A note on integral Graphs

Somnath Paul

Department of Applied Sciences
Tezpur University, Napaam, Sonitpur, Assam

16^{th} July, 2021

Outline

Introduction

Interrelation between Graphs and Matrices

- Simple graph.

Introduction

Interrelation between Graphs and Matrices

- Simple graph.
- Order and Size of a graph.

Introduction

Interrelation between Graphs and Matrices

- Simple graph.
- Order and Size of a graph.
- Neighbourhood of a vertex.

Introduction

Interrelation between Graphs and Matrices

- Simple graph.
- Order and Size of a graph.
- Neighbourhood of a vertex.
- Degree of a vertex; Regular Graph.

Introduction

Interrelation between Graphs and Matrices

- Simple graph.
- Order and Size of a graph.
- Neighbourhood of a vertex.
- Degree of a vertex; Regular Graph.
- Distance and diameter.

Introduction

Interrelation between Graphs and Matrices

- Simple graph.
- Order and Size of a graph.
- Neighbourhood of a vertex.
- Degree of a vertex; Regular Graph.
- Distance and diameter.
- Bipartite Graph.

Introduction

- Adjacency Matrix $A(G)$.

Introduction

- Adjacency Matrix $A(G)$.
- Matrix of vertex degrees $Deg(G)$.

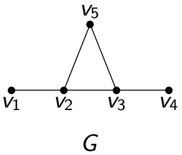
Introduction

- Adjacency Matrix $A(G)$.
- Matrix of vertex degrees $Deg(G)$.
- Laplacian Matrix $L(G) = Deg(G) - A(G)$.

Introduction

- Adjacency Matrix $A(G)$.
- Matrix of vertex degrees $Deg(G)$.
- Laplacian Matrix $L(G) = Deg(G) - A(G)$.
- Distance Matrix $D(G)$.

Introduction



$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$L(G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix} \qquad D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Figure: A graph G and various associated matrices.

Introduction

- Applicabilty of distance matrix, viz. Network Analysis, Graph embedding theory, Chemistry etc.

Introduction

- Applicabilty of distance matrix, viz. Network Analysis, Graph embedding theory, Chemistry etc.
- Applications in music theory [6], molecular biology [7], archeology [8], sociology [9] (see [19, 3]).

Introduction

- Applicabilty of distance matrix, viz. Network Analysis, Graph embedding theory, Chemistry etc.
- Applications in music theory [6], molecular biology [7], archeology [8], sociology [9] (see [19, 3]).
- Distance matrix as a more powerful structure discriminator than the adjacency matrix.

Introduction

- Transmission of a vertex $Tr_v(G)$.

Introduction

- Transmission of a vertex $Tr_v(G)$.
- Matrix of vertex transmissions $Tr(G)$.

Introduction

- Transmission of a vertex $Tr_v(G)$.
- Matrix of vertex transmissions $Tr(G)$.
- Distance Laplacian Matrix $D^L(G) = Tr(G) - D(G)$ (Aouchiche & Hansen, 2013).

Introduction

- Transmission of a vertex $Tr_v(G)$.
- Matrix of vertex transmissions $Tr(G)$.
- Distance Laplacian Matrix $D^L(G) = Tr(G) - D(G)$ (Aouchiche & Hansen, 2013).
- For the graph G shown in Fig. 1,

$$D^L(G) = \begin{bmatrix} 8 & -1 & -2 & -3 & -2 \\ -1 & 5 & -1 & -2 & -1 \\ -2 & -1 & 5 & -1 & -1 \\ -3 & -2 & -1 & 8 & -2 \\ -2 & -1 & -1 & -2 & 6 \end{bmatrix}.$$

Introduction

- Transmission of a vertex $Tr_v(G)$.
- Matrix of vertex transmissions $Tr(G)$.
- Distance Laplacian Matrix $D^L(G) = Tr(G) - D(G)$ (Aouchiche & Hansen, 2013).
- For the graph G shown in Fig. 1,

$$D^L(G) = \begin{bmatrix} 8 & -1 & -2 & -3 & -2 \\ -1 & 5 & -1 & -2 & -1 \\ -2 & -1 & 5 & -1 & -1 \\ -3 & -2 & -1 & 8 & -2 \\ -2 & -1 & -1 & -2 & 6 \end{bmatrix}.$$

- For latest results on $D^L(G)$, see [17, 18] and the references therein.

Introduction

Spectrum of a symmetric matrix M

$$\sigma_M = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix},$$

where $\mu_1, \mu_2, \ldots, \mu_p$ are the distinct eigenvalues of M and $m_1, m_2, \ldots, m_p$ are the corresponding multiplicities.

Background

- Integral Graphs (Harary & Schwenk, 1974).

Background

- Integral Graphs (Harary & Schwenk, 1974).
- *Which graphs have integral spectra*?

Background

- Integral Graphs (Harary & Schwenk, 1974).
- *Which graphs have integral spectra*?
- General problem is intractable. These are very rare and difficult to be found.

Background

- Integral Graphs (Harary & Schwenk, 1974).
- *Which graphs have integral spectra*?
- General problem is intractable. These are very rare and difficult to be found.
- Out of 164,059,830,476 connected graphs on 12 vertices, there exist exactly 325 integral graphs (Balińska et. al., 2001).

Background

- Integral Graphs (Harary & Schwenk, 1974).
- *Which graphs have integral spectra*?
- General problem is intractable. These are very rare and difficult to be found.
- Out of 164,059,830,476 connected graphs on 12 vertices, there exist exactly 325 integral graphs (Balińska et. al., 2001).
- Have applications in quantum networks allowing perfect state transfer (Saxena et. al., 2007).

Adjacency Integral Graphs

- Example-1:- The complete graph K_n, with $\sigma_A(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

Adjacency Integral Graphs

- Example-1:- The complete graph K_n, with
$$\sigma_A(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$
- Example-2:- The cocktail party graph $CP(n)$, with
$$\sigma_A(CP(n)) = \begin{pmatrix} 2n-2 & 0 & -2 \\ 1 & n & n-1 \end{pmatrix}.$$

Adjacency Integral Graphs

- Example-1:- The complete graph K_n, with $\sigma_A(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.
- Example-2:- The cocktail party graph $CP(n)$, with $\sigma_A(CP(n)) = \begin{pmatrix} 2n-2 & 0 & -2 \\ 1 & n & n-1 \end{pmatrix}$.
- Example-3:- A complete multipartite graph $K_{\frac{n}{k},\frac{n}{k},\dots,\frac{n}{k}}$ having k equal parts, with $\sigma_A(K_{\frac{n}{k},\frac{n}{k},\dots,\frac{n}{k}}) = \begin{pmatrix} n-\frac{n}{k} & 0 & -\frac{n}{k} \\ 1 & n-k & k-1 \end{pmatrix}$.

Adjacency Integral Graphs

- Example-1:- The complete graph K_n, with $\sigma_A(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.
- Example-2:- The cocktail party graph $CP(n)$, with $\sigma_A(CP(n)) = \begin{pmatrix} 2n-2 & 0 & -2 \\ 1 & n & n-1 \end{pmatrix}$.
- Example-3:- A complete multipartite graph $K_{\frac{n}{k},\frac{n}{k},\ldots,\frac{n}{k}}$ having k equal parts, with $\sigma_A(K_{\frac{n}{k},\frac{n}{k},\ldots,\frac{n}{k}}) = \begin{pmatrix} n-\frac{n}{k} & 0 & -\frac{n}{k} \\ 1 & n-k & k-1 \end{pmatrix}$.
- Complements of some disconnected regular graphs, viz. $K_n = \overline{nK_1}$, $CP(n) = \overline{nK_2}$, and $K_{\frac{n}{k},\frac{n}{k},\ldots,\frac{n}{k}} = \overline{kK_{\frac{n}{k}}}$.

Adjacency Integral Graphs

- If $P_{A(G)}(\lambda)$ denotes the adjacency characteristic polynomial of a graph G, then for a r-regular graph of order n,

$$P_{A(\overline{G})}(\lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} P_{A(G)}(-\lambda - 1). \qquad (2.1)$$

Adjacency Integral Graphs

- If $P_{A(G)}(\lambda)$ denotes the adjacency characteristic polynomial of a graph G, then for a r-regular graph of order n,

$$P_{A(\overline{G})}(\lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} P_{A(G)}(-\lambda - 1). \tag{2.1}$$

- Hence from (2.1), the complement of an integral regular graph must be integral, too.

Adjacency Integral Graphs

- Examples of some class of graphs having only finite number of integral graphs.

Adjacency Integral Graphs

- Examples of some class of graphs having only finite number of integral graphs.
  - The adjacency spectrum of P_n consists of the numbers $2\cos\left(\frac{\pi i}{n+1}\right)$, where $i = 1, 2, \ldots, n$. Thus, P_2 is the only integral path.

Adjacency Integral Graphs

- Examples of some class of graphs having only finite number of integral graphs.
  - The adjacency spectrum of P_n consists of the numbers $2\cos\left(\frac{\pi i}{n+1}\right)$, where $i = 1, 2, \ldots, n$. Thus, P_2 is the only integral path.
  - The eigenvalues of the cycle C_n are of the type $2\cos\left(\frac{2\pi i}{n}\right)$, where $i = 1, 2, \ldots, n$. Hence the only integral cycles are C_3, C_4 and C_6.

Adjacency Integral Graphs

- Research is restricted to cubic graphs, 4-regular graphs, complete multipartite graphs and circulant graphs [4, 23, 26].

Adjacency Integral Graphs

- Research is restricted to cubic graphs, 4-regular graphs, complete multipartite graphs and circulant graphs [4, 23, 26].
- One more approach is to create integral graphs applying some graph operations on the already known integral graphs [24, 10].

Adjacency Integral Graphs

- Research is restricted to cubic graphs, 4-regular graphs, complete multipartite graphs and circulant graphs [4, 23, 26].
- One more approach is to create integral graphs applying some graph operations on the already known integral graphs [24, 10].
- The sum $G_1 + G_2$ and the product $G_1 \times G_2$ has integral eigenvalues if G_1 and G_2 both are integral.

Adjacency Integral Graphs

- Research is restricted to cubic graphs, 4-regular graphs, complete multipartite graphs and circulant graphs [4, 23, 26].
- One more approach is to create integral graphs applying some graph operations on the already known integral graphs [24, 10].
- The sum $G_1 + G_2$ and the product $G_1 \times G_2$ has integral eigenvalues if G_1 and G_2 both are integral.
- In particular, the bipartite product $G \times K_2$ has eigenvalues $\pm\lambda_i$, where λ_i is an eigenvalue of G, and $i = 1, 2, \ldots, n$.

Adjacency Integral Graphs

- If G is an r-regular graph on n vertices and m edges with adjacency spectrum $(\lambda_1 = r, \lambda_2, \ldots, \lambda_n)$, then by [11],

$$\sigma_A(L_G) = \begin{pmatrix} 2r-2 & \lambda_2+r-2 & \cdots & \lambda_n+r-2 & -2 \\ 1 & 1 & \cdots & 1 & m-n \end{pmatrix}.$$

Adjacency Integral Graphs

- If G is an r-regular graph on n vertices and m edges with adjacency spectrum $(\lambda_1 = r, \lambda_2, \ldots, \lambda_n)$, then by [11],

$$\sigma_A(L_G) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_n + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & m-n \end{pmatrix}.$$

- Thus, the line graph of an integral graph is also integral.

Adjacency Integral Graphs

- If G is an r-regular graph on n vertices and m edges with adjacency spectrum $(\lambda_1 = r, \lambda_2, \ldots, \lambda_n)$, then by [11],

$$\sigma_A(L_G) = \begin{pmatrix} 2r-2 & \lambda_2+r-2 & \cdots & \lambda_n+r-2 & -2 \\ 1 & 1 & \cdots & 1 & m-n \end{pmatrix}.$$

- Thus, the line graph of an integral graph is also integral.
- Therefore, for each of the above mentioned class of integral graphs which are regular, we can obtain new classes of integral graphs by taking their line graphs.

Adjacency Integral Graphs

- An extensive research on integral graphs were done for trees.

Adjacency Integral Graphs

- An extensive research on integral graphs were done for trees.
- Unfortunately the majority of these papers were written in Chinese, as well as their authors were not always aware of the results of their colleagues of other countries, which led to some overlapping of results of Chinese and other authors.

Adjacency Integral Graphs

- An extensive research on integral graphs were done for trees.
- Unfortunately the majority of these papers were written in Chinese, as well as their authors were not always aware of the results of their colleagues of other countries, which led to some overlapping of results of Chinese and other authors.
- For a beautiful survey on integral graphs the reader can see [4].

Comparison with Laplacian Integral Graphs

- The situation in the case of Laplacian matrices is much better and as noted in [13], Laplacian integral graphs occurs more frequently.

Comparison with Laplacian Integral Graphs

- The situation in the case of Laplacian matrices is much better and as noted in [13], Laplacian integral graphs occurs more frequently.
- If G is r-regular, then $L(G) + A(G) = rI$.

Comparison with Laplacian Integral Graphs

- The situation in the case of Laplacian matrices is much better and as noted in [13], Laplacian integral graphs occurs more frequently.
- If G is r-regular, then $L(G) + A(G) = rI$.
- Hence λ is an eigenvalue of $L(G)$ iff $r - \lambda$ is an eigenvalue of $A(G)$.

Comparison with Laplacian Integral Graphs

- The situation in the case of Laplacian matrices is much better and as noted in [13], Laplacian integral graphs occurs more frequently.
- If G is r-regular, then $L(G) + A(G) = rI$.
- Hence λ is an eigenvalue of $L(G)$ iff $r - \lambda$ is an eigenvalue of $A(G)$.
- Thus for regular graphs, the theory of Laplacian integral graphs coincides with its adjacency counterpart.

Comparison with Laplacian Integral Graphs

- But, in other cases, there can be remarkable differences.

Comparison with Laplacian Integral Graphs

- But, in other cases, there can be remarkable differences.
- For example, out of 112 connected graphs of order 6, only 6 are adjacency integral, and out of them five are regular. Therefore, they are also Laplacian integral.

Comparison with Laplacian Integral Graphs

- But, in other cases, there can be remarkable differences.
- For example, out of 112 connected graphs of order 6, only 6 are adjacency integral, and out of them five are regular. Therefore, they are also Laplacian integral.
- But the sixth one is a tree obtained by joining the centers of two copies of P_3 with a new edge, is not Laplacian integral.

Comparison with Laplacian Integral Graphs

- But, in other cases, there can be remarkable differences.
- For example, out of 112 connected graphs of order 6, only 6 are adjacency integral, and out of them five are regular. Therefore, they are also Laplacian integral.
- But the sixth one is a tree obtained by joining the centers of two copies of P_3 with a new edge, is not Laplacian integral.
- Whereas there are 37 connected Laplacian integral graphs of order 6 (see [21]).

Comparison with Laplacian Integral Graphs

- Another difference involves trees. We have seen that some interesting work has been done on adjacency integral trees, whereas the same is not possible for Laplacian integral trees.

Comparison with Laplacian Integral Graphs

- Another difference involves trees. We have seen that some interesting work has been done on adjacency integral trees, whereas the same is not possible for Laplacian integral trees.
- Because, if we consider Laplacian spectrum of a tree, it turns out that the second smallest Laplacian eigenvalue is less than 1, unless we have a star $K_{1,n-1}$, where $\sigma_L(K_{1,n-1}) = \begin{pmatrix} n & 1 & 0 \\ 1 & n-2 & 1 \end{pmatrix}$.

Comparison with Laplacian Integral Graphs

- Another difference involves trees. We have seen that some interesting work has been done on adjacency integral trees, whereas the same is not possible for Laplacian integral trees.
- Because, if we consider Laplacian spectrum of a tree, it turns out that the second smallest Laplacian eigenvalue is less than 1, unless we have a star $K_{1,n-1}$, where
$$\sigma_L(K_{1,n-1}) = \begin{pmatrix} n & 1 & 0 \\ 1 & n-2 & 1 \end{pmatrix}.$$
- Thus, a tree is Laplacian integral if and only if it is a star.

Comparison with Laplacian Integral Graphs

- Another great difference concerns complements. Since $L(G) + L(\overline{G}) = nI - J$, the Laplacian eigenvalues of $L(\overline{G})$ are $\lambda_i(\overline{G}) = n - \lambda_{n-i}(G)$, where $1 \leq i \leq n-1$, and 0.

Comparison with Laplacian Integral Graphs

- Another great difference concerns complements. Since $L(G) + L(\overline{G}) = nI - J$, the Laplacian eigenvalues of $L(\overline{G})$ are $\lambda_i(\overline{G}) = n - \lambda_{n-i}(G)$, where $1 \leq i \leq n-1$, and 0.
- Therefore, $\overline{G}$ is Laplacian integral iff G is Laplacian integral.

Comparison with Laplacian Integral Graphs

- Another great difference concerns complements. Since $L(G) + L(\overline{G}) = nI - J$, the Laplacian eigenvalues of $L(\overline{G})$ are $\lambda_i(\overline{G}) = n - \lambda_{n-i}(G)$, where $1 \leq i \leq n-1$, and 0.
- Therefore, $\overline{G}$ is Laplacian integral iff G is Laplacian integral.
- This is one of the reasons that there are more Laplacian integral graphs compared to adjacency integral graphs.

Comparison with Laplacian Integral Graphs

- As already noted, the line graph of a regular adjacency integral graph is adjacency integral and a line graph of a regular graph is itself regular, this result carries over to Laplacian case too.

Comparison with Laplacian Integral Graphs

- As already noted, the line graph of a regular adjacency integral graph is adjacency integral and a line graph of a regular graph is itself regular, this result carries over to Laplacian case too.
- Hence the Peterson graph is Laplacian integral since it is the complement of the line graph of K_5.

Comparison with Laplacian Integral Graphs

- As already noted, the line graph of a regular adjacency integral graph is adjacency integral and a line graph of a regular graph is itself regular, this result carries over to Laplacian case too.
- Hence the Peterson graph is Laplacian integral since it is the complement of the line graph of K_5.
- The union and join of Laplacian integral graphs are Laplacian integral. Also the cartesian product of two Laplacian integral graphs is also Laplacian integral.

Comparison with Laplacian Integral Graphs

- As already noted, the line graph of a regular adjacency integral graph is adjacency integral and a line graph of a regular graph is itself regular, this result carries over to Laplacian case too.
- Hence the Peterson graph is Laplacian integral since it is the complement of the line graph of K_5.
- The union and join of Laplacian integral graphs are Laplacian integral. Also the cartesian product of two Laplacian integral graphs is also Laplacian integral.
- The process of finding new Laplacian integral graphs is still on.

Distance & Distance Laplacian integral graphs

- Very few results w.r.t. distance matrix till date.

Distance & Distance Laplacian integral graphs

- Very few results w.r.t. distance matrix till date.
- Some graph operations, such as the Cartesian product and the strong product may be used to generate new integral graphs from the given ones [14].

Distance & Distance Laplacian integral graphs

- Very few results w.r.t. distance matrix till date.
- Some graph operations, such as the Cartesian product and the strong product may be used to generate new integral graphs from the given ones [14].
- Recently in [20], the joined union (which can be seen as generalization of join and lexicographic product) is used to create distance Laplacian integral graphs.

Distance & Distance Laplacian integral graphs

- Very few results w.r.t. distance matrix till date.
- Some graph operations, such as the Cartesian product and the strong product may be used to generate new integral graphs from the given ones [14].
- Recently in [20], the joined union (which can be seen as generalization of join and lexicographic product) is used to create distance Laplacian integral graphs.
- Motivated by these, we consider two class of graph operations.

Distance & Distance Laplacian integral graphs

Definition 1

(Indulal & Vijaykumar, 2006)

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Take another copy of G with the vertices labelled by $\{u_1, u_2, \ldots, u_n\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$ in G, for each i. The resulting graph, denoted by D_2G, is called the double graph of G.

Distance & Distance Laplacian integral graphs

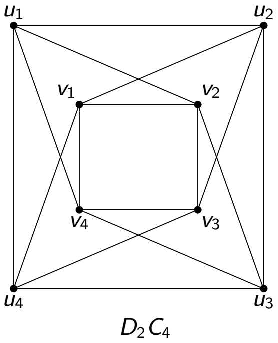


Figure: The double graph $D_2 C_4$ of cycle on 4 vertices.

Distance & Distance Laplacian integral graphs

- The distance spectrum of the double graph of G was derived from the distance spectrum of G ([15])

Theorem 2

(Indulal & Gutman, 2008). Let G be a graph with distance eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, then $\sigma_D(D_2G) = \begin{pmatrix} 2(\mu_i + 1) & -2 \\ 1 & n \end{pmatrix}$*, where $i = 1, 2, \ldots, n$.*

Distance & Distance Laplacian integral graphs

- The distance spectrum of the double graph of G was derived from the distance spectrum of G ([15])

Theorem 2

(Indulal & Gutman, 2008). Let G be a graph with distance eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, then $\sigma_D(D_2G) = \begin{pmatrix} 2(\mu_i + 1) & -2 \\ 1 & n \end{pmatrix}$, where $i = 1, 2, \ldots, n$.

- Therefore, if G is distance integral, then so does the double graph of G.

Distance & Distance Laplacian integral graphs

Lemma 3

(Davis, 1979; Nath & Paul, 2014)

Let $A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ *be a* 2×2 *block symmetric matrix. Then the eigenvalues of* A *are those of* $A_0 + A_1$ *together with those of* $A_0 - A_1$.

The double graph of G

Theorem 4

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the distance Laplacian eigenvalues of G. If Tr_i is the transmission of the i-th vertex, then

$$\sigma_{D^L}(D_2G) = \begin{pmatrix} 2\mu_i & 2Tr_i + 4 \\ 1 & 1 \end{pmatrix}, \ i = 1, 2, \ldots, n.$$

▸ Proof ▸▸ Skip Proof

The double graph of G

Proof.

By definition of D_2G, we have

$$
\begin{aligned}
d_{D_2G}(v_i, v_j) &= d_G(v_i, v_j)\\
d_{D_2G}(v_i, u_i) &= 2\\
d_{D_2G}(v_i, u_j) &= d_G(v_i, v_j)\\
d_{D_2G}(v_j, u_i) &= d_G(v_j, v_i)\\
d_{D_2G}(u_i, u_j) &= d_G(v_i, v_j)
\end{aligned}
$$

The double graph of G

Hence a suitable ordering of vertices yields the distance Laplacian matrix of D_2G of the form

$$
\begin{aligned}
& D^L(D_2G) \\
= & \begin{bmatrix} 2(Tr(G)+I) & 0 \\ 0 & 2(Tr(G)+I) \end{bmatrix} - \begin{bmatrix} D(G) & D(G)+2I \\ D(G)+2I & D(G) \end{bmatrix} \\
= & \begin{bmatrix} D^L(G)+Tr(G)+2I & -D(G)-2I \\ -D(G)-2I & D^L(G)+Tr(G)+2I \end{bmatrix}.
\end{aligned}
$$

Thus the theorem follows from Lemma 3. □

The double graph of G

Example 5

We have $\sigma_{D^L}(C_4) = \begin{pmatrix} 6 & 4 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ and $Tr_i = 4, \ \forall i$. For the double graph of C_4 (shown in Fig. 2), it can be seen that $\sigma_{D^L}(D_2C_4) = \begin{pmatrix} 12 & 8 & 0 \\ 6 & 1 & 1 \end{pmatrix}$, as stated in Theorem 4.

The double graph of G

Example 5

We have $\sigma_{D^L}(C_4) = \begin{pmatrix} 6 & 4 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ and $Tr_i = 4$, $\forall i$. For the double graph of C_4 (shown in Fig. 2), it can be seen that $\sigma_{D^L}(D_2C_4) = \begin{pmatrix} 12 & 8 & 0 \\ 6 & 1 & 1 \end{pmatrix}$, as stated in Theorem 4.

- Since the transmission Tr_i is always an integer, by Theorem 4, given a distance Laplacian integral graph G, the double graph of G is also distance Laplacian integral.

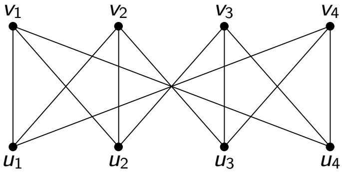
The extended double cover graph

Definition 6

(Alon, 1986)

Let G be a graph on the vertex set $\{v_1, v_2, \ldots, v_n\}$. Define a bipartite graph H with $V(H) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ in which v_i is adjacent to u_i for each $i = 1, 2, \ldots, n$ and v_i is adjacent to u_j if v_i is adjacent to v_j in G. The graph H is known as the extended double cover graph (EDC-graph) of G.

The extended double cover graph



$EDC-$ graph of C_4

Figure: The $EDC-$ graph of cycle on 4 vertices.

The EDC-graph of regular graphs of diameter 2

- The distance spectrum of the *EDC*-graph of a regular graph of diameter 2 has been obtained in [15].

Theorem 7

(Indulal & Gutman, 2008). Let G be an r-regular graph of diameter 2 of order n with adjacency eigenvalues $r = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. *Then*

$$\sigma_D(EDC-graph\ of\ G) = \begin{pmatrix} 5n-2r-4 & 2r-n & -2(\lambda_i+2) & 2\lambda_i \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

where $i = 2, 3, \ldots, n$.

The EDC-graph of regular graphs of diameter 2

- The distance spectrum of the *EDC*-graph of a regular graph of diameter 2 has been obtained in [15].

Theorem 7

(Indulal & Gutman, 2008). Let G be an r-regular graph of diameter 2 of order n with adjacency eigenvalues $r = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. *Then*

$$\sigma_D(EDC\!-\!graph\ of\ G) = \begin{pmatrix} 5n-2r-4 & 2r-n & -2(\lambda_i+2) & 2\lambda_i \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

where $i = 2, 3, \ldots, n$.

- Thus if G is an r-regular adjacency integral graph of diameter 2, then the EDC-graph of G is distance integral.

The EDC-graph of regular graphs of diameter 2

Theorem 8

Let G be an r-regular graph on n vertices with diameter 2, and $r = \lambda_1, \lambda_2, \ldots, \lambda_n$ be its adjacency eigenvalues. Then for $i = 2, 3, \ldots, n$,

$$\sigma_{D^L}(EDC - \text{graph of } G) = \begin{pmatrix} 0 & 5n-2r+2\lambda_i & 6n-4r-4 & 5n-2r-4-2\lambda_i \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Proof Skip Proof

The EDC-graph of regular graphs of diameter 2

Proof.

Let A and $\overline{A}$ be the adjacency matrices of G and $\overline{G}$, respectively. Then by the definition of $EDC-$ graph, its distance Laplacian matrix can be written as

$$
\begin{aligned}
& D^L(EDC - graph\ of\ G) \\
= & \begin{bmatrix} (5n-2r-2)I & 0 \\ 0 & (5n-2r-2)I \end{bmatrix} - \begin{bmatrix} 2(J-I) & A+3\overline{A}+I \\ A+3\overline{A}+I & 2(J-I) \end{bmatrix} \\
= & \begin{bmatrix} (5n-2r)I-2J & -3J+2A+2I \\ -3J+2A+2I & (5n-2r)I-2J \end{bmatrix},
\end{aligned}
$$

The EDC-graph of regular graphs of diameter 2

since $\overline{A} = J - I - A$. The theorem now follows from Lemma 3, and the fact that the all one vector $\mathbf{1}_n$ is the eigenvector of A corresponding to the eigenvalue r and the eigenvector of J corresponding to the eigenvalue n. $\square$

The EDC-graph of regular graphs of diameter 2

Example 9

Since C_4 is a 2-regular graph of diameter 2, we have $\sigma_A(C_4) = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \end{pmatrix}$. For the EDC-graph of C_4 (shown in Fig. 3), it can be seen that

$$\sigma_{D^L}(EDC - graph\ of\ C_4) = \begin{pmatrix} 16 & 12 & 0 \\ 3 & 4 & 1 \end{pmatrix},$$

as stated in Theorem 8.

Thus, if G is an r-regular adjacency integral graph of diameter 2, then by Theorem 8, the EDC-graph of G is distance Laplacian integral.

Conclusions

- Noted some important results on adjacency and Laplacian integral graphs.

Conclusions

- Noted some important results on adjacency and Laplacian integral graphs.
- Used two graph operations to create distance and distance Laplacian integral graphs.

Conclusions

- Noted some important results on adjacency and Laplacian integral graphs.
- Used two graph operations to create distance and distance Laplacian integral graphs.
- From Theorem 4, we see that by repeated application of the graph operation (double graph) one may obtain numerous infinite families of distance Laplacian integral graphs.

Conclusions

- Compared to Theorem 4, the scope of application of Theorem 8 is limited.

Conclusions

- Compared to Theorem 4, the scope of application of Theorem 8 is limited.
- We cannot repeat the graph operation (EDC) to obtain infinite families of distance Laplacian integral graphs (since the EDC-graph may not be a graph of diameter 2).

Conclusions

- Compared to Theorem 4, the scope of application of Theorem 8 is limited.
- We cannot repeat the graph operation (EDC) to obtain infinite families of distance Laplacian integral graphs (since the EDC-graph may not be a graph of diameter 2).
- But we can surely apply the operation at least once, to create a distance Laplacian integral graph from a given one.

References I

[1] N. Alon, Eigenvalues and expanders, *Combinatorica* 6 (1986) 83–96.

[2] M. Aouchiche, P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.*, 439 (2013) 21–33.

[3] M. Aouchiche, P. Hansen, Distance spectra of graphs: a survey, *Linear Algebra Appl.*, 458 (2014) 301–386.

[4] K.T. Balińska, D. Cvetković, Z. Radosavljević, S. K. Simić, D. Stevanović, A survey on integral graphs, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* 13 (2002) 42–65.

[5] K.T. Balińska, M. Kupczyk, S.K. Simić, K. T. Zwierzyński, On Generating all Integral Graphs on 12 Vertices, *Computer Science Center Re. No. 482,* Technical University of Poznań, Poznań, Poland, (2001) 1–36.

References II

[6] D.W. Bradley, R. A. Bradley, String edits and macromolecules in *Time Wraps,* D. Sankoff, J. B. Kruskal, Eds., Chapter 6, Addison-Wesley, Reading, Mass, USA, 1983.

[7] J. P. Boyd, K. N. Wexler, Trees with structure *Journal of Mathematical Psychology,* 10 (1973) 115–147.

[8] R. L. Graham, L. Lovász, Distance matrix polynomials of trees *Theory and Applications of Graphs,* 642 (1978) 186–190.

[9] M. S. Waterman, T. F. Smith, H. I. Katcher, Algorithms for restriction map comparisons *Nucleic Acids Research,* 12 (1984) 237–242.

[10] D. Cvetković, Cubic integral graphs *Univ. Beograd, Publ Elektrotehn. Fak. Ser. Mat. Fiz., Nos.* 498–541 (1975) 107–113.

References III

[11] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: Theory and Application, Third Edition,* Johann Abrosius Barth Verlag, Heidelberg- Leipzig, 1995.

[12] P. J. Davis, Circulant Matrices, *Wiley,* New York, 1979.

[13] R. Grone, R. Merris, The Laplacian spectrum of a graph II, *SIAM J Discrete Math,* 7 (1994) 221–229.

[14] F. Harary, A. J. Schwenk, Which graphs have integral spectra? in *Graphs and Combinatorics,* Lecture Notes in Mathematics, 406, A. Dold and B. Eckmann, eds., 45–51, Springer, Berlin, 1974.

[15] G. Indulal, I. Gutman, On the distance spectra of some graphs, *Mathematical Communications,* 13 (2008) 123–131.

References IV

[16] G. Indulal, A. Vijayakumar, On a pair of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* 55 (2006) 83–90.

[17] M. Nath, S. Paul, On the distance Laplacian spectra of graphs, *Linear Algebra Appl.* 460 (2014) 97–110.

[18] A. Niu, D. Fan, G. Wang, On the distance Laplacian spectral radius of bipartite graphs, *Discrete Appl.Math.* 186 (2015) 207–213.

[19] D. H. Rouvray, The role of the topological distance matrix in chemistry in *Mathematical and Computational Concepts in Chemistry*, N. Trinajstić, Ed., 295–306, Ellis Harwood, Chichester, UK, 1986.

[20] S. Paul, Distance Laplacian spectra of joined union of graphs, *Asian-European Journal of Mathematics* (2022) 2250039.

References V

[21] D. L. Powers, Graph Eigenvectors, unpublished tables.

[22] N. Saxena, S. Severini, I. E. Shparlinski, Parameters of integral circulant graphs and periodic quantum dynamics, *Int. J. Quant. Inf.* 5 (2007) 417–430.

[23] W. So, Integral circulant graphs, *Discrete Math.* 306 (2006) 153–158.

[24] L. Wang, C. Hoede, Constructing fifteen infinite classes of nonregular bipartite integral graphs, *Electron. J. Combin.* 15 (2008) #R8.

[25] L. Wang, X. Li, C. Hoede, Two classes of integral regular graphs, *Ars Combin.* 76 (2005) 303–319.

References VI

[26] L. Wang, X. Liu, Integral complete multipartite graphs, *Discrete Math.* 308 (2008) 3860–3870.

Thank you!

Queries & suggestions please