Seidel spectrum of threshold graphs

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Introduction

- Attention in threshold graphs gained momentum during last 40 years.
- It has many applications in various field.
- Threshold graphs are a special case of cographs and split graphs.
- The motivation for considering threshold graphs comes from the spectral graph theory.
- An important thing is that-a threshold graph with n vertices can always be represented by a finite binary string of length n.
- For $n \geq 2$ there are 2^{n-2} connected threshold graph on n vertices.

Continue...

The study of threshold graphs was started with the papers by Chvatal and Hammer [1] and Henderson and Zalcstein [2] in 1977.

- The most complete reference on this topic is the book by Mahadeb and Peled [3], and for basic knowledge of spectral graph theory we refer the book by Godsil [13].
- Bapat [5] proved that the number of negative, zero, and positive eigenvalues of a threshold graph can be find out directly from its binary code representation. He also calculated the determinant value of the adjacency matrix of a generalized threshold graph and calculated the inverse of the adjacency matrix of an antiregular graph.
- Sciriha *et al*. [4] explore the adjacency spectrum of antiregular graphs and show common properties of a connected threshold graph.

Continue...

- Jacobs *et al.* ([6, 7, 8]), focus on eigenvalue location, characteristic polynomial and energy of a threshold graph. They developed an algorithm for computing the characteristic polynomial of a threshold graph. They proved that-a threshold graph has no adjacency eigenvalue in the interval $(0, 1)$.
- Banerjee and Mehatari [9] derived some results on normalized spectrum of a threshold graphs. They characterized all the threshold graphs with at most five distinct adjacency eigenvalue.
- Lazzarin [11] *et al.* proved that no threshold graphs are cospectral with respect to its adjacency matrix.
- We use the concept of evaluating the determinant of a tri diagonal matrix which is given by Mikkawy [14].

Mathematical Setup

Suppose $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n\}$ is a real sequence and $A_B = (\alpha_{ij})$ is a symmetric matrix whose entries are

$$\alpha_{ij} = \begin{cases} \alpha_i, & \text{for } i > j, \\ \alpha_j, & \text{for } j > i, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_B = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_3 & \alpha_3 & 0 & \alpha_4 & \cdots & \alpha_n \\ \alpha_4 & \alpha_4 & \alpha_4 & 0 & \cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \alpha_n & \alpha_n & \alpha_n & \cdots & 0 \end{bmatrix}.$$

The matrix A_B is called the **threshold matrix** associated with the real sequence B.

Definition

A $\mathbf{P}_4$, $\mathbf{C}_4$ and $\mathbf{2K}_2$ free graph is called a ***Threshold graph***.

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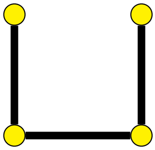

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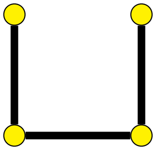

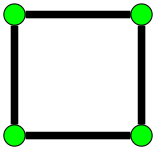

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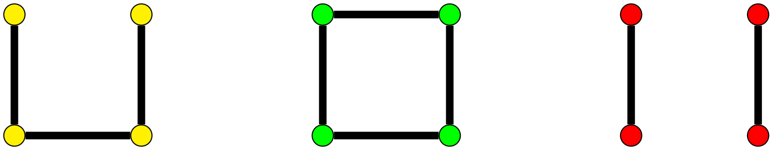

Seidel matrix

Let Γ be a threshold graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and binary string $b = \alpha_1\alpha_2\alpha_3\ldots\alpha_n$. Then Clearly A_B is the adjacency matrix of Γ. The **Seidel matrix** (S) is defined by

$$S = J - I - 2A_B.$$

$$\implies S = \begin{vmatrix} 0 & 1-2\alpha_2 & 1-2\alpha_3 & \ldots & 1-2\alpha_n \\ 1-2\alpha_2 & 0 & 1-2\alpha_3 & \ldots & 1-2\alpha_n \\ 1-2\alpha_3 & 1-2\alpha_3 & 0 & \ldots & 1-2\alpha_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1-2\alpha_n & 1-2\alpha_n & 1-2\alpha_n & \ldots & 0 \end{vmatrix},$$

Consider $1 - 2\alpha_i = \beta_i$ for $i = 2, 3, ..., n-1$, then S takes the form

$$S = \begin{vmatrix} 0 & \beta_2 & \beta_3 & \ldots & \beta_n \\ \beta_2 & 0 & \beta_3 & \ldots & \beta_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_n & \beta_n & \beta_n & \ldots & 0 \end{vmatrix}, \qquad (1)$$

Representation of Threshold Graph

Interesting Fact : **Binary Representation of threshold graphs**

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Notations:

- WLOG, we take $b = \alpha_1\alpha_2\alpha_3\alpha_4 \ldots \alpha_n = 0^{s_1}1^{t_1}0^{s_2}1^{t_2} \cdots 0^{s_k}1^{t_k}$.

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- $\alpha_1 = 0$, $\alpha_n = 1$.
- Set $s = \sum s_i$, $t = \sum t_i$,
  Then $s + t = n$.
- $Spec(W)$=spectrum of a matrix W.

Example: Construction of Threshold Graphs

$b = 0011100011 = 0^2 1^3 0^3 1^2$

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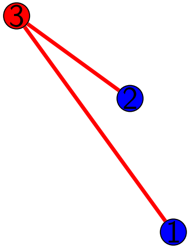

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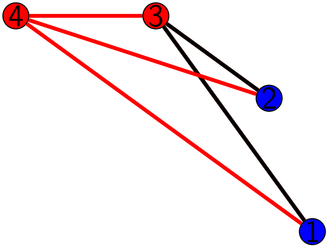

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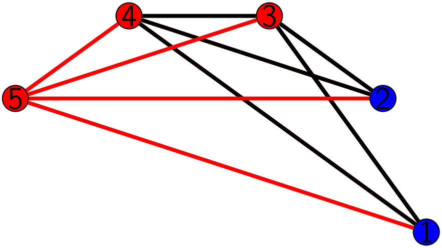

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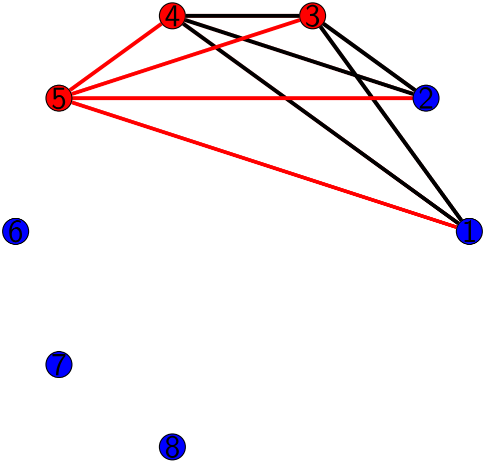

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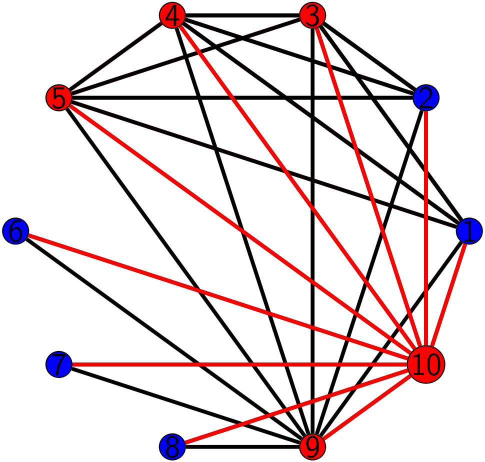

Adjacency matrix

The adjacency matrix A and the Seidel matrix S of a threshold graph Γ with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{t_k}$ are the square matrices of size n, given by

$$A = \begin{bmatrix} O_{s_1} & J_{s_1\times t_1} & O_{s_1\times s_2} & J_{s_1\times t_2} & \cdots & J_{s_1\times t_k} \\ J_{t_1\times s_1} & (J-I)_{t_1} & O_{t_1\times s_2} & J_{t_1\times t_2} & \cdots & J_{t_1\times t_k} \\ O_{s_2\times s_1} & O_{s_2\times t_2} & O_{s_2} & J_{s_2\times t_2} & \cdots & J_{s_2\times t_k} \\ J_{t_2\times s_1} & J_{t_2\times t_1} & J_{t_2\times s_2} & (J-I)_{t_2} & \cdots & J_{t_2\times t_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ J_{t_k\times s_1} & J_{t_k\times t_1} & J_{t_k\times s_2} & J_{t_k\times t_2} & \cdots & (J-I)_{t_k} \end{bmatrix},$$

Seidel matrix

$$S = \begin{bmatrix} (J-I)_{s_1} & -J_{s_1\times t_1} & J_{s_1\times s_2} & -J_{s_1\times t_2} & \cdots & -J_{s_1\times t_k} \\ -J_{t_1\times s_1} & (I-J)_{t_1} & J_{t_1\times s_2} & -J_{t_1\times t_2} & \cdots & -J_{t_1\times t_k} \\ J_{s_2\times s_1} & J_{s_2\times t_1} & (J-I)_{s_2} & -J_{s_2\times t_2} & \cdots & -J_{s_2\times t_k} \\ -J_{t_2\times s_1} & -J_{t_2\times t_1} & -J_{t_2\times s_2} & (I-J)_{t_2} & \cdots & -J_{t_2\times t_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -J_{t_k\times s_1} & -J_{t_k\times t_1} & -J_{t_k\times s_2} & -J_{t_k\times t_2} & \cdots & (I-J)_{t_k} \end{bmatrix}. \tag{2}$$

where, $O_{m\times n}$, $J_{m\times n}$ are all zero block matrix and all 1 block matrix respectively of size mn.

Quotient matrix

Let $\pi = \{C_1, C_2, \cdots, C_{2k}\}$ be an equitable partition of the vertex set of Γ, such that, C_1 contains first s_1 vertices, C_2 contains next t_1 vertices and so on. Then the adjacency quotient matrix(Q_A) and the Seidel quotient matrix (Q_S) corresponding to the same equitable partition π are the square matrices of size $2k$, given by

$$Q_A = \begin{bmatrix} 0 & t_1 & 0 & t_2 & \dots & t_k \\ s_1 & t_1 - 1 & 0 & t_2 & \dots & t_k \\ 0 & 0 & 0 & t_2 & \dots & t_k \\ \dots & \dots & \dots & & & \\ s_1 & t_1 & s_2 & t_2 & \dots & t_k - 1 \end{bmatrix},$$

Quotient matrix

$$Q_S = \begin{bmatrix} s_1 - 1 & -t_1 & s_2 & -t_2 & s_3 & \dots & -t_k \\ -s_1 & -(t_1 - 1) & s_2 & -t_2 & s_3 & \dots & -t_k \\ s_1 & t_1 & s_2 - 1 & -t_2 & s_3 & \dots & t_k \\ -s_1 & -t_1 - & -s_2 & -(t_2 - 1) & s_3 & \dots & -t_k \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -s_1 & -t_1 & -s_2 & -t_2 & -s_3 & \dots & -(t_k - 1) \end{bmatrix}.$$

Partition representation

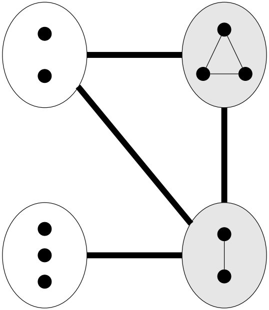

Figure: $b = 0011100011$

Contribution

We prove some important properties of Q_S and derive the formula for the multiplicity of the Seidel eigenvalues ± 1. We characterized all the threshold graphs which have at most five distinct Seidel eigenvalues. Finally, it is shown that the cospectrality may be occured if Seidel matrix is considered instead of adjacency matrix. Few results are given below.

Theorem

Q_S is diagonalizable.

Sketch of proof: Consider $D = diag(s_1, t_1, s_2, t_2, \ldots, s_k, t_k)$. Then Q_S is similar to $D^{\frac{1}{2}} Q_S D^{-\frac{1}{2}}$, a symmetric matrix.

Theorem

Q_A and Q_S are non singular.

Sketch of proof: By *Theorem* 2.3 of Banerjee and Mehatari [9],

Q_A and Q_S are non-singular

we have $det(Q_A) = (-1)^k s_1 t_1 s_2 t_2 \dots s_k t_k$.
Q_S is row equivalent to the following tridiagonal matrix.

$$T = \begin{bmatrix} 1-2s_1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2t_1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & -2s_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2t_2 & -1 & \dots & 0 & 0 \\ \dots & & \dots & & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & -2s_k & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2t_k-1 \end{bmatrix}$$

By *Algorithm* 2.1 of Mikkawy [14], we see that $det(T) \neq 0$.
Thus, $det(Q_S) \neq 0$.

Characteristic polynomial of S

Theorem

Let $b = \alpha_1\alpha_2\ldots\alpha_n$ be the binary string of a threshold graph and let $b_r = \alpha_1\alpha_2\alpha_3\ldots\alpha_r$. Suppose $\Phi_r(x)$ denote the characteristic polynomial of Seidel matrix of the threshold graph with binary string b_r, then the characteristic polynomial, $\Phi_n(x)$ of the Seidel matrix S given by (1) is obtained by the following recurrence formula

$$\Phi_r(x) = 2(x+\beta_{r-1})\Phi_{r-1}(x) - 2(x+\beta_{r-1})^2\Phi_{r-2}(x),$$

where $\Phi_1(x) = x$ and $\Phi_2(x) = x^2 - 1$.

Characteristic polynomial of S

Sketch of the proof: Here

$$\Phi_r(x) = \begin{vmatrix} x & -\beta_2 & -\beta_3 & \dots & -\beta_r \\ -\beta_2 & x & -\beta_3 & \dots & -\beta_r \\ -\beta_3 & -\beta_3 & x & \dots & -\beta_r \\ \dots & \dots & \dots & \dots & \dots \\ -\beta_r & -\beta_r & -\beta_r & \dots & x \end{vmatrix}.$$

We consider two cases.
Case I. $\beta_r = \beta_{r-1}$.
Case II. $\beta_r = -\beta_{r-1}$.

Spectrum of Q_S

Theorem

$-1 \notin Spec(Q_S)$ if $t_k > 1$.

Sketch of proof: Let us assume that Q_S has the eigenvalue -1. Then there exists a non zero vector $X = (x_1 \ x_2 \ x_3 \ \ldots \ x_{2k})^t$ such that $Q_S X = -X$. Solving the system $Q_S X = -X$, we have $x_i = 0$ for all $i = 1, 2, 3, \ldots, 2k$, if $t_k \neq 1$.

Corollary

If $t_k = 1$, then -1 is a simple eigenvalue of Q_S.

Sketch of proof: Here $X = (0 \ \ldots \ 0 \ 1 \ s_k)$ is an eigenvector corresponding to the eigenvalue -1.

Spectrum of Q_S

Theorem

$1 \notin Spec(Q_S)$ if $s_1 > 1$.

Sketch of proof: Let us assume that Q_S has the eigenvalue 1. Then there exists a non zero vector $X = (x_1 \ x_2 \ x_3 \ \ldots \ x_{2k})^t$ such that $Q_S X = X$. Solving the system $Q_S X = X$, we have $x_i = 0$ for all $i = 1, 2, 3, \ldots, 2k$, if $s_1 \neq 1$.

Corollary

If $s_1 = 1$, then 1 is a simple eigenvalue of Q_S.

Sketch of proof: Here $X = (t_1 \ -1 \ 0 \ \ldots \ 0)^t$ is an eigenvector corresponding to the eigenvalue 1.

Q_S has simple eigenvalue

Theorem

All eigenvalues of Q_S are simple.

Sketch of proof: Suppose λ is an eigenvalue of Q_S. Let $X = (x_1\ x_2\ x_3\ \ldots\ x_{2k})^t$ be an eigenvector corresponding to λ such that $x_l \neq 0$ and $x_m = 0$ for all $m < l$, where l is minimal. Then $l = 2p, 1 \leq p \leq k$. We already proved that $\lambda = \pm 1$ is a simple eigenvalue. Now we prove the theorem for $\lambda \neq \pm 1$. Then from the relation $Q_S X = \lambda X$, we have $x_{l+i} = c_{l+i} x_l$, where c_{l+i} are the constants depend on λ. Thus we can construct X as $X = x_l(0\ 0\ 0\ \ldots\ 0\ 1\ c_{l+1}\ c_{l+2}\ c_{l+3}\ \ldots\ c_{2k})^t$. Now if $X' = (x_1'\ x_2'\ x_3'\ \ldots\ x_{2k}')$ be the another eigenvector corresponding to λ, then we see that X' is a constant multiple of X. Hence the geometric multiplicity of λ is one. Again Q_S is diagonalizable. Hence algebraic multiplicity of λ is also one.

Properties of Q_S

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5. $\pm 1 \notin Spec(Q_S)$ if $s_1 > 1$, and $t_k > 1$.

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Properties of Q_S

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3. Every eigenvalue of Q_S is also an eigenvalue of S.
4. 0 is not an eigenvalue of Q_S.
5. $\pm 1 \notin Spec(Q_S)$ if $s_1 > 1$, and $t_k > 1$.
6. Eigenvalues of Q_S are simple.
7. Eigenvalues of Q_S are real.

Multiplicity of -1

Theorem

Let $n_{-1}(S)$ denotes the multiplicity of the eigenvalue -1 of S. Then

$$n_{-1}(S) = \begin{cases} \sum s_i - k, & \text{for } t_k > 1, \\ \sum s_i - k + 1, & \text{for } t_k = 1. \end{cases}$$

Sketch of proof: If $t_k > 1$, then $-1 \notin Spec(Q_S)$. Therefore the multiplicity of the eigenvalue -1 in S is exactly $\sum(s_i - 1)$.
If $t_k = 1$, then $-1 \in Spec(Q_S)$ and it is simple.
Thus,

$$n_{-1}(S) = \sum s_i - k + 1.$$

Multiplicity of $+1$

Theorem

Let $n_{+1}(S)$ denotes the multiplicity of the eigenvalue $+1$ of S. Then

$$n_{+1}(S) = \begin{cases} \sum t_i - k, & \text{for } s_1 > 1, \\ \sum t_i - k + 1, & \text{for } s_1 = 1. \end{cases}$$

Sketch of proof: If we take $s_1 > 1$. Then $1 \notin Spec(Q_S)$. Therefore the multiplicity of the eigenvalue 1 in S is exactly $\sum(t_i - 1)$.
If we take $s_1 = 1$. Then $1 \in Spec(Q_S)$ and it is simple.
Thus,

$$n_{+1}(S) = \sum t_i - k + 1.$$

Threshold graphs with at most five distinct Seidel eigenvalue

Corollary

Let Γ be a threshold graph with binary string $\boldsymbol{b}$. Then Γ has two distinct Seidel eigenvalues if and only if one of the following conditions hold.
(i) $\boldsymbol{b} = 0^{s_1}1, \ s_1 \geq 1$,
(ii) $\boldsymbol{b} = 01^{t_1}, \ t_1 \geq 1$.

Sketch of the proof: (i) Here Γ is a threshold graph with binary string $\mathbf{b} = 0^{s_1}1$ where $s_1 \geq 1$. Infact, this is a star graph K_{1,s_1}. It has two distinct Seidel eigenvalues s_1 with multiplicity 1 and -1 with multiplicity s_1.
(ii) Here Γ is a threshold graph with binary string $\mathbf{b} = 01^{t_1}$ where $t_1 \geq 1$. Clearly this is a complete graph K_{t_1+1}. It has two distinct Seidel eigenvalues $-t_1$ with multiplicity 1 and 1 with multiplicity t_1.

Threshold graphs with at most five distinct Seidel eigenvalue

Corollary

Let Γ be a threshold graph with binary string $\boldsymbol{b}$. Then Γ has four distinct Seidel eigenvalues if and only if one of the following conditions hold.
(i) $\boldsymbol{b} = 0^{s_1}1^{t_1},\ s_1 > 1,\ t_1 > 1,$
(ii) $\boldsymbol{b} = 01^{t_1}0^{s_2}1.$

Sketch of the proof: Let $b = 0^{s_1}1^{t_1}0^{s_2}\dots0^{s_k}1^{t_k}$ be the binary string of the threshold graph Γ. Here $k \leq 2$.
Case I. $k = 1$. Then $b = 0^{s_1}1^{t_1}$. If $s_1 > 1, t_1 > 1$, then the quotient matrix Q_S has two distinct eigenvalues other that ± 1. Therefore S has four distinct eigenvalues -1 with multiplicity $s_1 - 1$ and 1 with multiplicity $t_1 - 1$, and another two simple eigenvalues come from Q_S.

Threshold graphs with at most five distinct Seidel eigenvalue

Case II. Let $k = 2$ then $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}$. Now if Γ has 4 distinct eigenvalues, then S can not have eigenvalue ± 1 outside of the spectrum of Q_S. Therefore 4 distinct eigenvalues is possible only if $s_1 = 1 = t_2$, and in that case eigenvalues of S are 1^{t_1}, $(-1)^{s_2}$ and $\frac{(s_2-t_1)\pm\sqrt{(s_2-t_1)^2+4(1+t_1+s_2+2t_1s_2)}}{2}$.

Conversely if $b = 01^{t_1}0^{s_2}1$ or $b = 0^{s_1}1^{t_1}$, $s_1 > 1$, $t_1 > 1$, then S has four distinct eigenvalues.

Threshold graphs with at most five distinct Seidel eigenvalue

Corollary

No threshold graph can have three distinct Seidel eigenvalues.

Corollary

Let Γ be a threshold graph with binary string $\boldsymbol{b}$. Then Γ has five distinct Seidel eigenvalues if and only if $\boldsymbol{b} = 01^{t_1}0^{s_2}1^{t_2}$ with $t_2 > 1$ or $\boldsymbol{b} = 0^{s_1}1^{t_1}0^{s_2}1$ with $s_1 > 1$.

Sketch of the proof: Let $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\ldots 0^{s_k}1^{t_k}$. Then k must be equal to 2. Then binary string of Γ is $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}$. Then Q_S has four distinct eigenvalues. Now S will have five distinct Seidel eigenvalues if ± 1 are eigenvalues of S and exactly one of ± 1 belongs to spectrum of Q_S.

Threshold graphs with at most five distinct Seidel eigenvalue

Case I. $s_1 = 1, t_2 > 1$. Then $+1$ belongs to the spectrum of Q_S, where as -1 is not an eigenvalue of Q_S. Therefore spectrum of S is $\{-1^{s_2-1}, 1^{t_1+t_2}, \alpha_1, \beta_1, \gamma_1\}$, where α_1, β_1, and γ_1 are the distinct eigenvalues (other than 1) of Q_S.
Case II. $s_1 > 1, t_2 = 1$. Then -1 belongs to the spectrum of Q_S, where as 1 is not an eigenvalue of Q_S. Therefore spectrum of S is $\{-1^{s_1+s_2}, 1^{t_1-1}, \alpha_2, \beta_2, \gamma_2\}$, where α_2, β_2, and γ_2 are the distinct eigenvalues (other than -1) of Q_S.

Two threshold graphs may be cospectral

Theorem

Let us consider two threshold graphs Γ_1 and Γ_2 on n vertices with the binary string $b_1 = 0^{n-2}1^2$ and $b_2 = 010^{n-3}1$ respectively. Then Γ_1 and Γ_2 are always ***cospestral*** *on Seidel matrix.*

Sketch of proof: The characterastic equation of Q_S for the binary strings b_1 and b_2 are respectively

$$x^2 + (4-n)x + (7-3n) = 0,$$

$$(x^2-1)[x^2 + (4-n)x + (7-3n)] = 0.$$

Thus both the strings have same Seidel spectrum $\{-1^{n-3}, 1, \alpha, \beta\}$, where α, β are the roots of the equation $x^2 + (4-n)x + (7-3n) = 0$.

Example of cospectral threshold graphs

Example

If we take $n = 4$ in previous theorem, we get threshold graphs Γ_1 and Γ_2 with binary string $b_1 = 0011$ and $b_2 = 0101$ respectively. In that case, Γ_1 and Γ_2 are not isomorphic (see Figure 2) but they both have eigenvalues $\pm 1, \pm\sqrt{5}$.

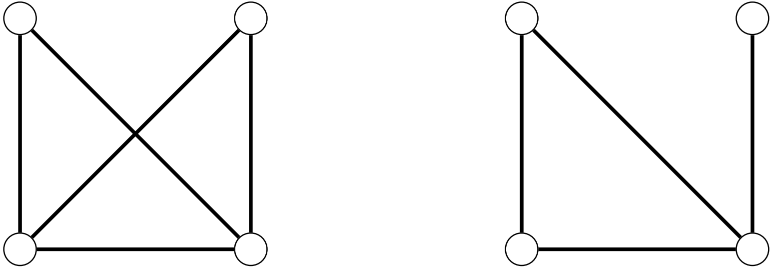

Figure: Non isomorphic ***cospestral*** threshold graphs with 4 vertices.

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That's all.