An introduction to Fan-Theobald-von Neumann systems

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Abstract: A Fan-Theobald-von Neumann system (FTvN system, for short) is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a norm-preserving map satisfying the inequality

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \quad (x, y \in \mathcal{V})$$

together with a condition for equality. One trivial example is $(\mathcal{V}, \mathcal{R}, ||\cdot||)$, where $\mathcal{V}$ is a real inner product space with norm $||\cdot||$. A nontrivial example is $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where λ takes a vector in $\mathcal{R}^n$ to its decreasing rearrangement. Other examples include: the space $\mathcal{S}^n$ $(\mathcal{H}^n)$ of all $n \times n$ real symmetric (respectively, complex Hermitian) matrices with eigenvalue map λ, Euclidean Jordan algebras, systems induced by certain hyperbolic polynomials, and normal decompositions systems (Eaton triples).

The aim of this talk is to (i) motivate the definition of FTvN system via various examples, (ii) to introduce the concept of commutativity in a FTvN system via the equality $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$ and show how it naturally appears as an optimality condition in linear optimization problems as well as in complementarity problems and variational inequality problems, (iii) introduce/describe the concepts of majorization and doubly stochastic maps in the general setting.