An introduction to Fan-Theobald-von Neumann systems

M. Seetharama Gowda

Department of Mathematics and Statistics

University of Maryland, Baltimore County

Baltimore, Maryland

gowda@umbc.edu

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This talk is based on articles

- M.S. Gowda, *Optimizing certain combinations of linear/distance functions over spectral sets,* arXiv:1902.06640v2, March 2019.
- M.S. Gowda, *Commutation principles for optimization problems on spectral sets in Euclidean Jordan algebras,* arXiv:2009.04874v1, Sept. 2020, to appear in Optimization Letters.
- M.S. Gowda and J. Jeong, *Commutativity, majorization, and reduction in Fan-Theobald-von Neumann systems*, under preparation.

Definition

Consider a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a nonlinear map. For any $u \in \mathcal{V}$, let

$[u] := \{x \in \mathcal{V} : \lambda(x) = \lambda(u)\}$

denote the so-called λ-orbit of u.

We say that the triple $(\mathcal{V}, \mathcal{W}, \lambda)$

is a **Fan-Theobald-von Neumann system** if

Definition of FTvN system

- λ is norm-preserving, so $||\lambda(x)|| = ||x||$ for all $x \in \mathcal{V}$,
- for all $x, y \in \mathcal{V}$, $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$, and
- for all $c, u \in \mathcal{V}$, $\max\Big\{\langle c, x\rangle : x \in [u]\Big\} = \langle \lambda(c), \lambda(u)\rangle.$

Note that the second item is a consequence of the third.

The third item specifies the attainment of the linear function $x \mapsto \langle c, x\rangle$ on any λ-orbit.

The inequality $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$
will be called *Fan-Theobald-von Neumann inequality* and

the equality $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$
(whenever holds) defines the *commutativity*
of x and y in this system.

A number of questions arise:

Why the big names?

What are some examples?

What concepts/results can be obtained in this general setting?

What are some applications? Open problems?

We will address these in the course of the talk.

Examples include spaces of real/complex Hermitian matrices, Euclidean Jordan algebras, spaces induced by hyperbolic polynomials, normal decomposition systems, Eaton triples,...

The new concepts deal with commutativity, majorization, doubly stochastic maps,...

Applications deal with optimality conditions, complementarity problems, variational inequality problems,...

Two elementary examples

Example 1: Let $\mathcal{V}$ be a real inner product space and

$\lambda : \mathcal{V} \to \mathbb{R}$, $\lambda(x) := ||x||$. Then,

- λ is norm-preserving,
- $\langle x, y\rangle \leq ||x||\,||y|| = \langle \lambda(x), \lambda(y)\rangle$ (Cauchy-Schwarz), and
- for all c,

$$\max\Big\{\langle c,\, x\rangle : ||x|| = 1\Big\} = ||c||.$$

By scaling, the last item becomes

- For all $c, u \in \mathcal{V}$,

$$\max\left\{\langle c, x\rangle : ||x|| = ||u||\right\} = ||c||\,||u||,$$

which translates to

$$\max\left\{\langle c, x\rangle : x \in [u]\right\} = \langle \lambda(c), \lambda(u)\rangle.$$

Thus, $(\mathcal{V}, \mathbb{R}, \lambda)$ is a FTvN system.

Here, x and y commute iff one of them is a nonnegative multiple of the other.

Example 2: Let $\mathcal{V}$ be a real inner product space,

$S : \mathcal{V} \to \mathcal{V}$ be a linear isometry

(=inner product preserving).

Then, with $\lambda(x) = Sx$,

$(\mathcal{V}, \mathcal{V}, \lambda)$ becomes a FTvN system.

Here, $[u] = \{x : Sx = Su\} = \{u\}$ and $\langle x, y\rangle = \langle Sx, Sy\rangle$.

So, any two elements commute.

The Euclidean space $\mathbb{R}^n$

Consider $\mathbb{R}^n$, where objects are column/row vectors.

$\mathbb{R}^n$ carries the usual inner product and norm.

For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$,

$x \circ y := (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$ (Hadamard/Jordan product).

Note that $\mathbb{R}^n_+ = \{x \circ x : x \in \mathbb{R}^n\}$ (nonnegative orthant).

For $x \in \mathbb{R}^n$, $x^\downarrow$ is the *decreasing rearrangement* of entries of x.

For example, in $\mathbb{R}^3$ with $x = (-1, 2, 0)$, $x^\downarrow = (2, 0, -1)$.

If vectors in $\mathbb{R}^n$ are viewed as column vectors, then $x^\downarrow = Px$, for some permutation matrix P.

Hardy-Littlewood-Pólya (1938):

In $\mathbb{R}^n$,

$$\langle x, y\rangle \leq \langle x^\downarrow, y^\downarrow\rangle,$$

with equality if and only if there exists a permutation matrix P with $Px = x^\downarrow$ and $Py = y^\downarrow$.

This immediately shows that for all $c, u \in \mathbb{R}^n$,

$$\max\left\{ \langle c,\, x\rangle :\; x^{\downarrow} = u^{\downarrow} \right\} = \langle c^{\downarrow}, u^{\downarrow}\rangle.$$

(Let $c = Pc^{\downarrow}$ for some P and $x := Pu^{\downarrow}$.)

Now, let $\lambda(x) = x^{\downarrow}$.

We may think of entries of x as eigenvalues of the diagonal matrix with x as its diagonal. Then, $\lambda(x)$ is the vector of eigenvalues of x written in the decreasing order. We call λ, the eigenvalue map.

Then, the λ-orbit of $u \in \mathbb{R}^n$ is

$[u] := \{x \in \mathbb{R}^n : x^{\downarrow} = u^{\downarrow}\} = \{Pu : \ P \in \Sigma_n\}$,

where Σ_n denotes the group of all $n \times n$ permutation matrices.

Aside: Σ_n is precisely the set of all real invertible matrices A with $A(x \circ y) = Ax \circ Ay$ for all $x \in \mathbb{R}^n$.

So, if $(\mathbb{R}^n, \circ)$ is viewed as an algebra, then Σ_n becomes its automorphism group.

The result of Hardy-Littlewood-Pólya can now be rewritten: For all $c, u \in \mathbb{R}^n$,

$$\max\Big\{\,\langle c,\,x\rangle\,:\,x \in [u]\Big\} = \langle \lambda(c), \lambda(u)\rangle.$$

As $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is norm-preserving, we see that $(\mathbb{R}^n, \mathbb{R}^n, \lambda)$ is a FTvN system.

In this system, vectors x and y commute if and only if a single permutation matrix takes x to $x^{\downarrow}$ and y to $y^{\downarrow}$.

Example: In $(\mathbb{R}^2, \mathbb{R}^2, \lambda)$, $(0, 1)$ and $(-1, 0)$ commute, but $(1, 0)$ and $(0, 1)$ do not.

However, viewed as diagonal matrices, any two vectors in $\mathbb{R}^n$ commute.

We shall see that the usual matrix commutativity is weaker than the commutativity in a corresponding FTvN system.

Majorization in $\mathbb{R}^n$

Given $u, v \in \mathbb{R}^n$, we say that u is *majorized* by v and write $u \prec v$ if for all natural numbers k, $1 \leq k \leq n$,

$$\sum_{i=1}^{k} u_i^{\downarrow} \leq \sum_{i=1}^{k} v_i^{\downarrow}$$

with equality for $k = n$. *A result of Hardy, Littlewood, and Pólya says that $u \prec v$ if and only if $u = Dv$, where D is a $n \times n$ doubly stochastic matrix* (i.e., a nonnegative matrix where each row/column sum is one).

Additionally, *a result of Birkhoff says that every doubly stochastic matrix D is a convex combination of permutation matrices.*

So, *in the setting of $\mathbb{R}^n$, $u \prec v$ if and only if*

$u \in$ *conv* $[v]$,

where 'conv' refers to the convex hull.

Fact: A real $n \times n$ matrix D is a doubly stochastic matrix if and only if $Dx \prec x$ for all $x \in \mathbb{R}^n$.

Polynomial interpretation

On the real vector space $\mathbb{R}^n$, consider the polynomial

$p(x) := x_1 x_2 \cdots x_n$, where $x = (x_1, x_2, \ldots, x_n)$.

Then p is a real homogeneous polynomial.

With $e = (1, 1, \ldots, 1)$,

for any fixed $x \in \mathbb{R}^n$, the roots of

the univariate polynomial

$t \mapsto p(te - x) = (t - x_1)(t - x_2) \cdots (t - x_n)$

are all **real**, namely, $x_1, x_2, \ldots, x_n$.

With these numbers, we can create a vector $\lambda(x)$ with decreasing components.

Note that $\lambda(x) = 0 \Rightarrow x = 0$.

This λ induces an inner product on $\mathbb{R}^n$:

$$\langle x, y \rangle := \frac{1}{4}\Big[||\lambda(x+y)||^2 - ||\lambda(x-y)||^2\Big].$$

Then, $\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle$.

The polynomial p along with e given above is an example of hyperbolic polynomial.

Spaces of Hermitian matrices

Let $\mathcal{S}^n$ denote the space of all $n \times n$ real symmetric matrices, and $\mathcal{H}^n$ denote the space of all $n \times n$ complex Hermitian matrices. We let $\mathcal{V}$ denote one of these spaces.

On $\mathcal{V}$, we define the inner product and Jordan product:

$\langle X, Y \rangle := trace(XY)$ and $X \circ Y := \frac{XY+YX}{2}$.

Note that

$\mathcal{V}_+ := \{X \circ X : X \in \mathcal{V}\}$ is the semidefinite cone.

Each $X \in \mathcal{V}$ has a spectral decomposition:

$$X = UDU^*,$$

where U is unitary (orthogonal when $X \in \mathcal{S}^n$) and D is a diagonal matrix consisting of (real) eigenvalues of X.

Let $\lambda(X)$ denote the vector of eigenvalues of X written in the decreasing order. Then, $\lambda : \mathcal{V} \to \mathbb{R}^n$ is single-valued and norm-preserving.

Theorem: In $\mathcal{V}$,

Richter 1958, Fan(?) 1949: $\langle X, Y \rangle \leq \langle \lambda(X), \lambda(Y) \rangle$.

Theobald 1975: $\langle X, Y \rangle = \langle \lambda(X), \lambda(Y) \rangle$

if and only if there is a unitary matrix U such that

$X = U\, diag(\lambda(X))\, U^*$ and $Y = U\, diag(\lambda(Y))\, U^*$.

(So, these two generalize Hardy-Littlewood-Pólya result to matrices.)

Orbits

In $\mathcal{V}$, X belongs to the λ-orbit of Y if and only if $\lambda(X) = \lambda(Y)$, or equivalently, $Y = UXU^*$ for some unitary matrix.

Thus, $[X] = \{UXU^* : U \text{ unitary}\}$.

Aside: Linear transformations of the form $X \mapsto UXU^*$ are the only invertible linear transformations on $\mathcal{V}$ that keep the Jordan product invariant.

Claim: $(\mathcal{V}, \mathcal{V}, \lambda)$ is a FTvN system.

As λ is norm-preserving, we need to show that for all $C, Z \in \mathcal{V}$, $\max\Big\{ \langle C,\, X\rangle :\ X \in [Z] \Big\} = \langle \lambda(C), \lambda(Z)\rangle.$

By Richter's result, $\langle C,\, X\rangle \leq \langle \lambda(C), \lambda(X)\rangle = \langle \lambda(C), \lambda(Z)\rangle$ for any $X \in [Z]$. To see the (max) equality above, write $C = U\, \text{diag}(\lambda(C))\, U^*$ for some unitary U.

Define $X := U\, \text{diag}(\lambda(Z))\, U^*$. Then, $X \in [Z]$ and $\langle C, X\rangle = \langle \lambda(C), \lambda(Z)\rangle$. Hence the claim.

Consider the FTvN system $(\mathcal{H}^n, \mathbb{R}^n, \lambda)$.

As observed above, X and Y commute in this system

if and only if there is a unitary matrix U such that

$X = U\, diag(\lambda(X))\, U^*$ and $Y = U\, diag(\lambda(Y))\, U^*$.

In particular, commutativity in this system

implies $XY = YX$, i.e., the usual matrix commutativity.

However, the converse is not true:

Let X and Y be 2×2 diagonal matrices with

diagonals $(1, 0)$ and $(0, 1)$.

Polynomial approach

For any $X \in \mathcal{V}$, let $p(X) := \det(X)$.

This is a real homogeneous polynomial on $\mathcal{V}$.

With $e = I$, the roots of the univariate polynomial

$t \mapsto p(te - X) = \det(tI - X)$ are all real.

With $\lambda(X)$ denoting the roots of this polynomial written in

the decreasing order, one can define the inner product

$$\langle X, Y \rangle := \frac{1}{4}\Big[||\lambda(X+Y)||^2 - ||\lambda(X-Y)||^2 \Big]$$

and verify that $\langle X, Y \rangle \leq \langle \lambda(X), \lambda(Y) \rangle$.

Majorization in $\mathcal{V}$

Given $A, B \in \mathcal{V}$, A is said to be majorized by B if $\lambda(A)$ is majorized by $\lambda(B)$ in $\mathbb{R}^n$.

In this setting, one writes $A \prec B$.

Ando 1989 has shown: $A \prec B$ *if and only if A is a convex combination of matrices of the form UBU^*, where U is a unitary matrix.*

In our language,

$A \prec B$ if and only if $A \in$ conv $[B]$.

Space of all $n \times n$ matrices

Let M_n denote the space of all $n \times n$ complex matrices.

This becomes a real inner product space under

$\langle X, Y \rangle := Re(trace(X^*Y))$.

For each $X \in M_n$, let $s(X)$ denote the vector of

singular values of X (=eigenvalues of $\sqrt{X^*X}$)

written in the decreasing order.

Recall **Singular Value Decomposition**:

$X = U\,\text{diag}(s(X))\,V$, where U and V are unitary matrices.

Theorem (von Neumann 1935): In M_n,

$$\langle X, Y \rangle \leq \langle s(X), s(Y) \rangle,$$

with equality if and only if $X = U\,\text{diag}(s(X))\,V$ and $Y = U\,\text{diag}(s(Y))\,V$ for some unitary matrices U and V.

As $s : M_n \to \mathbb{R}^n$ is norm-preserving, above result implies that $(M_n, \mathbb{R}^n, s)$ is a FTvN system.

Hyperbolic polynomials

Hyperbolic polynomials were introduced by Gärding 1959 in connection with partial differential equations.

Let $\mathcal{V}$ be a finite dimensional real vector space, $e \in \mathcal{V}$, and p be a real homogeneous polynomial on $\mathcal{V}$.

We say that p is *hyperbolic with respect to* e if $p(e) \neq 0$ and for every $x \in \mathcal{V}$, the roots of the univariate polynomial $t \to p(te - x)$ are all real.

For any $x \in \mathcal{V}$, let $\lambda(x)$ denote the vector of roots of this univariate polynomial with entries written in the decreasing order.

Gärding 1959 has shown that the set $\{x \in \mathcal{V} : \lambda(x) \geq 0\}$ is a convex cone.

This is the *hyperbolicity cone* corresponding to p.

Optimization relative to this cone is *hyperbolic programming.*

Examples of hyperbolic polynomials are:

On $\mathbb{R}^n$: $p(x) = x_1 x_2 \cdots x_n$ with $e = (1, 1, 1 \ldots, 1)$,

On $\mathcal{S}^n$: $p(X) = \det(X)$ with $e = I$.

On $\mathbb{R}^n$: $p(x) = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ with $e = (1, 0, \ldots, 0)$.

(The last example may have been the

reason for the use of the word

'hyperbolic' in this analysis.

The corresponding hyperbolicity cone is the so-called

Lorentz cone = second-order cone= ice-cream cone.)

When p is *complete* (which means that $\lambda(x) = 0 \Rightarrow x = 0$),

$\mathcal{V}$ becomes a inner product space under the

inner product $\langle x, y \rangle := \frac{1}{4}\left[||\lambda(x+y)||^2 - ||\lambda(x-y)||^2\right]$,

see Bauschke et al., 2001.

We say that p is *isometric*, if for

every $c \in \mathcal{V}$ and $q = \lambda(u)$ with $u \in \mathcal{V}$,

there exists $x \in \mathcal{V}$ such that $\lambda(x) = q$

and $\lambda(c + x) = \lambda(c) + \lambda(x)$.

It can be shown that when the hyperbolic polynomial p is complete and isometric, the triple $(\mathcal{V}, \mathbb{R}^n, \lambda)$ becomes a FTvN system. However, beyond the definition, no characterization is known. Also, not many examples are known.

Open problem: Construct/characterize complete and isometric hyperbolic polynomials.

So far, we have seen a number of examples of FTvN systems and the reason why we have used the names of Fan, Theobald, and von Neumann.

There are other important examples of FTvN systems, namely, Euclidean Jordan algebras, normal decomposition systems, and Eaton triples.

These will be mentioned at the end of the talk.

Commutativity in FTvN systems

Recall that in a FTvN system, two elements x and y commute if $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$.

We now introduce a number of results/concepts related to this commutativity.

In a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, the **center** consists of those elements in $\mathcal{V}$ that commute with every element of $\mathcal{V}$.

A nonzero element e in $\mathcal{V}$ is a **unit** element if center $=\mathbb{R}\, e$.

Example: Identity matrix is a unit element in $\mathcal{S}^n$ $(\mathcal{H}^n)$.

Theorem: *In a FTvN system, the following hold:*

- λ *is positive homogeneous and Lipschitz.*
- x *and* y *commute if and only if* $\lambda(x+y)=\lambda(x)+\lambda(y)$*.*
- *The center is a closed linear subspace of* $\mathcal{V}$ *and* λ *is linear on it.*

Decomposition

Recall that the center is a closed subspace of $\mathcal{V}$.

Theorem: *Let* $(\mathcal{V}, \mathcal{W}, \lambda)$ *be a FTvN system,*

where $\mathcal{V}$ *is a Hilbert space. With* $\mathcal{C}$ *denoting the center,*

- $\mathcal{V} = \mathcal{C} + \mathcal{C}^{\perp}$,
- $(C, \mathcal{W}, \lambda)$ *is a FTvN system whose center is* C.
- $(C^{\perp}, \mathcal{W}, \lambda)$ *is a FTvN system whose center is* $\{0\}$.

Our trivial Example 2 has full center, and

M_n has trivial center.

Spectral sets

In a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, a set E in $\mathcal{V}$ is a **spectral set** if it is of the form $E = \lambda^{-1}(Q)$ for some set $Q \subseteq \mathcal{W}$.

(So, E is just a union of λ-orbits.)

Example: In $(\mathbb{R}^n, \mathbb{R}^n, \lambda)$, a set is spectral iff it is permutation invariant.

Example: Semidefinite cone is spectral in $\mathcal{S}^n$.

The following result shows that the property of being spectral is invariant under certain algebraic and topological operations.

Theorem: *Let $(\mathcal{V}, \mathcal{W}\,\lambda)$ be a FTvN system. For a spectral set E, the following hold:*

- *$\overline{E}$, E°, and $\partial(E)$ are spectral.*
- *If $\mathcal{V}$ is finite dimensional, then conv(E) is a spectral set.*
- *If $\mathcal{V}$ is a Hilbert space, then $\overline{\text{conv}(E)}$ and (the polar) E^p are spectral.*

Linear optimization

The following result essentially says that maximizing a linear function over a spectral set in $\mathcal{V}$ is the same as maximizing a related linear function over a set in $\mathcal{W}$.

Theorem: *Fix $c \in \mathcal{V}$, let E be spectral.*

Let $f(v) := \langle c, v \rangle$ where $v \in \mathcal{V}$

and $f^*(w) := \langle \lambda(c), w \rangle$, where $w \in \mathcal{W}$.

Then,

$$\max\{f(v) : v \in E\} = \max\{f^*(w) : w \in \lambda(E)\}.$$

Commutation principles

Consider the key defining condition of a FTvN system: $\max\{\langle c, x\rangle : x \in [u]\} = \langle \lambda(c), \lambda(u)\rangle$.

So, when the objective function $\langle c, x\rangle$ attains its maximum on $[u]$, say at $\bar{x}$, we have

$\langle c, \bar{x}\rangle = \langle \lambda(c), \lambda(u)\rangle = \langle \lambda(c), \lambda(\bar{x})\rangle$.

This means that c and $\bar{x}$ commute in the FTvN system.

A similar statement can be made when the λ-orbit $[u]$ is replaced by a spectral set.

So, commutativity becomes an optimality condition for a linear maximization problem over a spectral set. There are a number of such commutation principles in the literature, especially for $\mathcal{S}^n$, $\mathcal{H}^n$, and Euclidean Jordan algebras.

We now state a **geometric commutation principle.**

Theorem: *Let E be a spectral set in a FTvN system and $a \in E$. Then, any outward normal to E commutes with a.*

Here, d is an outward normal to E at a if $\langle d, x-a \rangle \leq 0$ for all $x \in E$.

Proof: Assuming that d is an outward normal to E at a, we have $\langle d, x\rangle \leq \langle d, a\rangle$ for all $x \in [a]$. Taking the maximum over x, we get $\langle \lambda(d), \lambda(a)\rangle \leq \langle d, a\rangle$.

Since the reverse inequality always holds, we have $\langle \lambda(d), \lambda(a) = \langle d, a\rangle$.

So a and d commute.

An application to variational inequality problems:

Given a function $G : \mathcal{V} \to \mathcal{V}$ and $E \subseteq \mathcal{V}$, the *variational inequality problem* VI(G, E) is to find $a \in E$ such that $\langle G(a), x - a \rangle \geq 0$ for all $x \in E$.

When E is a closed convex cone, this reduces to a *complementarity problem.*

These problems appear in partial differential equations, optimization, game theory, economics, rigid-body dynamics, etc.

Theorem: *If E is spectral and a solves VI(G, E),*

then, a and $-G(a)$ commute.

Proof: If a solves VI(G, E),

then $-G(a)$ is an outward normal to E at a.

We can apply the geometric commutation principle.

Majorization

Motivated by equivalent formulations of majorization concepts in $\mathbb{R}^n$ and $\mathcal{S}^n$ ($\mathcal{H}^n$), we now define majorization in a general FTvN system.

Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a FTvN system and $x, y \in \mathcal{V}$.

We say that x is majorized by y and write $x \prec y$ if $x \in$ conv$[y]$.

Note: While this requires (only) the knowledge of λ-orbits, useful examples and results arise when all conditions of FTvN system are in force.

Doubly stochastic maps

In the setting of a FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$,

a linear map $D : \mathcal{V} \to \mathcal{V}$ is said to be

doubly stochastic if $Dx \prec x$ for all $x \in \mathcal{V}$.

Theorem: *Let D be a doubly stochastic map with adjoint D^*.*

- *For every convex spectral set K, $D(K) \subseteq K$.*
- *If e is a unit element, then $De = e$ and $D^*e = e$.*
- *If $\mathcal{V}$ is finite dimensional, then $D^*x \prec x$ for all $x \in \mathcal{V}$.*

We note that when $\mathcal{V}$ is finite dimensional,

Item (a) in the above proposition is

equivalent to D being doubly stochastic.

This is because, when $\mathcal{V}$ is finite dimensional,

for any $x \in \mathcal{V}$, $K := \text{conv}\,[x]$ is a convex spectral

set (by an earlier result); so $D(K) \subseteq K$

implies that $Dx \in \text{conv}\,[x]$, or equivalently, $Dx \prec x$.

The condition $De = e$ is usually referred to as 'unital' and $D^* e = e$ as 'trace-preserving'.

In certain settings (such as $\mathcal{S}^n$, $\mathcal{H}^n$, Euclidean Jordan algebras), apart from $De = e$ and $D^*(e) = e$, it is enough to require D to keep only one spectral convex set (example, semidefinite cone) invariant.

Whether this is true in the case of hyperbolic polynomials and hyperbolicity cone is an open problem.

Euclidean Jordan algebra

$(\mathcal{V}, \langle \cdot, \cdot \rangle, \circ)$ is a **Euclidean Jordan algebra** (EJA) if $\mathcal{V}$ is a finite dimensional real inner product space and the bilinear Jordan product $x \circ y$ satisfies:

- $x \circ y = y \circ x$
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$

$K = \{x \circ x : x \in \mathcal{V}\}$ is a closed convex cone.

It is called a *symmetric cone*.

Examples EJAs

Any EJA is a product of the following:

- $\mathcal{S}^n = \text{Herm}(\mathcal{R}^{n\times n})$ - $n \times n$ real symmetric matrices.
- $\text{Herm}(\mathcal{C}^{n\times n})$ - $n \times n$ complex Hermitian matrices.
- $\text{Herm}(\mathcal{Q}^{n\times n})$ - $n \times n$ quaternion Hermitian matrices.
- $\text{Herm}(\mathcal{O}^{3\times 3})$ - 3×3 octonion Hermitian matrices.
- $\mathcal{L}^n$ $(n \geq 3)$- Jordan spin algebra.

The above algebras are (the only) *simple* algebras.

Thanks to the so-called spectral theorem,

every EJA can be regarded as a FTvN system.

Normal decomposition system

Motivated by optimization and convex analysis considerations, **Lewis** 1996 introduced normal decomposition systems.

Let $\mathcal{V}$ be a real inner product space, $\mathcal{G}$ be a closed subgroup of the orthogonal group of $\mathcal{V}$, and $\gamma : \mathcal{V} \to \mathcal{V}$ be a map satisfying the following conditions:

- $\gamma(Ax) = \gamma(x)$ for all $x \in \mathcal{V}$ and $A \in \mathcal{G}$.
- For each $x \in \mathcal{V}$, there exists $A \in \mathcal{G}$ such that $x = A\gamma(x)$.
- For all $x, y \in \mathcal{V}$, we have $\langle x, y \rangle \le \langle \gamma(x), \gamma(y) \rangle$.

Then, $(\mathcal{V},\, \mathcal{G},\, \gamma)$ is called a

normal decomposition system.

In this system, for any two elements x and y in $\mathcal{V}$,

we have $\quad \max_{A \in \mathcal{G}} \langle Ax, y \rangle = \langle \gamma(x), \gamma(y) \rangle.$

Also, $\langle x, y\rangle = \langle \gamma(x), \gamma(y)\rangle$

holds for two elements x and y if and only

if there exists an $A \in \mathcal{G}$ such that

$x = A\gamma(x)$ and $y = A\gamma(y)$.

It can be shown that every normal

decomposition system is a FTvN system.

It is known that every simple EJA

is a normal decomposition system.

Eaton triples

These were introduced and studied by Eaton et al., from the perspective of majorization techniques in probability.

Let $\mathcal{V}$ be a finite dimensional real inner product space, $\mathcal{G}$ be a closed subgroup of the orthogonal group of $\mathcal{V}$, and F be a closed convex cone in $\mathcal{V}$ satisfying the following conditions:

- $Orb(x) \cap F \neq \emptyset$ for all $x \in \mathcal{V}$, where $Orb(x) := \{Ax : A \in \mathcal{G}\}$.
- $\langle x, Ay\rangle \leq \langle x, y\rangle$ for all $x, y \in F$ and $A \in \mathcal{G}$.

Then, $(\mathcal{V}, \mathcal{G}, \mathcal{F})$ is called an *Eaton triple*.

It has been observed that Eaton triple is a

particular instance of a normal decomposition system.

Hence, every Eaton triple is a FTvN system.

References:

For articles cited in the talk, refer to
the three papers mentioned at the beginning.