E-seminar talk

Resistance distance in directed cactus graphs

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Introduction

- Let $G = (V, E)$ be a simple directed graph. (i.e. There are no loops and in one direction there is at most one edge connecting a pair of vertices.)
- Let V be written $\{1, \ldots, n\}$.
- $(i, j) \in E$ if there is a directed edge from vertex i to vertex j.

Adjacency matrix in a digraph

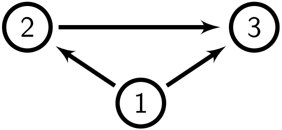
- Define

$$a_{ij} := \begin{cases} 1 & (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

- $A := [a_{ij}]$ is the adjacency matrix of G.

Example

A directed graph and its adjacency matrix.



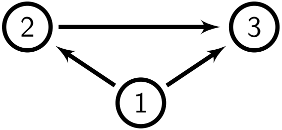
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Laplacian matrix

- The **Laplacian** of G is defined by $L := \mathrm{Diag}(A\mathbf{1}) - A$.

Example

A directed graph and its Laplacian matrix.



$$L = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

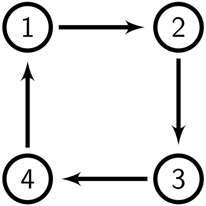
Properties of the Laplacian

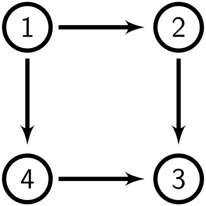
- If L is the Laplacian matrix of a directed graph, then
  - L need not be symmetric.
  - All off-diagonal entries of L are non-positive.
  - $L\mathbf{1} = 0$ (i.e. Row sums are equal to 0)
  - $L'\mathbf{1}$ need not be 0. (i.e. Column sums need not be 0).
  - $\mathrm{rank}(L)$ need not be $n-1$.

Strongly connected digraph

- A directed graph G is **strongly connected**, if each pair of vertices is connected by a directed path.

Example





- For a strongly connected graph G, $\mathrm{rank}(L) = n - 1$.

Balanced digraphs

- Indegree of vertex i is the total number of edges coming into i. $(= \sum_j a_{ji} = (A'\mathbf{1})_i)$.
- Outdegree of vertex i is the total number of edges going out of i. $(= \sum_j a_{ij} = (A\mathbf{1})_i)$.
- Vertex i is *balanced*, if

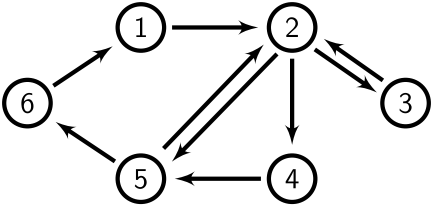
$$\text{Indegree of } i = \text{Outdegree of } i.$$

- Digraph G is **balanced** if all the vertices are balanced.

Balanced digraphs

Example

A balanced digraph



Indegree/Outdegree of vertices 1, 3, 4 and 6 $= 1$.

Indegree/Outdegree of vertex 2 $= 3$.

Indegree/Outdegree of vertex 5 $= 2$.

Balanced digraph

Example

The adjacency and Laplacian matrices of G are:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- For a balanced graph G, $L'\mathbf{1} = 0$.

Resistance

- Let $J := \mathbf{1}\mathbf{1}'$.

We define the resistance in digraphs.

Definition (Resistance)

The resistance between any two vertices i and j in V is defined by

$$r_{ij} := l^{\dagger}_{ii} + l^{\dagger}_{jj} - 2l^{\dagger}_{ij},$$

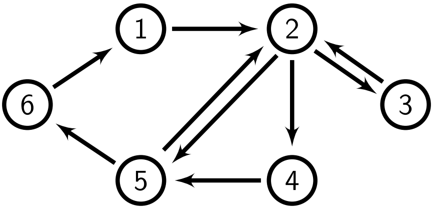
where $l^{\dagger}_{ij}$ is the $(i,j)^{\mathrm{th}}$ entry in the Moore-Penrose inverse of L.

- $R := [r_{ij}]$ is called the resistance matrix of G.

Resistance matrix

Example

The directed graph G is strongly connected and balanced.



Resistance matrix

Example

The Moore-Penrose inverse of L is:

$$L^{\dagger} = \begin{bmatrix} \frac{5}{9} & \frac{1}{18} & -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{2}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & -\frac{1}{9} \\ -\frac{4}{9} & \frac{1}{18} & \frac{8}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{5}{18} \\ -\frac{7}{36} & -\frac{7}{36} & -\frac{13}{36} & \frac{23}{36} & \frac{5}{36} & -\frac{1}{36} \\ -\frac{1}{36} & -\frac{1}{36} & -\frac{7}{36} & -\frac{7}{36} & \frac{11}{36} & \frac{5}{36} \\ \frac{7}{18} & -\frac{1}{9} & -\frac{5}{18} & -\frac{5}{18} & -\frac{5}{18} & \frac{5}{9} \end{bmatrix}.$$

Resistance matrix

Example

The resistance matrix is:

$$R = [r_{ij}] = [l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}] = \begin{bmatrix} 0 & \frac{2}{3} & \frac{5}{3} & \frac{17}{12} & \frac{13}{12} & \frac{5}{3} \\ \frac{4}{3} & 0 & 1 & \frac{3}{4} & \frac{5}{12} & 1 \\ \frac{7}{3} & 1 & 0 & \frac{7}{4} & \frac{17}{12} & 2 \\ \frac{19}{12} & \frac{5}{4} & \frac{9}{4} & 0 & \frac{2}{3} & \frac{5}{4} \\ \frac{11}{12} & \frac{7}{12} & \frac{19}{12} & \frac{4}{3} & 0 & \frac{7}{12} \\ \frac{1}{3} & 1 & 2 & \frac{7}{4} & \frac{17}{12} & 0 \end{bmatrix}.$$

Properties of the resistance

Let $G = (V, E)$ be a simple, strongly connected and balanced directed graph with vertex set $V = \{1, \ldots, n\}$ and edge set E. If $R := [r_{ij}]$ is the resistance matrix of G, then

Theorem (R.Balaji, R. B. Bapat and Shivani Goel. Resistance matrices of balanced directed graphs, Linear and Multilinear Algebra,(2020).)

(A) $r_{ij} = 0$ *iff* $i = j$.

(B) $r_{ij} \geq 0$

i.e. Resistance distance is non-negative.

(C) *For* $i, j, k \in V$, $r_{ij} \leq r_{ik} + r_{kj}$

i.e. Resistance distance satisfies triangle inequality.

Distance matrix

- For each distinct pair of vertices i and j in V, let d_{ij} be the length of the shortest directed path from i to j and define $d_{ii} := 0$.
- The non-negative real number d_{ij} is the classical distance between i and j.
- By numerical experiments, we noted that the inequality $r_{ij} \leq d_{ij}$ always holds.

Example

Consider the graph below.

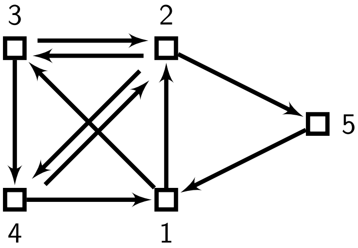


Figure: A strongly connected and balanced digraph on 5 vertices.

Example

The resistance and distance matrices of G are:

$$R = [r_{ij}] = \begin{bmatrix} 0 & \frac{16}{35} & \frac{18}{35} & \frac{27}{35} & \frac{44}{35} \\ \frac{24}{35} & 0 & \frac{22}{35} & \frac{3}{5} & \frac{4}{5} \\ \frac{32}{35} & \frac{18}{35} & 0 & \frac{19}{35} & \frac{46}{35} \\ \frac{23}{35} & \frac{19}{35} & \frac{31}{35} & 0 & \frac{47}{35} \\ \frac{26}{35} & \frac{6}{5} & \frac{44}{35} & \frac{53}{35} & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 3 & 0 \end{bmatrix}.$$

It is easily seen that $r_{ij} \leq d_{ij}$ for each i, j.

• Given a general strongly connected and balanced digraph, we do not know how to prove the above inequality.

• In this talk, when G is a directed cactus graph, we discuss a proof for this inequality.

Directed cycle

- A directed cycle graph is a directed version of a cycle graph with all edges being oriented in the same direction.

Example

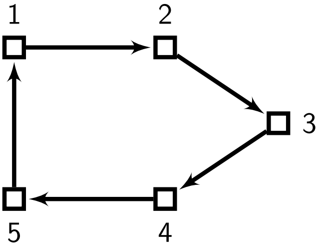


Figure: Directed cycle Graph on 5 vertices.

Directed cactus graph

- A directed cactus graph is a strongly connected digraph in which each edge is contained in exactly one directed cycle.

OR

- A digraph G is a directed cactus if and only if any two directed cycles of G share at most one common vertex.

Example

The graph G given in Figure 3 is a directed cactus graph.

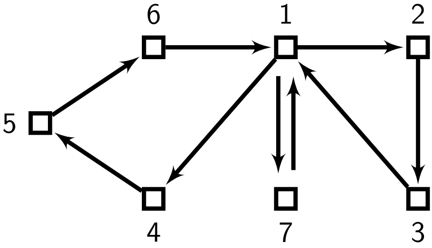


Figure: A directed cactus graph on 7 vertices.

- In a directed cactus, for each vertex i, $\delta_i^{in} = \delta_i^{out}$ and hence it is balanced.

Spanning tree rooted at a vertex

Suppose $G = (V, \mathcal{E})$ is a digraph with vertex set $V = \{1, 2, ..., n\}$ and Laplacian matrix L. A spanning *tree* of G rooted at vertex i is a connected subgraph T with vertex set V such that

(i) Every vertex of T other than i has indegree 1.

(ii) The vertex i has indegree 0.

(iii) T has no directed cycles.

Example

The graph H has two spanning trees rooted at 1.

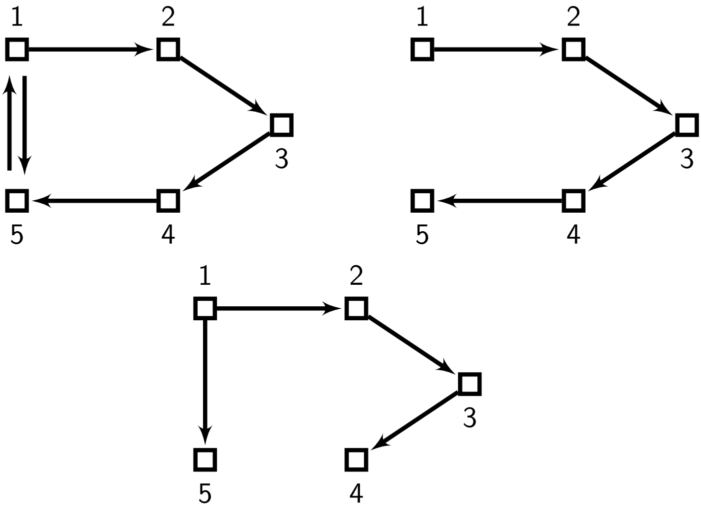


Figure: (a) Digraph H (b) Spanning trees of H rooted at 1.

Notations

- Let Δ_1 and Δ_2 are non-empty subsets of $\{1, \dots, n\}$ and $\pi : \Delta_1 \to \Delta_2$ be a bijection.
- The pair $\{i, j\} \subset \Delta_1$ is called an inversion in π if $i < j$ and $\pi(i) > \pi(j)$.
- Let $n(\pi)$ denote the number of inversions in π.
- For a matrix A, $A[\Delta_1, \Delta_2]$ will denote the submatrix of A obtained by choosing rows and columns corresponding to Δ_1 and Δ_2, respectively.
- For $\Delta \subseteq \{1, 2, \dots, n\}$, we define $\alpha(\Delta) = \sum_{i \in \Delta} i$.

All minors matrix tree theorem (AMMTT)

Let $G = (V, E)$ be a digraph with vertex set $V = \{1, 2, \dots, n\}$ and Laplacian matrix L. Let $\Delta_1, \Delta_2 \subset V$ be such that $|\Delta_1| = |\Delta_2|$. Then

$$\det(L[\Delta_1^c, \Delta_2^c]) = (-1)^{\alpha(\Delta_1)+\alpha(\Delta_2)} \sum_F (-1)^{n(\pi)}.$$

where the sum is over all spanning forests F such that

(a) F contains exactly $|\Delta_1| = |\Delta_2|$ trees.

(b) each tree in F contains exactly one vertex in Δ_2 and exactly one vertex in Δ_1.

(c) each directed edge in F is directed away from the vertex in Δ_2 of the tree containing that directed edge. (i.e. each vertex in Δ_2 is the root of the tree containing it.)

F defines a bijection $\pi : \Delta_1 \rightarrow \Delta_2$ such that $\pi(j) = i$ if and only if i and j are in the same oriented tree of F.

- Let $\kappa(G, i)$ be the number of spanning trees of G rooted at i.

- By AMMTT, it immediately follows that

$$\kappa(G, i) = \det(L[\{i\}^c, \{i\}^c]). \tag{1}$$

- Suppose G is a strongly connected and balanced directed graph. Let L be the Laplacian matrix of G.

- Since $\mathrm{rank}(L) = n - 1$ and $L\mathbf{1} = L'\mathbf{1} = 0$, all the cofactors of L are equal.

- From (1), we see that $\kappa(G, i)$ is independent of i.

- From here on, we shall denote $\kappa(G, i)$ simply by $\kappa(G)$.

Notation

Let $i, j, k \in V$. We introduce the following two notation.

1. Let $\#(F[\{i \to\}, \{j \to\}])$ denote the number of spanning forests F of G such that (i) F contains exactly 2 trees, (ii) each tree in F contains either i or j, and (iii) vertices i and j are the roots of the respective trees containing them.
2. Let $\#(F[\{k \to\}, \{j \to, i\}])$ denote the number of spanning forests F of G such that (i) F contains exactly 2 trees, (ii) each tree in F exactly contains either k or both i and j, and (iii) vertices k and j are the roots of the respective trees containing them.

From AMMTT, we deduce the following proposition which will be used to prove the main result.

Proposition (1)

Let $i, j \in V$ be two distinct vertices. Then

(a)

$$\det(L[\{i,j\}^c, \{i,j\}^c]) = \#(F[\{i \rightarrow\}, \{j \rightarrow\}]).$$

Proof: Substituting $\Delta_1 = \Delta_2 = \{i,j\}$ in AMMTT, we have

$$\det(L[\{i,j\}^c, \{i,j\}^c]) = (-1)^{2i+2j} \sum_{F} (-1)^{n(\pi)} \qquad (2)$$

where the sum is over all forests F such that

(i) F contains exactly 2 trees,

(ii) each tree in F contains either i or j, and

(iii) vertices i and j are the roots of the respective trees containing them.

Since for each such forest F, $\pi(i) = i$ and $\pi(j) = j$, there are no inversions in π. Thus $n(\pi) = 0$.

Hence from (2), we have

$$\det(L[\{i,j\}^c, \{i,j\}^c]) = \#(F[\{i \to\}, \{j \to\}]).$$

This completes the proof of (a).

(b) If $i \neq n$ and $j \neq n$, then

$$\det(L[\{n,i\}^c,\{n,j\}^c]) = (-1)^{i+j}\#(F[\{n\rightarrow\},\{j\rightarrow, i\}]).$$

Proof: Substitute $\Delta_1 = \{n, i\}$ and $\Delta_2 = \{n, j\}$ in AMMTT to obtain

$$\det(L[\{n,i\}^c,\{n,j\}^c]) = (-1)^{2n+i+j}\sum_F(-1)^{n(\pi)} \qquad (3)$$

where the sum is over all forests F such that

(i) F contains exactly 2 trees,

(ii) each tree in F exactly contains either n or both i and j,and

(iii) vertices n and j are the roots of the respective trees containing them.

For each such forest F, $\pi(n) = n$ and $\pi(i) = j$.

Since $i, j < n$, there are no inversions in π and so $n(\pi) = 0$.

From (3), we have

$$\det(L[\{n, i\}^c, \{n, j\}^c]) = (-1)^{i+j} \#(F[\{n \to\}, \{j \to, i\}]).$$

Hence (b) is proved.

(c) If $i \neq 1$ and $j \neq 1$, then

$$\det(L[\{1,i\}^c, \{1,j\}^c]) = (-1)^{i+j} \#(F[\{1 \to\}, \{j \to, i\}]).$$

Proof: The proof of (c) is similar to the proof of (b).

Lemma (1)

Let L be a Z-matrix such that $L\mathbf{1} = L'\mathbf{1} = 0$ and $\mathrm{rank}(L) = n-1$.

If e is the vector of all ones in $\mathbb{R}^{n-1}$, then L can be partitioned as

$$L = \begin{bmatrix} B & -Be \\ -e'B & e'Be \end{bmatrix},$$

where B is a square matrix of order $n-1$ and

$$L^{\dagger} = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{1}\mathbf{1}'.$$

Let $G = (V, E)$ be a strongly connected and balanced digraph with vertex set $V = \{1, 2, \ldots, n\}$, Laplacian matrix L and resistance matrix $R = (r_{ij})$.

Lemma (2)

Let $i, j \in V$. If $(i, j) \in E$ or $(j, i) \in E$, then

$$\det(L[\{i, j\}^c, \{i, j\}^c]) \leq \kappa(G).$$

- As G is balanced, we know that $\delta_i^{in} = \delta_i^{out}$ for any i.
- We call this common value to be the degree of i.

Lemma (3)

Let $(i, j) \in E$. If either i or j has degree 1, then $r_{ij} \leq 1$.

Proof.

Without loss of generality, let $i = 1$ and $j = n$.

Let $B = L[\{n\}^c, \{n\}^c]$. Then

$$L^\dagger = \begin{bmatrix} B^{-1} - \frac{1}{n}ee'B^{-1} - \frac{1}{n}B^{-1}ee' & -\frac{1}{n}B^{-1}e \\ & \\ -\frac{1}{n}e'B^{-1} & 0 \end{bmatrix} + \frac{e'B^{-1}e}{n^2}\mathbf{11}'.$$

Let $C = B^{-1}$, $C = (c_{ij})$, $x = Ce$ and $y = C'e$. □

By a well-known result on Z-matrices, C is a non-negative matrix.

Using (36), we have

$$\begin{aligned} r_{1n} &= l^{\dagger}_{11} + l^{\dagger}_{nn} - 2l^{\dagger}_{1n} \\ &= c_{11} - \frac{1}{n}y_1 - \frac{1}{n}x_1 + \frac{2}{n}x_1 \\ &= c_{11} - \frac{1}{n}(y_1 - x_1). \end{aligned}$$

We claim that $x_1 \leq y_1$.

To see this, we consider the following cases:

(i) degree of vertex 1 is one.

(ii) degree of vertex n is one.

Case (i): For $k \in \{2, 3, \ldots, n-1\}$,

$$\begin{aligned} c_{1k} &= \frac{(-1)^{1+k}}{\det(B)} \det(B[\{k\}^c, \{1\}^c]) \\ &= \frac{(-1)^{1+k}}{\det(L[\{n\}^c, \{n\}^c])} \det(L[\{n, k\}^c, \{n, 1\}^c]). \end{aligned} \tag{4}$$

Using (1) and Proposition 1(b) in (4), we get

$$c_{1k} = \frac{\#(F[\{n \to\}, \{1 \to, k\}])}{\kappa(G)},$$

As degree of vertex 1 is one, $(1, n)$ is the only edge directed away from 1.

So, it is not possible for a forest to have a tree such that the tree does not contain the vertex n but contains both the vertices 1 and k with 1 as the root.

Therefore, no such forest F exists and hence , $c_{1k} = 0$ for each $k \in \{2, 3, \ldots, n-1\}$.

Using the fact that C is a non-negative matrix, we have

$$x_1 = \sum_{k=1}^{n-1} c_{1k} = c_{11} \leq \sum_{k=1}^{n-1} c_{k1} = y_1.$$

Hence $x_1 \leq y_1$.

Case (ii): *R. Balaji, R. B. Bapat and shivani Goel. Resistance distance in directed cactus graphs, The Electronic Journal of Linear Algebra,* 36*(2020).*

We now obtain

$$r_{1n} \leq c_{11} = \frac{\det(L[\{1,n\}^c, \{1,n\}^c])}{\kappa(G)}.$$

By Lemma 2, it follows that $r_{1n} \leq 1$. The proof is complete.

Lemma (4)

Let $G = (V, E)$ be a directed cactus graph on n vertices. Then there is a unique directed path from i to j.

Lemma (5)

Let $V := \{1, \ldots, n\}$ and $G = (V, E)$ be a directed cactus graph. Suppose $(i, j) \in E$. If both i and j have degree greater than one, then V can be partitioned into three disjoint sets

(a) $\{i, j\}$

(b) $V_j(i \rightarrow)$

(c) $V_i(j \rightarrow)$,

where $V_\nu(\delta \rightarrow) = \{k \in V \setminus \{\delta, \nu\} : \exists$ a directed path from δ to k which does not pass through $\nu\}$.

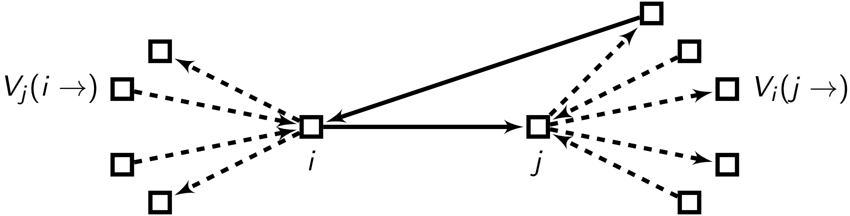


Figure: Partition of a directed cactus graph.

Main result

Theorem

Let $G = (V, E)$ be a directed cactus graph with $V = \{1, 2, \ldots, n\}$. If $R = (r_{ij})$ and $D = (d_{ij})$ are the resistance and distance matrices of G, respectively, then $r_{ij} \leq d_{ij}$ for each $i, j \in \{1, 2, \ldots, n\}$.

Proof.

By triangle inequality, it suffices to show that if $(i, j) \in E$, then $r_{ij} \leq 1$.

In view of Lemma 3, it suffices to show this inequality when both i and j have degree greater than one.

Without loss of generality, assume $i = 1$ and $j = n$. □

We know that

$$r_{1n} = c_{11} - \frac{1}{n}(y_1 - x_1).$$

As before, it is sufficient to show that $x_1 \leq y_1$.

Let $k \in \{2, 3, \ldots, n-1\}$. Then we already know

$$c_{1k} = \frac{\#(F[\{n \to\}, \{1 \to, k\}])}{\kappa(G)},$$

Also

$$\begin{aligned} c_{k1} &= \frac{(-1)^{1+k}}{\det(B)} \det(B[\{1\}^c, \{k\}^c]) \\ &= \frac{(-1)^{1+k}}{\det(L[\{n\}^c, \{n\}^c])} \det(L[\{n, 1\}^c, \{n, k\}^c]). \end{aligned} \tag{5}$$

Using (1) and Proposition 1(b) in (5), we get

$$c_{k1} = \frac{\#(F[\{n \rightarrow\}, \{k \rightarrow, 1\}])}{\kappa(G)}$$

Recall that the vertex set V can be partitioned into three disjoint sets

(a) $\{1, n\}$

(b) $V_n(1 \rightarrow)$

(c) $V_1(n \rightarrow)$.

Observations:

(i) for each $k \in V_n(1 \to)$,

$$\#(F[\{n \to\}, \{1 \to, k\}]) = 1$$

(ii) for every $k \notin V_n(1 \to)$,

$$\#(F[\{n \to\}, \{1 \to, k\}]) = 0.$$

(iii) for each $k \in V_n(1 \to)$,

$$\#(F[\{n \to\}, \{k \to, 1\}]) \geq 1.$$

Thus, we have

$$c_{1k} = \begin{cases} \frac{1}{\kappa(G)} & \text{if } k \in V_n(1 \rightarrow) \\ 0 & \text{otherwise.} \end{cases}$$

and

$$c_{k1} \geq \frac{1}{\kappa(G)}, \quad \text{whenever } k \in V_n(1 \rightarrow).$$

Since C is a non-negative matrix, we have

$$
\begin{aligned}
x_1 &= \sum_{k=1}^{n-1} c_{1k} \\
&= c_{11} + \sum_{k \in V_n(1\to)} c_{1k} \\
&= c_{11} + \sum_{k \in V_n(1\to)} \frac{1}{\kappa(G)} \\
&\leq c_{11} + \sum_{k \in V_n(1\to)} c_{k1} \leq \sum_{k=1}^{n-1} c_{k1} = y_1.
\end{aligned}
$$

Hence, $r_{1n} \leq 1$. This completes the proof.

References

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Thank You!