Adjacency matrices of complex unit gain graphs

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Outline

- Adjacency matrices of graphs
- Spectral properties
- Perron-Frobenius theorem
- Adjacency matrices of complex unit gain graphs
- Characterization of bipartite graphs and trees

Adjacency matrix

Definition (Adjacency matrix)

The adjacency matrix of a graph G with n vertices, $V(G) = \{v_1, \ldots, v_n\}$ is the $n \times n$ matrix, denoted by $A(G) = (a_{ij})$, is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

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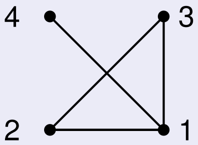

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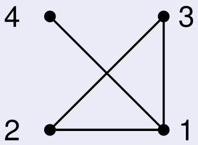

The **adjacency matrix** of G is

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.
3. $(i, j)^{th}$ entry of the matrix A^k equals the number of walks of length k from the vertex i to the vertex j.

Spectrum of adjacency matrix

Let G be a graph with n vertices and with eigenvalues of its adjacency matrices, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We denote by $\Delta(G)$ and $\delta(G)$, the maximum and the minimum of the vertex degrees of G, respectively.

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Irreducible matrices

An $n \times n$ matrix, $n \geq 2$, is *reducible* its rows and columns can be simultaneously permuted to

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

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Working definition: A is irreducible if and only if $G(A)$ is strongly connected.

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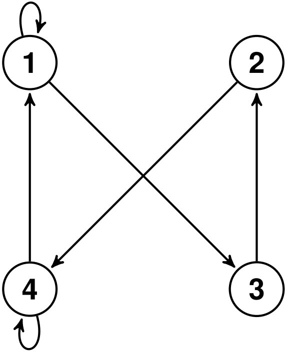

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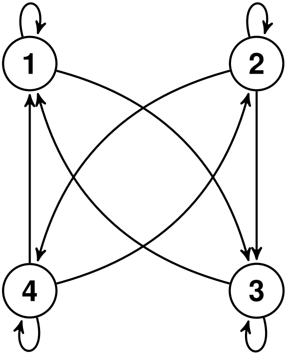

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If A is nonnegative and irreducible, then

a) $\rho(A) > 0$, *where* $\rho(A)$ *is the maximum of absolute value of all the eigenvalues of* A*,*
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Let $A, B \in \mathbb{C}^{n\times n}$ *and suppose that* A *is nonnegative. If* $A \geq |B|$, *then* $\rho(A) \geq \rho(|B|) \geq \rho(B)$.

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Gain graphs

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- Assign a weight (gain) $g \in \mathfrak{G}$ for each directed edge $e_{jk} \in \overrightarrow{E(G)}$, such that the weight of e_{kj} is g^{-1}. Let us denote this assignment by φ.

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Definition (Thomas Zaslavsky)

A $\mathfrak{G}$-gain graph is a graph G in which each orientation of an edge is given a gain which is the inverse of the gain assigned to the opposite orientation.

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- $\mathfrak{G} = \{\pm 1, \pm i\}$[D. Kalita and S. Pati(2014)] ,
- $\mathfrak{G} = \{1, \pm i\}$ [K. Guo and B. Mohar(2017), J. Liu and X. Li(2015)],
- $\mathfrak{G} = \{1, \frac{1 \pm i\sqrt{3}}{2}\}$. [B. Mohar(2020)]
- $\mathfrak{G} = \{1, e^{\pm i\theta}\}, \theta \in \mathbb{R}$. [S. Kubota, E. Segawa and T. Taniguchi(2019)]
- $\mathfrak{G} = \mathbb{C}^*$(with nonnegative imaginary part)[R. B. Bapat, D. Kalita and S. Pati(2012)].

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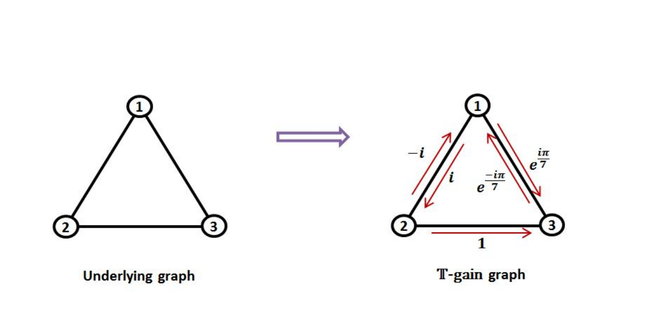

Figure: $\mathbb{T}$-gain graph Φ and its underlying graph

On $\mathbb{T}$-gain adjacency matrix

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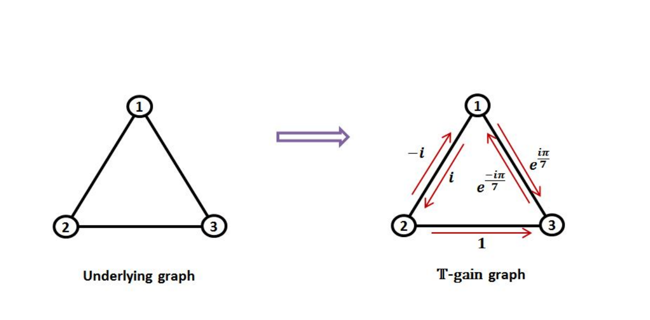

Figure: $\mathbb{T}$-gain graph Φ and its underlying graph

$$A(\Phi) = \begin{pmatrix} 0 & i & e^{i\frac{\pi}{7}} \\ -i & 0 & 1 \\ e^{-i\frac{\pi}{7}} & 1 & 0 \end{pmatrix}$$

Definition

- **The gain of a cycle** $C = v_1 v_2, \ldots v_l v_1$, denoted by $\varphi(C)$, is defined as the product of the gains of its edges, that is $\varphi(C) = \varphi(e_{12})\varphi(e_{23}) \ldots \varphi(e_{(l-1)l})\varphi(e_{l1})$.

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- A function from the vertex set of G to the complex unit circle $\mathbb{T}$ is called a **switching function**.
- We say that, two gain graphs $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ are said to be **switching equivalent, written as** $\Phi_1 \sim \Phi_2$, if there is a switching function $\zeta : V \to \mathbb{T}$ such that $\varphi_2(e_{ij}) = \zeta(v_i)^{-1}\varphi_1(e_{ij})\zeta(v_j)$.

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Let $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ be two $\mathbb{T}$-gain graph. If $\Phi_1 \sim \Phi_2$, then $A(\Phi_1)$ and $A(\Phi_2)$ have the same spectrum.

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Key theorem

Theorem (R. Mehatari,M.-, A.Samanta)

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain (connected) graph, then $\rho(A(\Phi)) = \rho(A(G))$ if and only if either Φ or $-\Phi$ is balanced.

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(ii) *If Φ is balanced implies $-\Phi$ is balanced for some gain Φ, then the graph is bipartite.*

Proof.

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Gains and bipartite graphs

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Proof.

(i) Suppose G is bipartite and Φ is balanced. Then due to the absence of odd cycles, $-\Phi$ is balanced.
(ii) Let Φ be a balanced cycle such that $-\Phi$ is balanced. Suppose that G is not bipartite. Then, any odd cycle in G can not be balanced with respect to $-\Phi$, which contradicts the assumption. Thus G must be bipartite. □

Converse of Reff's theorem

Theorem (R. Mehatari,M.-, A.Samanta)

Let $\Phi = (G, \varphi)$ *be a* $\mathbb{T}$*-gain(connected) graph. Then,* $\sigma(A(\Phi)) = \sigma(A(G))$ *if and only if* Φ *is balanced.*

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Characterization of bipartite graphs

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Let G be a connected graph. Then, G is bipartite if and only if $\rho(A(\Phi)) = \rho(A(G))$ implies $\sigma(A(\Phi)) = \sigma(A(G))$ for every gain φ.

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Invariance of gain spectrum and gain spectral radius

Theorem (A.Samanta, M.-)

Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. Then G is a tree if and only if $\sigma(A(G)) = \sigma(A(\Phi))$ for all φ.

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Let $\Phi = (G, \varphi)$ be a $\mathbb{T}$-gain graph. TFAE,

1. *G is tree,*
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Thank you !