A generalization of Fiedler's lemma and the spectra of H-join of graphs

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E-Seminar
Organized By
IIT Kharagpur

August 13, 2021

M. Saravanan, S. P. Murugan, G. Arunkumar, *A generalization of Fiedler's lemma and the spectra of H-join of graphs*, Lin. Alg. Appl. 625(8) 20-43

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H-join operation of graphs

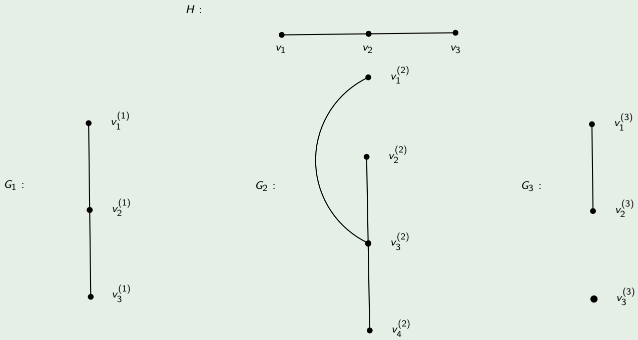
Let H be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs. In [4], the H-join operation of the graphs $G_1, G_2, \ldots, G_k$, denoted by $\bigvee_H \mathcal{F}$, is obtained by replacing the vertex v_i of H by the graph G_i for $1 \leq i \leq k$ and every vertex of G_i is made adjacent with every vertex of G_j, whenever v_i is adjacent to v_j in H.

Precisely, $\bigvee_H \mathcal{F}$ is the graph with vertex set $V(\bigvee_H \mathcal{F}) = \bigcup_{i=1}^{k} V(G_i)$ and edge set

$$E(\bigvee_H \mathcal{F}) = (\bigcup_{i=1}^{k} E(G_i)) \cup (\bigcup_{v_i v_j \in E(H)} \{xy : x \in V(G_i), y \in V(G_j)\}).$$

Example

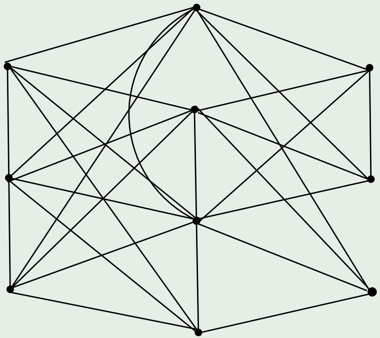
Consider the graphs $H = P_3$, $G_1 = P_3$, $G_2 = K_{1,3}$ and $G_3 = K_2 \cup K_1$ as follows.



Example

Let $\mathcal{F} = \{G_1, G_2, G_3\}$. Then the H-join graph $G = \bigvee_H \mathcal{F}$ is given as

G :



In [21], the H-join operation of the graphs was initially introduced as **generalized composition by Schwenk**, denoted by $H[G_1, G_2, \ldots, G_k]$. Also, the same operation is studied in some other names as generalized lexicographic product and joined union in [23, 19, 22]. When all G_i's are equal to the same graph G, it is called the lexicographic product[15], denoted by $H[G]$.

In [21] it is remarked by Schwenk, that **"In general, it does not appear likely that the characteristic polynomial of the generalized composition can always be expressed in terms of the characteristic polynomials of $H, G_1, G_2, \ldots, G_k$".**

In this work, we prove that it is possible to express the characteristic polynomial of H-join operation of graphs (i.e. generalized composition) in terms of

- **the characteristic polynomials of** $G_1, G_2, \ldots, G_k$
- **the 'main' functions of** $G_1, G_2, \ldots, G_k$
- **and another function obtained from the adjacency matrix of** H.

Moreover for the H-join operation of any graphs, we obtain the characteristic polynomial and the spectrum of its **universal adjacency matrix**.

Lemma [12, Lemma 2.2]

Let A be a symmetric $m \times m$ matrix with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_m$ and B be a symmetric $n \times n$ matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. let u be an eigenvector of A corresponding to α_1 and v be an eigenvector of B corresponding to β_1 such that $\|u\| = \|v\| = 1$. Then for any constant ρ the matrix

$$C = \begin{bmatrix} A & \rho u v^t \\ \rho v u^t & B \end{bmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2$ where γ_1 and γ_2 are the eigenvalues of the matrix

$$\widehat{C} = \begin{bmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{bmatrix}.$$

In [3, 4], this lemma is called Fiedler's lemma. It is easy to see that, this lemma can be used to find the spectrum of H-join of regular graphs when $H = K_2$.

Theorem[4]

Let M_i be a symmetric matrix of order n_i and u_i be an eigenvector of M_i corresponding to the eigenvalue α_i, such that $\|u_i\| = 1$ for $1 \leq i \leq k$. Let $\rho_{i,j}$ be a collection of arbitrary scalars such that $\rho_{i,j} = \rho_{j,i}$ for $1 \leq i < j \leq k$. Considering

$$\mathbf{M} = (M_1, M_2, \ldots, M_k), \mathbf{u} = (u_1, u_2, \ldots, u_k)$$

as k-tuples, and

$$\rho = (\rho_{12}, \ldots, \rho_{1,k}, \rho_{23}, \ldots, \rho_{2,k}, \cdots \rho_{k-1k})$$

as $\frac{k(k-1)}{2}$-tuple, the following matrices are defined.
(To be Cont.)

Theorem[4](Cont.)

$$A(\mathbf{M},\mathbf{u},\rho) := \begin{bmatrix} M_1 & \rho_{1,2}u_1u_2^t & \cdots & \rho_{1,k}u_1u_k^t \\ \rho_{2,1}u_2u_1^t & M_2 & \cdots & \rho_{2,k}u_2u_k^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}u_ku_1^t & \rho_{k,2}u_ku_2^t & \cdots & M_k \end{bmatrix} \text{ and } \widehat{A}(\mathbf{M},\mathbf{u},\rho) :=$$

$$\begin{bmatrix} \alpha_1 & \rho_{1,2} & \cdots & \rho_{1,k} \\ \rho_{2,1} & \alpha_2 & \cdots & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1} & \rho_{k,2} & \cdots & \alpha_k \end{bmatrix}.$$

Then $\text{spec}(A(\mathbf{M},\mathbf{u},\rho)) = \left(\bigcup_{i=1}^{k} (\text{spec}(M_i)\setminus\{\alpha_i\}) \right) \cup \text{spec}(\widehat{A}(\mathbf{M},\mathbf{u},\rho))$.

Adjacency matrix of H-join of graphs

Consider a graph H on k vertices and a family of graphs $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$. Let $G = \bigvee_H \mathcal{F}$ be the H-join of graphs in $\mathcal{F}$, and let n_i, A_i and D_i be the number of vertices, the adjacency matrix and the degree matrix of the graph G_i respectively, for $1 \leq i \leq k$. Also let $\rho_{i,j}$ be the scalars defined by $\rho_{i,j} = \rho_{j,i} = 1$ if $ij \in E(H)$ and 0 otherwise, for $1 \leq i, j \leq k$ and $i \neq j$. Then the adjacency matrix of the graph G can be written as

$$A(G) = \begin{bmatrix} A_1 & \rho_{1,2}\mathbf{1}_{n_1}\mathbf{1}^t_{n_2} & \cdots & \rho_{1,k}\mathbf{1}_{n_1}\mathbf{1}^t_{n_k} \\ \rho_{2,1}\mathbf{1}_{n_2}\mathbf{1}^t_{n_1} & A_2 & \cdots & \rho_{2,k}\mathbf{1}_{n_2}\mathbf{1}^t_{n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}\mathbf{1}_{n_k}\mathbf{1}^t_{n_1} & \rho_{k,2}\mathbf{1}_{n_k}\mathbf{1}^t_{n_2} & \cdots & A_k \end{bmatrix}. \tag{1}$$

Theorem[4] Spectrum of H-join of regular graphs

Let H be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs. Let $G = \bigvee_H \mathcal{F}$ be the H-join of graphs in $\mathcal{F}$. Suppose the graph G_i is r_i-regular for $1 \leq i \leq k$ on n_i vertices. Then

$$\operatorname{spec}(A(G)) = \left(\bigcup_{i=1}^{k} \left(\operatorname{spec}(G_i) \setminus \{r_i\} \right) \right) \cup \operatorname{spec}(\widetilde{A}(G))$$

where $\widetilde{A}(G) = \begin{bmatrix} r_1 & \sqrt{n_1 n_2}\rho_{1,2} & \cdots & \sqrt{n_1 n_k}\rho_{1,k} \\ \sqrt{n_2 n_1}\rho_{2,1} & r_2 & \cdots & \sqrt{n_2 n_k}\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_k n_1}\rho_{k,1} & \sqrt{n_k n_2}\rho_{k,2} & \cdots & r_k \end{bmatrix}$.

Notations

- I, I_n- Identity matrix
- J, J_n- All one matrix
- $\mathbf{1}_n$- All one vector

Main eigenvalues and Main angles

Suppose $A(G)$ has spectral decomposition $A(G) = \Sigma_{i=1}^{k} \theta_i E_{\theta_i}$, where θ_i's are distinct eigenvalues of G and E_{θ_i} is the orthogonal projection on the eigenspace of θ_i, $\mathcal{E}(\theta_i) = ker(A(G) - \theta_i I_n)$.

- An eigenvalue θ_i is called a **main eigenvalue** if the corresponding eigenspace $\mathcal{E}(\theta_i)$ is not orthogonal to $\mathbf{1}_n$.
- The cosines of the angles between $\mathbf{1}_n$ and the eigenspaces of A are known as **main angles** of G, given by $\beta_i = \frac{1}{\sqrt{n}} \|E_{\theta_i} \mathbf{1}_n\|$, for $1 \leq i \leq k$.
- So θ_i is a main eigenvalue if and only if $\beta_i \neq 0$.

For more on the main angles and main eigenvalues, we refer [20] and references therein.

Consider the field of rational functions $\mathbb{C}(\lambda)$. The $\det(\lambda I - A)$ is a non-zero element of $\mathbb{C}(\lambda)$ and hence the matrix $\lambda I - A$ is invertible over $\mathbb{C}(\lambda)$.

In [17], the function $\mathbf{1}_n^t(\lambda I_n - A(G))^{-1}\mathbf{1}_n$ is introduced in the name of coronal of G and is used to find the characteristic polynomial of the corona of two graphs. Since $E_{\theta_i}^2 = E_{\theta_i}$, it is easy to see that

$$\mathbf{1}_n^t(\lambda I_n - A(G))^{-1}\mathbf{1}_n = \Sigma_{i=1}^k \frac{\mathbf{1}_n^t E_{\theta_i}\mathbf{1}_n}{\lambda - \theta_i} = \Sigma_{i=1}^k \frac{\|E_{\theta_i}\mathbf{1}_n\|^2}{\lambda - \theta_i} = \Sigma_{i=1}^k \frac{n\beta_i^2}{\lambda - \theta_i}, \qquad (2)$$

in which only non-vanishing terms are those terms corresponding to main eigenvalues.

Because of this relationship among main eigenvalues, main angles, and coronal of the graph G, we prefer to call $\mathbf{1}_n^t(\lambda I_n - A(G))^{-1}\mathbf{1}_n$, **the main function of the graph** G, and denote as $\Gamma_G(\lambda)$.
Moreover for any vectors u and v, and a matrix M of the same dimension, we introduce the following notions.

Main function of a matrix

Let M be an $n \times n$ complex matrix, and let u and v be $n \times 1$ complex vectors. The main function associated to the matrix M corresponding to the vectors u and v, denoted by $\Gamma_M(u, v)$, is defined to be $\Gamma_M(u, v; \lambda) = v^t(\lambda I - M)^{-1}u \in \mathbb{C}(\lambda)$. When $u = v$, we denote $\Gamma_M(u, v; \lambda)$ by $\Gamma_M(u; \lambda)$.

A generalization of Main eigenvalue

Let M be an $n \times n$ normal matrix over $\mathbb{C}$ and let u be an $n \times 1$ complex vector. An eigenvalue λ of M is called as u-main eigenvalue if the corresponding eigenspace $\mathcal{E}_M(\lambda)$ is not orthogonal to the vector u. In the case of $u = \mathbf{1}_n$, the all-one vector, we don't specify the vector and call eigenvalue λ of M as the main eigenvalue of M.

Some results on Main function

- Let M be a complex normal matrix of order n and let u be any $n \times 1$ vector. Then the poles of $u^t(\lambda I - M)^{-1}u$ are the u-main eigenvalues of M and are simple.
- Let M be a matrix of order n with an eigenvector u corresponding to the eigenvalue μ. Then $\Gamma_M(u; \lambda) = \dfrac{\|u\|^2}{\lambda - \mu}$.

Now we can state our main result, a new generalization of Fiedler's lemma.

Main theorem: Generalization of Fiedler's lemma

- Let M_i be a complex matrix of order n_i, and let u_i and v_i be arbitrary complex vectors of size $n_i \times 1$ for $1 \leq i \leq k$. Let $n = \Sigma_{i=1}^{k} n_i$. Let $\rho_{i,j}$ be arbitrary complex numbers for $1 \leq i, j \leq k$ and $i \neq j$.
- For each $1 \leq i \leq k$, let $\phi_i(\lambda) = \det(\lambda I_{n_i} - M_i)$ be the characteristic polynomial of the matrix M_i and $\Gamma_i(\lambda) = \Gamma_{M_i}(u_i, v_i; \lambda) = v_i^t(\lambda I - M_i)^{-1} u_i$.
- Let $\mathbf{M}$ be the k-tuple $(M_1, M_2, \ldots, M_k)$, $\mathbf{u}$ be the $2k$-tuple $(u_1, v_1, u_2, v_2 \ldots, u_k, v_k)$ and ρ be the $k(k-1)$-tuple $(\rho_{1,2}, \rho_{1,2} \ldots, \rho_{1,k}, \rho_{2,1}, \rho_{23}, \ldots, \rho_{2,k}, \ldots, \rho_{k,1}, \rho_{k,2}, \ldots, \rho_{k-1k})$.
(To be Contd.)

Main theorem: Generalization of Fiedler's lemma(Contd.)

Considering $\mathbf{M}$, $\mathbf{u}$ and ρ, the following matrices are defined:

$$A(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} M_1 & \rho_{1,2} u_1 v_2^t & \cdots & \rho_{1,k} u_1 v_k^t \\ \rho_{2,1} u_2 v_1^t & M_2 & \cdots & \rho_{2,k} u_2 v_k^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1} u_k v_1^t & \rho_{k,2} u_k v_2^t & \cdots & M_k \end{bmatrix}$$

and $$\widetilde{A}(\mathbf{M}, \mathbf{u}, \rho) := \begin{bmatrix} \frac{1}{\Gamma_1(\lambda)} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2(\lambda)} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k(\lambda)} \end{bmatrix}.$$

(To be Contd.)

Main theorem: Generalization of Fiedler's lemma(Contd.)

Then the characteristic polynomial of $A(\mathbf{M}, \mathbf{u}, \rho)$ is given as

$$\det(\lambda I_n - A(\mathbf{M}, \mathbf{u}, \rho)) = \left(\Pi_{i=1}^{k} \phi_i(\lambda) \Gamma_i(\lambda) \right) \det(\widetilde{A}(\mathbf{M}, \mathbf{u}, \rho)). \tag{3}$$

Previous generalization as Corollary of Main theorem

Consider the notations defined in Main theorem. Suppose $u_i = v_i$ is an eigenvector of M_i corresponding to an eigenvalue α_i with $\|u_i\| = 1$, then the characteristic polynomial of $A(\mathbf{M}, \mathbf{u}, \rho)$ is

$$\phi(A(\mathbf{M}, \mathbf{u}, \rho)) = \frac{\phi_1}{\lambda-\alpha_1} \frac{\phi_2}{\lambda-\alpha_2} \cdots \frac{\phi_k}{\lambda-\alpha_k} \det(\widetilde{A}(\mathbf{M}, \mathbf{u}, \rho))$$

$$\text{where } \widetilde{A}(\mathbf{M}, \mathbf{u}, \rho) = \begin{bmatrix} \lambda-\alpha_1 & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \lambda-\alpha_2 & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \lambda-\alpha_k \end{bmatrix}.$$

Proof of Main theorem by induction

- A Lemma:
  $\det(\lambda I - A + \alpha uv^t) = (1 + \alpha\Gamma)\det(\lambda I - A) = (1 + \alpha\Gamma)\phi_A(\lambda)$
- We prove the result of main theore by using induction on k.
- For convenience, we take $\Gamma_i = \Gamma_i(\lambda)$. The base case $k = 1$ is clear.
- We prove the result also for $k = 2$ for the sake of understanding.

$$\begin{vmatrix} \lambda I_{n_1} - M_1 & -\rho_{1,2}u_1v_2^t \\ -\rho_{2,1}u_2v_1^t & \lambda I_{n_2} - M_2 \end{vmatrix}$$

$$= \det(\lambda I_{n_2} - M_2)\det(\lambda I_{n_1} - M_1 - \rho_{1,2}\rho_{2,1}\Gamma_2 u_1 v_1^t)$$

$$= \phi_1\phi_2(1 - \rho_{1,2}\rho_{2,1}\Gamma_2\Gamma_1)$$

$$= \phi_1\phi_2 \begin{vmatrix} 1 & -\rho_{1,2}\Gamma_1 \\ -\rho_{2,1}\Gamma_2 & 1 \end{vmatrix}$$

Spectrum

Suppose the matrices M_i's are normal and $\{\theta_1, \theta_2, \ldots, \theta_{m_i}\}$ is the set of distinct u_i-main eigenvalues of M_i, for $1 \leq i \leq k$. Then we can write

$$\Gamma_i = \frac{f_i}{g_i} \text{ where } g_i = \prod_{j=1}^{m_i} (\lambda - \theta_j). \tag{4}$$

Hence by the Main theorem,

$$\det(\lambda I - A(\mathbf{M}, \mathbf{u}, \rho)) = (\frac{\phi_1}{g_1}) \ldots (\frac{\phi_k}{g_k})\Phi(\lambda) \tag{5}$$

where $\Phi(\lambda) = \begin{vmatrix} g_1(\lambda) & -\rho_{1,2} f_1(\lambda) & \cdots & -\rho_{1,k} f_1(\lambda) \\ -\rho_{2,1} f_2(\lambda) & g_2(\lambda) & \cdots & -\rho_{2,k} f_2(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} f_k(\lambda) & -\rho_{k,2} f_k(\lambda) & \cdots & g_k(\lambda) \end{vmatrix}$

So we can describe the spectrum of $A(\mathbf{M}, \mathbf{u}, \rho)$ as follows.

Main Theorem(Spectrum version)

Consider the notations defined above. Suppose the matrices M_i's are normal, then

- Every eigenvalue, which is not a u_i-main eigenvalue of M_i, say λ with multiplicity $m(\lambda)$ is an eigenvalue of $A(\mathbf{M}, \mathbf{u}, \rho)$ with multiplicity $m(\lambda)$.
- Every u_i-main eigenvalue of M_i, say λ with multiplicity $m(\lambda)$ is an eigenvalue of $A(\mathbf{M}, \mathbf{u}, \rho)$ with multiplicity $m(\lambda) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

The universal adjacency matrix of a graph G is defined as follows:

Universal adjacency matrix

- Let $A(G)$, I, J, and $D(G)$ be the adjacency matrix of G, the identity matrix, the all-one matrix, and the degree matrix of G, respectively.
- Any matrix of the form $U(G) = \alpha A + \beta I + \gamma J + \delta D$ where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha \neq 0$ is called the universal adjacency matrix of G.

Many interesting and important matrices associated to a graph can be obtained as special cases of $U(G)$. For example, we get
adjacency matrix $A(G)$,
Laplacian matrix $L(G) = D(G) - A(G)$,
signless Laplacian matrix $Q(G) = D(G) + A(G)$, and
Seidel matrix $S(G) = J - I - 2A(G)$
by taking appropriate values for α, β, γ, and δ.

Degree of vertex in H-join

Let H be a graph with vertex set $\{v_1, \ldots, v_k\}$ and $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of k graphs such that $V(G_i) = \{v_1^{(i)}, \ldots, v_{n_i}^{(i)}\}$ for $1 \leq i \leq k$. Then the degree of the vertex $v_j^{(i)}$ in $G = \bigvee_H \mathcal{F}$ is given by

$$\deg_G(v_j^{(i)}) = \deg_{G_i}(v_j^{(i)}) + w_i, 1 \leq i \leq k, 1 \leq j \leq n_i$$

where $w_i = \sum_{v_l \in N_H(v_i)} n_l$.

UA matrix of H-join

Let H be a graph on k vertices and $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of any graphs. Consider the graph $G = \bigvee_H \mathcal{F}$. Let $\phi_i(\lambda)$ be the characteristic polynomial of U_i and $\Gamma_i(\lambda) = \Gamma_{U_i}(\mathbf{1}_{n_i}; \lambda)$. Then we have the following.

i) The characteristic polynomial of the universal adjacency matrix $U(G)$ is

$$\phi_{U(G)}(\lambda) = \left(\Pi_{i=1}^{k} \phi_i(\lambda - \delta w_i)\Gamma_i(\lambda - \delta w_i)\right) \det(\widetilde{U}(G))$$

UA matrix of H-join

$$\text{where } \widetilde{U}(G) = \begin{bmatrix} \frac{1}{\Gamma_1(\lambda-\delta w_1)} & -(\rho_{1,2}\alpha+\gamma) & \cdots & -(\rho_{1,k}\alpha+\gamma) \\ -(\rho_{2,1}\alpha+\gamma) & \frac{1}{\Gamma_2(\lambda-\delta w_2)} & \cdots & -(\rho_{2,k}\alpha+\gamma) \\ \vdots & \vdots & \ddots & \vdots \\ -(\rho_{k,1}\alpha+\gamma) & -(\rho_{k,2}\alpha+\gamma) & \cdots & \frac{1}{\Gamma_k(\lambda-\delta w_k)} \end{bmatrix} \quad (6)$$

ii) We define f_i, g_i, and $\Phi(\lambda)$ corresponding to the main eigenvalues of U_i for $1 \leq i \leq k$. Then the universal spectrum of G is given as below.

- For every eigenvalue μ of U_i with multiplicity $m(\mu)$, which is not a main eigenvalue, $\mu + \delta w_i$ is a universal eigenvalue of G with multiplicity $m(\mu)$.
- For every main eigenvalue μ of U_i with multiplicity $m(\mu)$, $\mu + \delta w_i$ is a universal eigenvalue of G with multiplicity $m(\mu) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

Apart from adjacency matrix, we can deduce many results from the previous theorem. So our work can be considered as a generalization of following works.

Generalization

- In [4, Theorem 8], the authors obtained the Laplacian spectra of H-join of any graphs.
- In [23, Theorem 2.4], the authors obtained the characteristic polynomial of Lexicographic product $H[G']$ and investigated the spectrum in various cases.

Generalization

- The generalized characteristic polynomial of a graph G is introduced in [9], as the bivariate polynomial defined by $\phi_G(\lambda, t) = \det(\lambda I - (A(G) - tD(G)))$ where $A(G)$ and $D(G)$ are the adjacency and the degree matrix associated to the graph G.
- In [8, Theorem 3.1] the authors obtained a generalization of Fiedlers lemma, for the matrices with fixed row sum and as an application, they obtained the generalized characteristic polynomial of H-join of **regular** graphs.
- In [16] the universal adjacency spectra of the disjoint union of **regular** graphs is obtained.

Generalized Corona

In [13, Theorem 3.1], the generalized corona product is defined as below and its characteristic polynomial is obtained. We can deduce this result also. This is done by viewing the corona product as the H-join of suitably chosen graphs.

Definition

Let H' be a graph on k vertices. Let $G_1, G_2, \ldots, G_k$ be graphs of order $n_1, n_2, \ldots, n_k$ respectively. The generalized corona product of H' with $G_1, G_2, \ldots, G_k$, denoted by $H' \tilde{\circ} \Lambda_{i=1}^{k} G_i$, is obtained by taking one copy of graphs H', $G_1, G_2, \ldots, G_k$, and joining the i^{th} vertex of H' to every vertex of G_i.

Generalized Corona as H-join

Let $H = H' \circ K_1$. Let v_{k+i} be the new vertex in H attached with the vertex v_i in the copy of H', for $1 \leq i \leq k$. Let $\mathcal{F} = \{K_1, K_1, \ldots, K_1, G_1, G_2, \ldots, G_k\}$. Then we get the following visualization of generalized corona as H-join of graphs in $\mathcal{F}$.

$$(H' \tilde{\circ} \Lambda_{i=1}^{k} G_i) = (\bigvee_{H} \mathcal{F})$$

That is, each v_i is replaced by K_1 and v_{k+i} is replaced by G_i in H, to form the H-join.

Characteristic polynomial of Generalized Corona

Let H' be a graph with vertex set $V(H') = \{v_1, v_2, \ldots, v_k\}$. Let $G_1, G_2, \ldots, G_k$ be any graphs. $\rho_{i,j} = 1$ if $v_i v_j \in E(H')$ and 0 otherwise. The characteristic polynomial of the generalized corona product $G = H' \tilde{\circ} \Lambda_{i=1}^{k} G_i$ is given by

$$\phi_G(\lambda) = \left(\Pi_{i=1}^{k} \phi_{G_i}(\lambda) \right) \det(\widetilde{A}(H'))$$

where $\widetilde{A}(H') = \begin{bmatrix} \lambda - \Gamma_{G_1}(\lambda) & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \lambda - \Gamma_{G_2}(\lambda) & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \lambda - \Gamma_{G_k}(\lambda) \end{bmatrix}$

Cospectral graphs

- Two graphs are said to be cospectral if their (adjacency) spectrum are equal. In general, for any matrix say $M(G)$ for a given graph G, two graphs are said to be M-cospectral if their M-spectrum is equal.
- In [2], the author questioned the existence of non-regular graphs, which are cospectral with respect to the adjacency, the Laplacian, the signless Laplacian, and the normalized Laplacian spectrum simultaneously.
- In [8, Theorem 3.7], the authors affirmatively answered the question by the construction of such graphs using the H-join of regular graphs. We prove that those graphs are U-cospectral too. In particular, those graphs are cospectral with respect to the Seidel spectrum also.

Cospectral graphs

Let $\mathcal{F} = \mathcal{F}_1 = \{G_1, G_2, \ldots, G_k\}$ and $\mathcal{F}_2 = \{G'_1, G'_2, \ldots, G'_k\}$ be two families of graphs.

(i) If G_i and G'_i are cospectral regular graphs on n_i vertices for $1 \leq i \leq k$, and H is an arbitrary graph on k vertices, then $\bigvee\limits_{H} \mathcal{F}_1$ and $\bigvee\limits_{H} \mathcal{F}_2$ are U-cospectral.

(ii) If H_1 and H_2 are cospectral r_1-regular graphs on k vertices and every G_i is r_2-regular on m vertices for $1 \leq i \leq k$, then $\bigvee\limits_{H_1} \mathcal{F}$ and $\bigvee\limits_{H_2} \mathcal{F}$ are U-cospectral.

(iii) If H_1 and H_2 are cospectral r_1-regular graphs on k vertices and, G_i and G'_i are cospectral r_2-regular graphs on m vertices for $1 \leq i \leq k$ then $\bigvee\limits_{H_1} \mathcal{F}_1$ and $\bigvee\limits_{H_2} \mathcal{F}_2$ are U-cospectral.

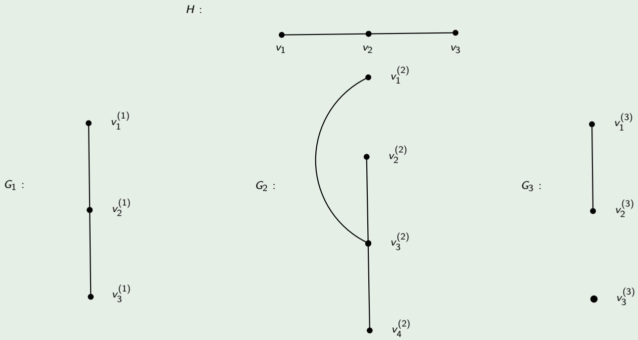
Let H be a graph with vertex set $\{v_1, v_2, \ldots, v_k\}$ and let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs. Now by considering a family of vertex subsets $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ where $S_i \subset V(G_i)$ for each $1 \leq i \leq k$, a generalization of H-join operation, known as H-generalized join operation constrained by vertex subsets, $\bigvee_{H,\mathcal{S}} \mathcal{F}$ is introduced in [5] as follows:

$$V(\bigvee_{H,\mathcal{S}} \mathcal{F}) = \bigcup_{i=1}^{k} V(G_i) \text{ and } E(\bigvee_{H,\mathcal{S}} \mathcal{F}) = (\bigcup_{i=1}^{k} E(G_i)) \cup (\bigcup_{v_i v_j \in E(H)} \{xy : x \in S_i, y \in S_j\}).$$

If we take $S_i = V(G_i)$ for each $1 \leq i \leq k$, then the H-generalized join operation $\bigvee_{H,\mathcal{S}} \mathcal{F}$ coincides with the H-join operation of the graphs $G_1, G_2 \ldots, G_k$.

Example

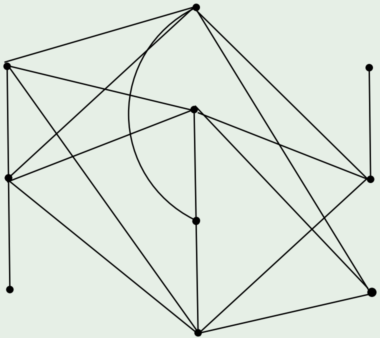
Consider the graphs $H = P_3$, $G_1 = P_3$, $G_2 = K_{1,3}$ and $G_3 = K_2 \cup K_1$ as follows.



Example

Consider H and $\mathcal{F}$ as in Example 3. Let $S_1 = \{v_1^{(1)}, v_2^{(1)}\}$, $S_2 = \{v_1^{(2)}, v_2^{(2)}, v_4^{(2)}\}$ and $S_3 = \{v_2^{(3)}, v_3^{(3)}\}$. Then the H-generalized join graph $G = \bigvee_{H,S} \mathcal{F}$ is given as

G :



The characteristic vector of a subset

Let G be any graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. For any subset $S \subset V(G)$, the characteristic vector of S, denoted by χ_S, is defined as the 0-1 vector such that i^{th} place of χ_S is 1 if and only if the vertex $v_i \in S$.

By the definition of $\bigvee_{(H,S)} \mathcal{F}$, its adjacency matrix can be given as

$$A(G) = \begin{bmatrix} A_1 & \rho_{1,2}\chi_{S_1}\chi_{S_2}^t & \cdots & \rho_{1,k}\chi_{S_1}\chi_{S_k}^t \\ \rho_{2,1}\chi_{S_2}\chi_{S_1}^t & A_2 & \cdots & \rho_{2,k}\chi_{S_2}\chi_{S_k}^t \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}\chi_{S_k}\chi_{S_1}^t & \rho_{k,2}\chi_{S_k}\chi_{S_2}^t & \cdots & A_k \end{bmatrix}.$$

Theorem on H-generalized join

Consider a graph H of order k and a family of graphs $\mathcal{F} = \{G_1, \ldots, G_k\}$. Consider also a family of vertex subsets $\mathcal{S} = \{S_1, \ldots, S_k\}$, such that $S_i \subset V(G_i)$ for $1 \leq i \leq k$. Let $G = \bigvee_{H,\mathcal{S}} \mathcal{F}$. Let n_i and A_i be the number of vertices and the adjacency matrix of the graph G_i respectively for $1 \leq i \leq k$. For $1 \leq i, j \leq k$, let $\rho_{i,j}$ be the scalars defined by $\rho_{i,j} = 1$ if $ij \in E(H)$ and 0 otherwise. Then we have the following.

Theorem on H-generalized join

i) The characteristic polynomial of G is

$$\phi_G(\lambda) = \left(\Pi_{i=1}^{k} \phi_i(\lambda)\Gamma_i(\lambda)\right) \det(\widetilde{A}(G))$$

$$\text{where } \widetilde{A}(G) = \begin{bmatrix} \frac{1}{\Gamma_1} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k}. \end{bmatrix} \quad (7)$$

where $\phi_i(\lambda) = \det(\lambda I_{n_i} - A(G_i))$ and $\Gamma_i(\lambda) = \Gamma_{A_i}(\chi_{S_i}; \lambda)$

ii) Analogous to the Equations (4) and (5), we define f_i, g_i and $\Phi(\lambda)$ corresponding to the χ_{S_i}-main eigenvalues of G_i for $1 \leq i \leq k$. Then the spectrum of G is given as below.

- Every eigenvalue μ of A_i with multiplicity $m(\mu)$, which is not χ_{S_i}-main eigenvalue, is an eigenvalue of G with multiplicity $m(\mu)$.
- Every χ_{S_i}-main eigenvalue μ of A_i with multiplicity $m(\mu)$, is an eigenvalue of G with multiplicity $m(\mu) - 1$.
- Remaining eigenvalues are the roots of the polynomial $\Phi(\lambda)$.

A Recent work

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Thank You...