Expander Graphs: Constructions
(and Applications)

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Introduction

"These graphs - expanders - are highly connected sparse graphs that play an important role in combinatorics and computer science. Loosely speaking, they are "fat and round": one cannot cut them into two large subsets without cutting a lot of edges. Or, equivalently, for every subset A of the vertices of the graphs, its boundary, i.e., the vertices outside A that are connected to A, form a fairly large set compared to A. Such graphs were known to exist by random consideration, but explicit constructions are desired." - Alex Lubotzky

Lecture 1: Definition ((n, k, ϵ)-Expander)

For $0 < \epsilon \in \mathbb{R}$, a finite k-regular graph $\mathcal{G} = (V, E)$ with $|V| = n$ (i.e. $|E| = \frac{kn}{2}$) is called an (n, k, ϵ)-expander if for all $A \subset V$:

$$|\partial A| \geq \epsilon(1 - \frac{|A|}{n})|A|,$$

where $\partial A = \{y \in V \setminus A : (x, y) \in E, x \in A\}$ is the *boundary of A*.

- Each finite connected regular graph is (trivially) an expander for some $\epsilon > 0$.
- This notion is meaningful only when one considers an infinite family of (n, k, ϵ)-expander, where $n \to \infty$ with k and ϵ fixed.
- Expect k as small as possible (sparse) and ϵ as large as possible (highly connected).
- The basic idea behind all the definitions of expanders is always that every set is guaranteed to 'expand' by some fixed amount.

Definition ((n,k,c')-Expander: Bi-partite version)

For $0 < \epsilon' \in \mathbb{R}$, an (n,k,ϵ')-expander $\mathcal{G}$ is a bi-partite k-regular graph with two sets I and O with $|I| = n = |O|$, if for any $A \subset I$ with $|A| \leq \frac{n}{2}$, we have:

$$|\partial A| \geq (1+\epsilon')|A|,$$

where $\partial A = \{y \in V \setminus A : (x,y) \in E, x \in A\}$ is the *boundary of A*.

- Starting with an expander as in the previous definition, we get a bi-partite expander by taking a double cover of $\mathcal{G} = (V,E)$, where I and O are disjoint copies of V. A vertex in I is joined by an edge with a 'twin' vertex of O, and to the twins in O, of all its neighbours in $\mathcal{G}$.
- Converse is obtained by identifying I and O 'suitably'.
- Expansion coefficients ϵ and ϵ' will change accordingly.

Existence of an Expander

It is easier to show that expanders exist, than to actually construct one. Now by some counting argument (following Sarnak) we will show that bi-partite (n, k, ϵ)-expanders exist. We will sketch a proof for $k \geq 5$ and $\epsilon = \frac{3}{2}$ and this can be extended to the general case.

- Let $I = O = \{1, 2, \cdots, n\}$. Construct a k-regular bi-partite graph X_π by taking k permutations $\pi = (\pi_1, \pi_2, \cdots, \pi_k)$, $\pi_i \in S_n$ and joining each j to $\pi_r(j)$ for $r = 1, \ldots, k$.
- There are total $(n!)^k$ many choices of π, and hence of X_π (not all distinct).
- Call $\pi = (\pi_1, \ldots, \pi_k)$ *bad* if for some $A \subset I$ with $|A| \leq \frac{n}{2}$ there is $B \subset O$ with $|B| \leq \frac{3}{2}|A|$ for which $\pi_r(A) \subset B$, $r = 1, \ldots, k$. We want to bound the number of such bad π's.
- Let, $|A| = t \leq \frac{n}{2}$ and $t \leq |B| = m \leq \frac{3}{2}t$ be as above. The number of bad π's corresponding to such A and B is:

$$([m(m-1)\cdots(m-t+1)](n-t)!)^k = (\frac{m!(n-t)!}{(m-t)!})^k.$$

Existence of an Expander: Contd.

- Hence the total number of bad π's (denoted $\#BAD$) is atmost

$$\sum_{t \leq \frac{n}{2}} \sum_{t \leq m \leq \frac{3t}{2}} \binom{n}{t}\binom{n}{m}\left(\frac{m!(n-t)!}{(m-t)!}\right)^k.$$

  Let the summand be denoted by $R(t)$.

- $$\sum_{t \leq \frac{n}{2}} \sum_{t \leq m \leq \frac{3t}{2}} R(t) = \sum_{t \leq \frac{n}{3}} \sum_{t \leq m \leq \frac{3t}{2}} R(t) + \sum_{\frac{n}{3} \leq t \leq \frac{n}{2}} \sum_{t \leq m \leq \frac{3t}{2}} R(t) = I + II$$

- $I \leq n^4((n-1)!)^k$ and hence, for $k \geq 5$, $\frac{I}{(n!)^k} \to 0$. Also, $\frac{II}{(n!)^k} \to 0$ as $n \to \infty$.
- Hence, for $k \geq 5$, $\frac{\#BAD}{(n!)^k} \to 0$ as $n \to \infty$.
- 'Most' k-regular bi-partite graphs X_π constructed as above are expanders.
- Hence, expanders certainly exist.

Definition: Cheeger - constant

The optimal $c > 0$ for a finite k-regular graph $\mathcal{G}$, denoted by $h(\mathcal{G})$ and called the *(discrete) Cheeger constant* (or the expanding constant/isoperimetric constant, inspired by analogous notion in Riemannian Geometry), is a measure of edge expansion in $\mathcal{G}$.

$$h(\mathcal{G}) = \inf_{A \subset V, |A| \leq \frac{1}{2}|V|} \frac{|\partial A|}{|A|}$$

.

1. Let $\mathcal{G} = K_n$ be the complete graph with n vertices. Then $h(K_n) = 1$.
2. Let $\mathcal{G} = T_d$ be the regular tree of degree $d \geq 2$. Then $h(T_d) = d - 2$.

Let $\mathcal{G}$ be a k-regular graph with n vertices.

- If $\mathcal{G}$ is an (n, k, c)-expander, then $h(\mathcal{G}) \geq \frac{c}{2}$.
- $\mathcal{G}$ is an $(n, k, \frac{h(\mathcal{G})}{k})$-expander.

Definition (Combinatorial Laplacian)

- On a finite, connected, k-regular graph $\mathcal{G}$, the Markov operator (or averaging operator) P on the set of functions on $\mathcal{G}$ is defined by

$$(Pf)(x) := \frac{1}{k} \sum_{x \sim y} f(y)$$

- P defines a self-adjoint contraction ($||Pf||_2 \leq ||f||_2$) on $l^2(\mathcal{G})$. So, the spectrum of P is real and contained in $[-1, 1]$. In decreasing order eigenvalues of P are $\mu_0 \geq \mu_1 \geq \mu_2 \cdots \geq \mu_{|\mathcal{G}|}$.
- The highest eigenvalue of P is 1 (since $\mathbf{1}$ is an eigenfunction). Since $\mathcal{G}$ is connected, (upto scalars) $\mathbf{1}$ is the only eigenfunction with eigenvalue 1 (Maximum modulus principle). $\mu_0 = 1$, $\mu_1 < 1$.
- The operator $\Delta := \mathbf{Id} - P$ is self-adjoint with non-negative eigenvalues denoted as $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|\mathcal{G}|}$ where $\lambda_i(\mathcal{G}) = 1 - \mu_i(\mathcal{G})$. This operator Δ is called the *combinatorial Laplacian* and λ_1 is called the *spectral gap*.

Discrete Cheeger-Buser inequality

Given a connected k-regualr graph $\mathcal{G}$, we have:

$$\frac{1}{2}\lambda_1(\mathcal{G}) \leq \frac{1}{k} h(\mathcal{G}) \leq \sqrt{2\lambda_1(\mathcal{G})}.$$

Definition: Expander Family

Let $k \in \mathbb{N}$ be a fixed integer and let $0 < \epsilon \in \mathbb{R}$ be a constant. A family $\{\mathcal{G}_n\}_{n\in\mathbb{N}} = \{(V_n, E_n)\}_{n\in\mathbb{N}}$ of finite, connected, k-regular graphs is a family of (k, ϵ)-expanders if

(i) $\lim_{n\to\infty} |V_n| = \infty$

(ii) $\lambda_1(\mathcal{G}_n) \geq \epsilon$ for all $n \in \mathbb{N}$.

Equivalently,

Let $k \in \mathbb{N}$ be a fixed integer and let $0 < \epsilon' \in \mathbb{R}$ be a constant. A family $\{\mathcal{G}_n\}_{n\in\mathbb{N}} = \{(V_n, E_n)\}_{n\in\mathbb{N}}$ of finite, connected, k-regular graphs is a family of (k, ϵ')-expanders if

(i) $\lim_{n\to\infty} |V_n| = \infty$

(ii) $h(\mathcal{G}_n) \geq \epsilon'$ for all $n \in \mathbb{N}$.

Kazhdan's Property (T)

- Unitary representation of a discrete group G consists of a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and $\pi : G \to B(\mathcal{H})$ is a homomorphism such that $\pi(g)$ is unitary for all $g \in G$.
- $(\pi, \mathcal{H})$ is reducible if there exists a proper closed subspace $\mathcal{H}' \subset \mathcal{H}$ such that $\pi(g)(\mathcal{H}') \subset \mathcal{H}'$ for all $g \in G$. π is irreducible otherwise.

Property (T)

A finitely generated discrete group G with some finite generating set S has property (T) if there exists $\epsilon > 0$ such that for all non-trivial irreducible representations $(\pi, \mathcal{H})$ of G, there exists $s \in S$ such that $||\pi(s)(v) - v|| \geq \epsilon$ for all unit vectors in $\mathcal{H}$. In other words this means 'some neighbourhood' of the one-dimensional trivial representation ρ_0 contains only ρ_0.

Construction I: à la Margulis

Margulis was the first to explicitly construct a family of expanders, using Property (T).

Theorem (Margulis): Suppose Γ is a group with Kazhdan's property (T) and S is a symmetric generating set such that $|S| = k$. Let $\Gamma_n \leq \Gamma$ be a family of finite index normal subgroups such that the index $[\Gamma : \Gamma_n]$ tends to infinity as $n \to \infty$. Then the family of k-regular Cayley graphs $\mathcal{G}_n = \mathcal{G}(\Gamma/\Gamma_n, S)$ forms a family of expanders.

Proof:

- Let $l_0^2(\Gamma/\Gamma_n) := \{f \in l^2(\Gamma/\Gamma_n) : \sum_{x \in \Gamma/\Gamma_n} f(x) = 0\}$
- $\Gamma \curvearrowright l_0^2(\Gamma/\Gamma_n)$, π_n denotes the resulting unitary representation
- Property (T) gives the Kazhdan's constant $\epsilon = \epsilon(S) > 0$, such that for any unitary representation π without an invariant vector $\max_{s \in S} ||\pi(s)(f) - f|| \geq \epsilon ||f||$
- π_n does not have any non-zero Γ-invariant vector in $l_0^2(\Gamma/\Gamma_n)$

Construction I: à la Margulis (Contd.)

- Let $A \subset \mathcal{G}_n$, with $|A| = a$ and let $B = \mathcal{G}_n \setminus A$ with $|B| = b = |\mathcal{G}_n| - a$.
- Let $f = b\chi_A - a\chi_B$. Then $f \in l_0^2(\Gamma/\Gamma_n)$ and $||f||^2 = ab^2 + ba^2 = |\mathcal{G}_n|ab$
- For every $s \in S$, $||\pi_n(s)f - f||^2 = (a+b)^2|E_s(A,B)|$, where $E_s(A,B) = \{x \in \mathcal{G}_n : x \in A \text{ and } xs \in B \text{ or } x \in B \text{ and } xs \in A\}$
- There exists $s \in S$ such that $|\partial A| \geq \frac{1}{2}|E_s(A,B)| = \frac{||\pi_n(s)f - f||^2}{2|\mathcal{G}_n|^2} \geq \frac{\epsilon^2||f||^2}{2|\mathcal{G}_n|^2}$
- Therefore, $|\partial A| \geq \frac{\epsilon^2 ab}{2|\mathcal{G}_n|} = \frac{\epsilon^2}{2} \cdot (1 - \frac{a}{|\mathcal{G}_n|})a = \frac{\epsilon^2}{2} \cdot (1 - \frac{|A|}{|\mathcal{G}_n|})|A|$
- $\{\mathcal{G}_n\}$ is an expander family.

Remarks

1. In the above proof, the assumption of Γ_n being a normal subgroup in Γ is superfluous. Γ_n can be arbitrary finite index subgroup. In that case one needs to consider the Schreier graph $\mathcal{S}(\Gamma/\Gamma_n, S)$ instead of the Cayley graph $\mathcal{G}(\Gamma/\Gamma_n, S)$. Vertices in $\mathcal{S}(\Gamma/\Gamma_n, S)$ are left cosets of Γ modulo Γ_n and two cosets $a\Gamma_n$ and $b\Gamma_n$ are connected by $s \in S$ if $sa\Gamma_n = b\Gamma_n$.

2. This proof also shows that to construct expander families we do not need to use the full power of property (T). We only needed the natural representations of Γ in $l^2(\Gamma/\Gamma_n)$ to be bounded away from the trivial representation.

3. Let Γ be an amenable group generated by a finite set S. Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be an infinite family of finite index subgroups with $|\Gamma/\Gamma_n| \to \infty$ as $n \to \infty$. Still, $\mathcal{G}(\Gamma/\Gamma_n, S)$ is not a family of expanders. The problem here would be existence of almost invariant vectors. This highlights the importance of the property (T) assumption.

Examples

1. Consider $\Gamma = SL_3(\mathbb{Z})$ with generating set given by the elementary matrices. $SL_3(\mathbb{Z})$ satisfies property (T) and $\epsilon = \frac{1}{960}$ is a Kazhdan's constant for this group. For every prime integer p, let

$$\Gamma(p) = \{a \in \Gamma : A \equiv I \mod p\}$$

which is the kernel of the surjective homomorphism $SL_3(\mathbb{Z}) \to SL_3(\mathbb{Z}/p\mathbb{Z})$ (given by reduction modulo p) be the principal congruence subgroups. Since, $\Gamma/\Gamma(p) \simeq SL_3(\mathbb{Z}/p\mathbb{Z})$, $\Gamma(p)$ has finite index in Γ. The Schreier graphs $\{\mathcal{G}(\Gamma/\Gamma(p), S)\}_p$ form an expander family.

2. (Selberg's $\frac{3}{16}$ Theorem and Expanders) $SL_2(Z)$ does not satisfy property (T). But, if $\Gamma(m) = \ker\{SL_2(\mathbb{Z} \to SL_2(\mathbb{Z}/m\mathbb{Z}\}$, a celebrated theorem of Selberg sates that $\lambda_1(\mathbb{H}^2/\Gamma(m)) \geq \frac{3}{16}$. It can be shown that the Cheeger constant of the dual graph $\mathcal{G}(\mathbb{H}^2/\Gamma(m))$ of a triangulation for the quotient $\mathbb{H}^2/\Gamma(m)$ is bounded below. It follows that $\mathcal{G}(\mathbb{H}^2/\Gamma(m))$ forms an expander family.

Property (τ)

A finitely generated group Γ with finite symmetric generating set S is said to have property (τ) with respect to a family of finite index (normal) subgroups $\{\Gamma_n\}_n$ if the family of Cayley graphs $\mathcal{G}(\Gamma/\Gamma_n, S_n)$ where S_n is the projection of S to Γ/Γ_n is a family of expanders. If $\{\Gamma_n\}_n$ runs over all finite index normal subgroups of Γ, then we say that Γ has property (τ).

Easy to observe that property (T) implies property (τ), but the other implication is not true.

Lecture 2: Recap

- Expander Family: Let $k \in \mathbb{N}$ and let $0 < \epsilon \in \mathbb{R}$. A family of finite, connected, k-regular graphs $\{\mathcal{G}_n\}_{n\in\mathbb{N}} = \{(V_n, E_n)\}_{n\in\mathbb{N}}$ is a family of (k,ϵ)-expanders if: (i) $\lim_{n\to\infty} |V_n| = \infty$, (ii) $\lambda_1(\mathcal{G}_n) \geq \epsilon$ for all $n \in \mathbb{N}$.
- Construction I (à la Margulis): Let Γ has Kazhdan's property (T) and S is a symmetric generating set with $|S| = k$. Let $\Gamma_n \leq \Gamma$ be a family of finite index subgroups such that $[\Gamma : \Gamma_n]$ tends to infinity as $n \to \infty$. Then $\mathcal{G}_n = \mathcal{G}(\Gamma/\Gamma_n, S)$ forms a family of expanders.
- Property (τ): A group Γ with finite symmetric generating set S has property (τ) with respect to a family of finite index subgroups $\{\Gamma_n\}_n$, if $\mathcal{G}(\Gamma/\Gamma_n, S_n)$ (where S_n is the projection of S to Γ/Γ_n) is an expander family. If $\{\Gamma_n\}_n$ runs over all finite index normal subgroups of Γ, then we say that Γ has property (τ).

Alon-Boppana Theorem

"To have good quality expanders, the spectral gap has to be as large as possible. However, the spectral gap cannot be too large..."

- $X = X_{n,k}$: undirected, k-regular graph on n vertices.
- $A = A_X$ defined as $(Af)(i) := \sum_{j=1}^{n} A_{ij} f(i)$ for $i \in V(X)$ and $f \in L^2(X)$, is a symmetric matrix with real eigenvalues. k is an eigenvalues of A.
- $\lambda(X) := \max\{|\lambda| : \lambda$ is an eigenvalue of A, $|\lambda| \neq k\}$

(Alon-Boppana Theorem)

Let $X_{n,k}$ be an infinite family of k-regular connected graphs on n vertices with k fixed and $n \to \infty$. Then, $\liminf_{n\to\infty} \lambda(X_{n,k}) \geq 2\sqrt{k-1}$ or $\lambda(X_{n,k}) \geq 2\sqrt{k-1} - o(1)$.

i.e. for a large k-regular graph X the strongest upper bound for $\lambda(X)$ is $2\sqrt{k-1}$.

Definition: A k-regular finite graph X is Ramanujan if $\lambda(X) \leq 2\sqrt{k-1}$.

In some sense, Ramanujan graphs, are the optimal expanders.

Construction II: Ramanujan Graphs $X^{p,q}$

One of the major developments in the study of expanders was the construction of Ramanujan Graphs by Lubotzky, Philips and Sarnak (also, independently by Margulis).

- Let p be a prime congruent to 1 modulo 4. Let, $H(\mathbb{Z})$ denote the integral quaternions: $H(\mathbb{Z}) = \{\alpha = a_0 + a_1 i + a_2 j + a_3 k : a_r \in \mathbb{Z}\}$. Let, $\bar{\alpha} = a_0 - a_1 i - a_2 j - a_3 k$ and $N(\alpha) = \alpha\bar{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2$.
- $S_p = \{\alpha \in H(\mathbb{Z}) : N(\alpha) = p, \alpha \equiv 1(\text{mod } 2), a_0 > 1\}$. $|S_p| = p + 1$ (follows from a theorem of Jacobi).
- Consider $\tau_q : H(\mathbb{Z}) \to H(\mathbb{Z}/q\mathbb{Z})$ (reduction modulo q)
- There exists isomorphism $\psi_q : H(\mathbb{Z}/q\mathbb{Z}) \to M_2(\mathbb{Z}/q\mathbb{Z})$ such that (i) $N(\alpha) = \det \psi_q(\alpha)$ and (ii) if $\alpha \in H(\mathbb{Z}/q\mathbb{Z})$ is real ($\alpha = \bar{\alpha}$), $\psi_q(\alpha)$ is a scalar matrix.
- For $\alpha \in S_p$, $\psi_q(\tau_q(\alpha)) \in GL(2,q) \subset M_2(\mathbb{Z}/q\mathbb{Z})$, since $p \neq q$.
- Also, $\psi_q(\tau_q(\alpha\bar{\alpha})) = \psi_q(\tau_q(\bar{\alpha}\alpha))$ is a non-zero scalar matrix in $GL(2,q)$.
- Now consider $\phi : GL(2,q) \to PGL(2,q)$ whose kernel is the scalar matrices.

Construction II: Ramanujan Graphs $X^{p,q}$ - Contd.

- Set, $S_{p,q} = (\phi \circ \psi_q \circ \tau_q)(S_p)$. Can be checked easily that $S_{p,q}^{-1} = S_{p,q}$.
- If q is 'large enough' w.r.t. p, $|S_{p,q}| = p + 1$.
- If p is congruent to a perfect square modulo q (or if p is a quadratic residue modulo q) denoted $(\frac{p}{q}) = 1$, $S_{p,q}$ is actually contained in $PSL(2,q)$. Then we define $X^{p,q} = \mathcal{G}(PSL(2,q), S_{p,q})$, the Cayley graph of $PSL(2,q)$ w.r.t. $S_{p,q}$.
- Else, p is not congruent to a square modulo q, denoted $(\frac{p}{q}) = -1$, $S_{p,q}$ is actually contained in $PGL(2,q) \backslash PSL(2,q)$. Then we define $X^{p,q} = \mathcal{G}(PGL(2,q), S_{p,q})$, the Cayley graph of $PGL(2,q)$ w.r.t. $S_{p,q}$. In this case $X^{p,q}$ is bipartite with edges between $PSL(2,q)$ and its complement.
- More interesting is the situation when $(\frac{p}{q}) = 1$. In this case $X^{p,q}$ is a $(p+1)$-regular graph with $|X^{p,q}| = \frac{q(q^2-1)}{2}$ and it can be shown that $\lambda(X^{p,q}) = 2\sqrt{p}$.
- For a fixed p, $\{X^{p,q}\}_{(q \text{ is prime})}$ forms an expander family. This is an example of an 'optimally expanding' expander.

Construction III: Zig-Zag Product

'In a breakthrough work Reingold, Vadhan and Wigderson showed that there is an elementary combinatorial way to build expanders via the 'Zig-Zag product' of graphs, which they introduced.'

- Given an (n, m) graph (i.e. m regular with n vertices) X and an (m, d) graph Y, the Zig-Zag product is a method that produces an (mn, d^2) graph $X \circ Y$.
- Expansion in $X \circ Y$ can be bounded by expansions in X and Y.
- Start with a (d^4, d) graph $X = X_0$ with a good spectral gap. (Exists!)
- For a graph Y, let Y^2 be the graph with the same vertex set as Y, with edges between end-points of any path of length 2.
- Define $X_1 = X^2$ (a (d^4, d^2) graph), and inductively, $X_n = X_{n-1}^2 \circ X$, for $n \geq 1$.
- Therefore, X_n is a (d^{4n}, d^2) graph.
- $\{X_n\}_{n=1}^{\infty}$ is a family of d^2-regular graphs with $|X_n| \to \infty$ and the Cheeger constant $h(X_n) \geq \frac{d^2}{4}$, for all n. Hence, $\{X_n\}_n$ forms an expander family.

Expander Cayley graphs: survey of examples and non-examples

1. (Symmetric groups as expanders) M. Kassabov proved the following remarkable result proving certain Cayley graphs of symmetric groups S_n expanders. This is important because there exists generating sets of S_n such that the Cayley graphs are not expanders.

Theorem(Kassabov): There exists $k \in \mathbb{N}$ and $0 < \epsilon \in \mathbb{R}$ such that for every $n \geq 5$, the symmetric group S_n has a symmetric generating subset Σ with $|\Sigma| < k$ for which $\{\mathcal{G}(S_n, \Sigma)\}_n$ is an ϵ-expander family.

2. (Finite solvable groups are never expanders) Fix $l, k \in \mathbb{N}$. Suppose $\mathcal{G}_n$ is a family of k-regular Cayley graphs of finite solvable groups G_n with derived lenth $\leq l$. Then $\{\mathcal{G}\}$ is not a family of expanders.

3. (Simple groups as expanders) There is $k \geq 2$ and $\epsilon > 0$ such that every finite simple group G has a k-regular Cayley graph which is an ϵ-expander. Kassabov, Lubotzky and Nikolov proved it except for Suzuki groups $\{\text{Suz}(2^{2n+1})\}_n$, which was later proved by Breuillard, Green, Tao.

4. (Bourgain-Gamburd Theorem) Given $k \geq 1$ and $\tau > 0$ there is $\epsilon = \epsilon(k, \tau) > 0$ such that every Cayley graph $\mathcal{G}(SL(2,p), S)$ of $SL(2,p)$ with symmetric generating set S of size $2k$ and girth at least $\tau \log p$ is an expander.

A few applications

- Counter examples to long-standing Baum-Connes conjecture was constructed by M. Gromov and others, using 'random groups' constructed via expanders.
- Expanders do not 'coarsely embed' into Hilbert spaces. Due to which it is interesting to find examples of groups with expanders in their Cayley graphs, in order to find counter examples to groups satisfying Property (A) or related notions.
- A. Valette used Ramanujan graphs to study different possible norms on certain tensor product C^*-algebra of bounded linear operators on a Hilbert space.
- Proof of the fact that pseudo-Anosov elements in mapping class group $MCG(\Sigma_g)$ are exponentially generic, uses expanders.

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Thank you!