Weekly e-seminar on "Graphs, Matrices and Applications"-IIT Kharagpur

Some Graphs Determined By Their Spectra

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Friday 27^{th} August, 2021

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Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$ be the degree sequence of G.

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Graph Matrices:

Adjacency matrix: $A(G) := [a_{ij}]_{n \times n}$, $a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G); \\ 0 & \text{otherwise.} \end{cases}$

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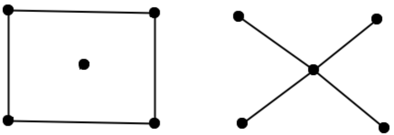

Figure 1: Adjacency copsectral graphs of smallest order with adjacency spectrum $\{2, 0, 0, 0, -2\}$

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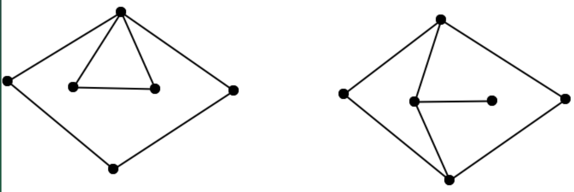

Figure 2: Laplacian copsectral graphs of smallest order with Laplacian spectrum $\{5.236, 3, 3, 2, 0.764, 0\}$

Signless Laplacian Cospectral Graphs (Q-cospectral graph): Two graphs are signless Laplacian cospectral if they share the same signless Laplacian spectrum.

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Figure 3: Signless Laplacian copsectral graphs of smallest order with Q-spectrum $\{4, 1, 1, 0\}$

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- A graph G is said to be *determined by the signless Laplacian spectrum* (simply, *DQS*) if G has no Q-cospectral mate up to isomorphism.

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Developments on spectral characterizations of graphs with respect to adjacency spectrum and (signless) Laplacian spectrum until 2008 are reported in the following two survey articles by Dam and Haemers.

- “Which graphs are determined by their spectra” Linear Algebra Appl., vol. 373, pp. 241–272, 2003.

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Linear Algebra and its Applications 373 (2003) 241–272
LINEAR ALGEBRA AND ITS APPLICATIONS
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Which graphs are determined by their spectrum?

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Received 30 April 2002; accepted 10 March 2003

Submitted by B. Shader

Abstract

For almost all graphs the answer to the question in the title is still unknown. Here we survey the cases for which the answer is known. Not only the adjacency matrix, but also other types of matrices, such as the Laplacian matrix, are considered.

Keywords: Spectra of graphs; Eigenvalues; Cospectral graphs; Distance-regular graphs

1. Introduction

Consider the two graphs with their adjacency matrices, shown in Fig. 1. It is easily checked that both matrices have spectrum

$$\{[2]^1, [0]^3, [-2]^1\}$$

(exponents indicate multiplicities). This is the usual example of non-isomorphic cospectral graphs first given by Cvetković [19]. For convenience we call this couple the *Saltire pair* (since the two pictures superposed give the Scottish flag: Saltire). For graphs on less than five vertices, no pair with cospectral adjacency matrices exists, so each of these graphs is determined by its spectrum.

We abbreviate ‘determined by the spectrum’ to DS. The question ‘which graphs are DS?’ goes back for about half a century, and originates from chemistry. In 1956

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[1] The research of E.R. van Dam has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

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doi:10.1016/S0024-3795(03)00483-X

Discrete Mathematics 309 (2009) 576–586

Contents lists available at ScienceDirect

ELSEVIER

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Developments on spectral characterizations of graphs

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ARTICLE INFO

Article history:
Received 10 May 2007
Accepted 23 July 2007
Available online 21 September 2008

Keywords:
Spectra of graphs
Cospectral graphs
Generalized adjacency matrices
Distance-regular graphs

ABSTRACT

In [E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum? Linear Algebra Appl. 373 (2003), 241–272] we gave a survey of answers to the question of which graphs are determined by the spectrum of some matrix associated to the graph. In particular, the usual adjacency matrix and the Laplacian matrix were addressed. Furthermore, we formulated some research questions on the topic. In the meantime, some of these questions have been (partially) answered. In the present paper we give a survey of these and other developments.

1. Introduction

Since [13] was published, the study of spectral characterizations of graphs has developed significantly. Therefore, we believe that a second survey has become worthwhile. In this survey, we focus on new developments. Most of the mentioned results have been published, whereas some other results are new; obtained either by the authors themselves, or through personal communications.

We do not only consider the spectrum of the adjacency matrix, but also deal with the Laplacian matrix, the (so-called) signless Laplacian, and the generalized adjacency matrices. As in [13], we abbreviate ‘determined by the spectrum’ by DS.

An important development is the new method of Wang and Xu (see Section 5) for finding graphs that are DS with respect to the generalized adjacency matrix. Their approach often works for randomly generated graphs, and this strengthens our believe that the statement ‘almost all graphs are not DS’ (which is true for trees) is false.

Another result deals with cospectrality of generalized adjacency matrices, in particular an answer is given to the question (posed in [13]): ‘when can regularity of a graph be deduced from the spectrum of a generalized adjacency matrix?’ (see Section 4).

Several families of graphs are shown to be DS with respect to the adjacency matrix (see Section 2), the Laplacian matrix (see Section 3), or both (see Section 6.1). For the signless Laplacian we know of one new result (see Section 3). However, the remark made in [13] that, with respect to the signless Laplacian, graphs tend to be more often DS than with respect to the Laplacian, or (generalized) adjacency matrix, motivated Cvetković, Rowlinson, and Simić [12] to (re)start investigations of this rather unusual matrix.

For many other graphs, cospectral mates have been found. This includes some special bipartite graphs (see Section 2.3), and many distance-regular graphs (see Section 6.4). One such family of graphs cospectral with distance-regular graphs turned out to be a new infinite family of distance-regular graphs. Important methods for constructing cospectral graphs are Godsil-McKay switching [23] and the partial-linear-space technique, which have been explained in our previous survey [13]. We assume the reader to be familiar with the methods and results from that paper.

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doi:10.1016/j.disc.2008.08.019

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For the Laplacian matrix the following follows from the spectrum:

- The number of components.
- The number of spanning trees.

Let M be a Hermitian matrix of order m and let $\theta_1(M) \geq \theta_2(M) \geq \cdots \geq \theta_m(M)$ be its eigenvalues.

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Theorem [17] Let M be a Hermitian matrix of order n.

- Interlacing: If M_k is a principal submatrix of M of order k with $1 \leq k \leq n$, then for $1 \leq i \leq k$, $\theta_{n-k+i}(M) \leq \theta_i(M_k) \leq \theta_i(M)$.

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- Weyl's inequality: If $M = N + P$, where N and P are Hermitian matrices of order n. Then for $1 \leq i, j \leq n$, we have
  - $\theta_i(N) + \theta_j(P) \leq \theta_{i+j-n}(M)$ $(i + j > n)$;
  - $\theta_{i+j-1}(M) \leq \theta_i(N) + \theta_j(P)$ $(i + j - 1 \leq n)$.

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Proposition [9] The complete graph K_n,the complete bipartite graph $K_{m,m}$ and the Cycle graph C_n are determined by the adjacency spectrum.

Note that the graph $K_{4,4} \cup 2K_1$ and $K_{8,2}$ are cospectral.

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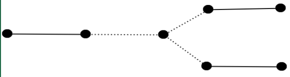

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The following result is due to Doob and Haemers.

Theorem [12] The complement of the path graph is determined by its adjacency spectrum.

Graphs with small spectral radius that are DAS

• In [33], Smith determined all connected graphs with spectral radius at most 2. This includes the cycle C_n, P_n and the graphs shown in the following figure.

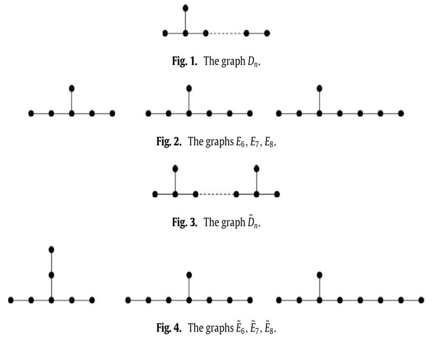

Fig. 1. The graph D_n.

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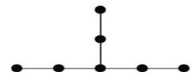

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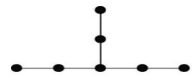

In [5], Brouwer and Neumaier classified all graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$ and in [13], Ghareghani et al. showed that all these graphs are *DAS*. i.e.,

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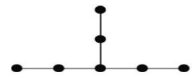
Theorem [9] All connected graphs with spectral radius at most 2 are DAS, except for the graphs shown in the following figure.



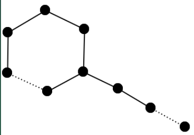
In [5], Brouwer and Neumaier classified all graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$ and in [13], Ghareghani et al. showed that all these graphs are *DAS*. i.e.,

Theorem [13] All connected graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$ are DAS.

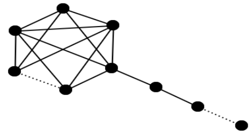
Some special graphs that are DAS

Theorem (R. Boulet and B. Jouve [3]) The lollipop graph is DAS.

Adjacency spectral characterization of lollipop graphs were first consider by Heamers, Liu and Zhang in [15] and it was shown that the lollipop graph with odd cycle is *DAS*.



Theorem [34] The kite graph is DAS.



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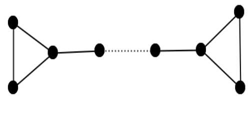

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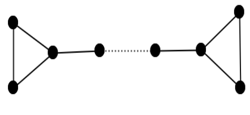
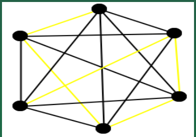


Theorem [21] The graph $K_n \backslash P_k$ is DAS.

In 2014, Cámara and Haemers [6] conjectured that $K_n \backslash P_k$ is DAS and they succeeded in proving it for $2 \leq k \leq 6$.



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Lemma [7] Let G be a graph on n vertices then the Laplacian spectra of $\overline{G} = \{n - \mu_1(G), n - \mu_2(G), \ldots, n - \mu_{n-1}(G), 0\}$.

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Corollary The complement graph of path graph is *DLS*.

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Lemma [19] Let G be a graph on n vertices. Then

$$\Delta(G) + 1 \leq \mu_1 \leq \max\left\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}, uv \in E(G)\right\}$$

where $\Delta(G)$ denotes the maximum vertex degree of G, μ_1 denotes the largest Laplacian eigenvalue of G, m_v denotes the average of the degrees of the vertices adjacent to vertex v in G.

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- Let x, y and z be the number of vertices of G of degree 1, 2, and 3, respectively. Since the order n, the edges (=n) and the sum $\sum_{i=1}^{n} d_i^2 (= 4(n-2)+9+1)$ are determined by the Laplacian spectrum, we must have

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Some other classes of graphs that are studied for Laplacian spectral characterization are Starlike trees [29]; double starlike trees [24]; multi-fan graphs in [22]; complete-split graph [11], butter-fly graph [20], etc.

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- Therefore by above lemma G is L-cospectral with P_n.
- Hence $G \cong P_n$. (Since P_n is DLS). This completes the proof.

In literature some special graphs are proved to be determined by the spectra for example the lollipop graph [37], short kite graph [34], complete split graph [11], sun graph [28], fan graph [23], etc.

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Recently, (signless) Laplacian spectral characterization of disjoint union of graphs have been studied and also the problem of characterizing join graphs which are determined by their (signless) Laplacian spectra is considered, see [16, 26, 30, 36, 25, 1, 31] for more details.

Recent studies:

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- Which graphs are determined by the distance spectra? [18]

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Thank you