & D(G)^{-1} = -\mathcal{L}_G + \frac{1}{\lambda_G}\mu_G \mu_G^t. \end{aligned}$$

Given a connected graph G, we are looking for a tuple $(D(G), \mathcal{L}_G, \mu_G, \lambda_G)$ satisfies the above conditions.

$$G \to (D(G), \mathcal{L}_G, \mu_G, \lambda_G).$$

A Rough Sketch of the Main Result

Theorem[Zhou et. al., 2017]

Let G be a connected graph with blocks $G_1, G_2, \cdots, G_b$. For $1 \leq t \leq b$,, we search of

$$G_t \rightarrow (D(G_t), \mathcal{L}_{G_t}, \mu_{G_t}, \lambda_{G_t}) \text{ with } \mathbf{1}^t \mu_{G_t} = 1.$$

Then

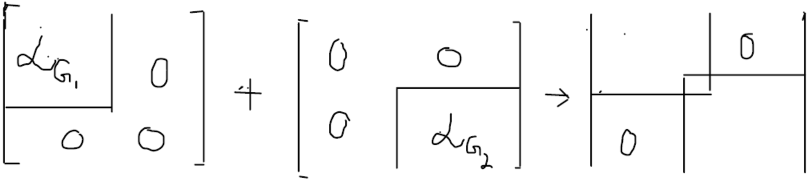
$$G \rightarrow (D(G), \mathcal{L}_G, \mu_G, \lambda_G),$$

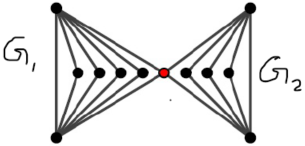
where

$$\lambda_G = \sum_{t=1}^{b} \lambda_{G_t},$$

$$\mu_G(v) = \sum_{t=1}^{b} \mu_{G_t}(v) - (k-1), \text{ if vertex } v \text{ belongs to } k \text{ many blocks of } G.$$

$$\mathcal{L}_G = \sum_{t=1}^{b} \mathcal{L}_{G_t}.$$





$$\mu_{G_1} = (\mu_1^{(1)}, \ldots, \mu_7^{(1)}), \quad \mu_{G_2} = (\mu_1^{(2)}, \ldots, \mu_6^{(2)})$$

$$\mu_G = (\mu_1^{(1)}, \ldots, \mu_6^{(1)}, \mu_7^{(1)} + \mu_1^{(2)} - 1, \mu_2^{(2)}, \ldots, \mu_6^{(2)})$$

A Rough Sketch of the Main Result

Theorem (Graham et. al., 1977)

Let G be a connected graph with blocks $G_1, G_2, \cdots, G_b$. Then

$$\text{cof } D(G) = \prod_{i=1}^{b} \text{cof } D(G_i),$$

$$\det D(G) = \sum_{i=1}^{b} \det D(G_i) \prod_{j \neq i} \text{cof } D(G_j).$$

Observe that, if $\text{cof } D(G_t) \neq 0$ for all $t = 1, 2, \ldots, b$, then

$$\det D(G) = \left[\sum_{t=1}^{b} \frac{\det D(G_t)}{\text{cof } D(G_t)} \right] \prod_{t=1}^{b} \text{cof } D(G_t) = \left[\sum_{t=1}^{b} \lambda_{G_t} \right] \times \text{cof } D(G).$$

Define $\lambda_G = \sum_{t=1}^{b} \lambda_{G_t}$ with $\lambda_{G_t} = \frac{\det D(G_t)}{\text{cof } D(G_t)}$.

A Rough Sketch of the Main Result

Theorem

Let $D(K_{n_1,n_2,\cdots,n_m})$ be the distance matrix of complete m-partite graph $K_{n_1,n_2,\cdots,n_m}$ on $|V| = \sum_{i=1}^{m} n_i$ vertices. Then

$$\det D(K_{n_1,n_2,\cdots,n_m}) = (-2)^{|V|-m}\left[\sum_{i=1}^{m}\left(n_i\prod_{j\neq i}(n_j-2)\right) + \prod_{i=1}^{m}(n_i-2)\right].$$

$$\operatorname{cof} D(K_{n_1,n_2,\cdots,n_m}) = (-2)^{|V|-m}\left[\sum_{i=1}^{m}\left(n_i\prod_{j\neq i}(n_j-2)\right)\right].$$

Let $G = K_{n_1,n_2,\cdots,n_m}$. Then

$$\lambda_G = \frac{\det D(G)}{\operatorname{cof} D(G)}, \text{ whenever } \operatorname{cof} D(G) \neq 0.$$

A Rough Sketch of the Main Result

Let $n_i \in \mathbb{N}$, $1 \leq i \leq m$ and let us denote

$$\begin{cases} \beta_{n_1 n_2 \cdots n_m} = \sum_{i=1}^{m} n_i \prod_{j \neq i} (n_j - 2) + \prod_{i=1}^{m} (n_i - 2), \\ \beta_{\widehat{n_i}} = \beta_{n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_m}. \end{cases}$$

and

$$\begin{cases} \gamma_{n_1 n_2 \cdots n_m} = \sum_{i=1}^{m} n_i \prod_{j \neq i} (n_j - 2). \\ \gamma_{\widehat{n_i}} = \gamma_{n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_m} \end{cases}$$

The inverse in $m \times m$ block form is given by $D(K_{n_1, n_2, \cdots, n_m})^{-1} = [\widetilde{D}_{ij}]$, where

$$\widetilde{D}_{ij} = \begin{cases} \left(\dfrac{2\beta_{\widehat{n_i}} - \gamma_{\widehat{n_i}}}{2\beta_{n_1 n_2 \cdots n_m}} \right) J_{n_i} - \dfrac{1}{2} I_{n_i} & \text{if } i = j; \\ \\ -\dfrac{\prod_{l \neq i, j} (n_l - 2)}{\beta_{n_1 n_2 \cdots n_m}} J_{n_i \times n_j} & \text{if } i \neq j. \end{cases}$$

A Rough Sketch of the Main Result

Let $G = K_{n_1,n_2,\cdots,n_m}$; $m \geq 2$. Let V_{n_i}; $1 \leq i \leq m$ denote the m-partitions of the vertex set V of G.

- We define a matrix $\mathcal{L}_G = [\mathcal{L}_{uv}]$, called Laplacian-like matrix of $K_{n_1,n_2,\cdots,n_m}$, where

$$\mathcal{L}_{uv} = \begin{cases} \dfrac{(n_i - 1)\beta_{\widehat{n_i}} - 2\gamma_{\widehat{n_i}}}{2\gamma_{n_1 n_2 \cdots n_m}} & \text{if } u = v, u \in V_{n_i}, \text{for } 1 \leq i \leq m; \\ -\dfrac{\beta_{\widehat{n_i}}}{2\gamma_{n_1 n_2 \cdots n_m}} & \text{if } u \neq v, u, v \in V_{n_i}, \text{for } 1 \leq i \leq m; \\ \dfrac{\prod_{l \neq i,j}(n_l - 2)}{\gamma_{n_1 n_2 \cdots n_m}} & \text{if } u \sim v, u \in V_{n_i}, v \in V_{n_j}, \text{ for } 1 \leq i, j \leq m. \end{cases}$$

- We define a $|V|$-dimensional column vector μ_G as follows:

$$\mu_G(v) = \frac{1}{\gamma_{n_1 n_2 \cdots n_m}} \sum_{i=1}^{m} \sum_{v \in V_{n_i}} \prod_{j \neq i} (n_j - 2)$$

Other Results

Theorem

Let $D(K_{n_1,n_2,\cdots,n_m})$ be the distance matrix of complete m-partite graph $K_{n_1,n_2,\cdots,n_m}$ on $|V| = \sum_{i=1}^{m} n_i$ vertices. Then

$$\det D(K_{n_1,n_2,\cdots,n_m}) = (-2)^{|V|-m}\left[\sum_{i=1}^{m}\left(n_i\prod_{j\neq i}(n_j-2)\right)+\prod_{i=1}^{m}(n_i-2)\right].$$

$$\operatorname{cof} D(K_{n_1,n_2,\cdots,n_m}) = (-2)^{|V|-m}\left[\sum_{i=1}^{m}\left(n_i\prod_{j\neq i}(n_j-2)\right)\right].$$

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1. If $n_i > 2$, for all $i = 1, 2, \ldots, m$, then both $\det D(G)$ and cof $D(G) \neq 0$
2. For $1 \leq i \leq m$, if atleast two n_i's are 2, then $\det D(G) = \text{cof } D(G) = 0$.
3. For $1 \leq i \leq m$, if exactly one n_i is 2, then $\det D(G) = \text{cof } D(G) \neq 0$.
4. If $n_i = 1$, for all $i = 1, 2, \ldots, m$, then $G = K_m$ and for $m > 1$, $\det D(G), \text{cof } D(G) \neq 0$.

Other Results

Theorem

Let $m \geq 2$ and $G = K_{n_1, n_2, \cdots, n_m}$. Then, $\det D(G) = 0$ if and only if either of the following holds:

(1) at least two n_i's are 2 for $1 \leq i \leq m$,

(2) there exists $l \in \mathbb{N}$ with $\frac{m+1}{2} < l \leq \frac{3m+1}{4}$ such that $n_i = 1$ for $1 \leq i \leq l$ and $n_i > 2$ for $l+1 \leq i \leq m$ with

$$2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - (m+1).$$

Theorem

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$$2 \sum_{i=l+1}^{m} \frac{1}{n_i - 2} = 2l - m.$$

Other Results

- There are infinitely many complete multipartite graphs G with cof $D(G) \neq 0$ satisfying $\lambda_G < 0$.
- Similar assertion is true for $\lambda_G > 0$ and as well as for $\lambda_G = 0$.

Other Results

Given a multi-block graph G with blocks G_t; $1 \leq t \leq b$. Recall that, if cof $D(G_t) \neq 0$; $1 \leq t \leq b$. , then

$$\lambda_G = \sum_{t=1}^{b} \lambda_{G_t},$$

and

$$\det D(G) \neq 0 \text{ iff } \lambda_G \neq 0.$$

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- We find multi-block graph G with blocks G_t with cof $D(G_t) \neq 0$ and $\det D(G_t) \neq 0$; $1 \leq t \leq b$, but $\det D(G) = 0$.

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- Infinitely many such multi-block graphs can be constructed.

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Thank You