Tree Laplacian Immanantal polynomials and the GTS poset

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31 July 2020, IIT KGP Webinar

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Joint work with Mukesh Nagar

Optimization problems on Trees

Several optimization problems on trees on n vertices have as extrema, either the path tree or the star tree.

Some examples:

	Parameter	Max	Min
1	largest eigenvalue of adjacency matrix	star	path
2	Number of closed walks of fixed length ℓ	star	path
3	Number of walks of fixed length ℓ	star	path
4	Coefficients of adjacency characteristic polynomial	path	star
5	Coefficients of laplacian characteristic polynomial	path	star

Csikvári has shown more than 15 such parameters which exhibit an identical behaviour.

Why does this happen?

Is this the whole story or is there something deeper here? Firstly, note that all the properties above have nothing to do with the labelling of the vertex set of the tree T.

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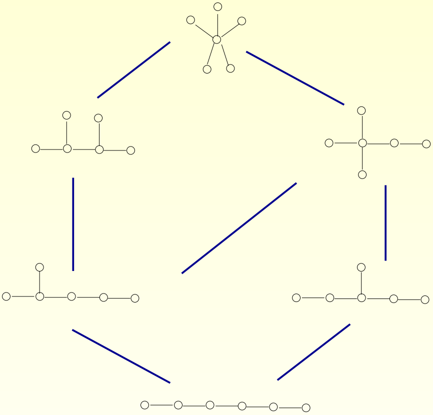
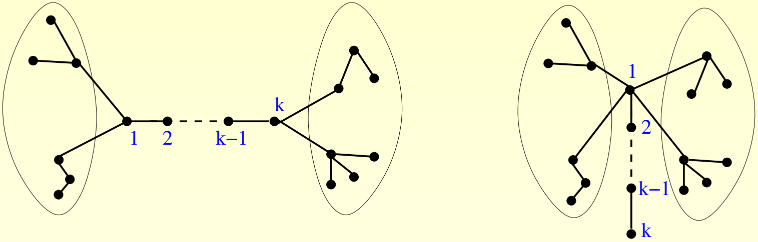
Is this the whole story or is there something deeper here? Firstly, note that all the properties above have nothing to do with the labelling of the vertex set of the tree T.

There is a poset denoted GTS_n on the set of unlabelled trees on n vertices where these properties are monotonic and the maximal and minimal elements are the star and the path trees respectively. This poset was defined by Csikvári.

The GTS poset on trees of order 6




1
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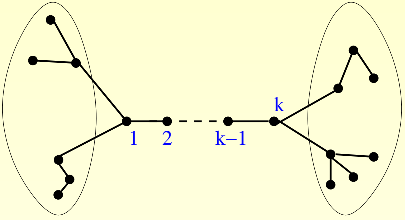
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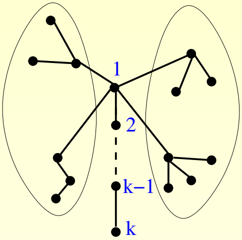
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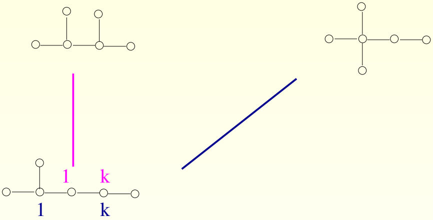

1
k
1
k

The main lemma of Csikvári is the following.

Lemma 1 (Csikvári)

Every tree T with n vertices other than the path, lies above some other tree T' on GTS_n. The star tree on n vertices is the maximal element and the path tree on n vertices is the minimal element of GTS_n.

We wish to generalise the following result.

	Parameter	Max	Min
1	largest eigenvalue of adjacency matrix	star	path
2	Number of closed walks of fixed length ℓ	star	path
3	Number of walks of fixed length ℓ	star	path
4	Coefficients of adjacency characteristic polynomial	path	star
5	Coefficients of laplacian characteristic polynomial	path	star

We will change laplacian to q-laplacian and characteristic to immanantal.

Immanants: relatives of determinants

Let $\mathfrak{S}_n$ be the symmetric group containing permutations of $[n] = \{1, 2, \ldots, n\}$. Given a function $f : \mathfrak{S}_n \to \mathbb{Z}$ and $A = (a_{i,j})_{1 \leq i,j \leq n}$, an $n \times n$ matrix with entries from a commutative ring, we define its generalized matrix function as

Definition 2

$$\det_f(A) = \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

This definition is inspired by Laplace's expansion formula for the determinant.

Representation theory of $\mathfrak{S}_n$ over $\mathbb{C}$ gives us several such functions f. It gives us $p(n)$ many of them. Thus, such functions are indexed by partitions λ of n. By this theory, corresponding to the partition (n), we get the permanent and corresponding to the partition 1^n, we get the determinant.

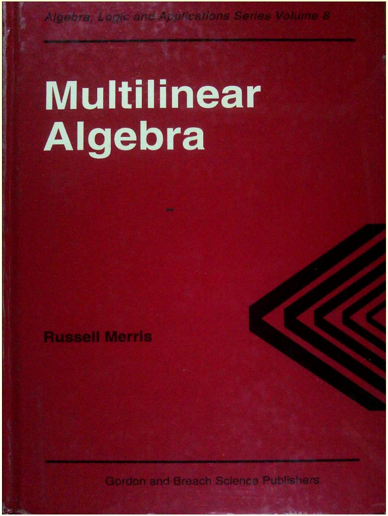
We only consider functions $f : \mathfrak{S}_n \to \mathbb{Z}$ that arise as characters of irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$. $\det_f(A)$ is called an immanant if f arises in this manner.

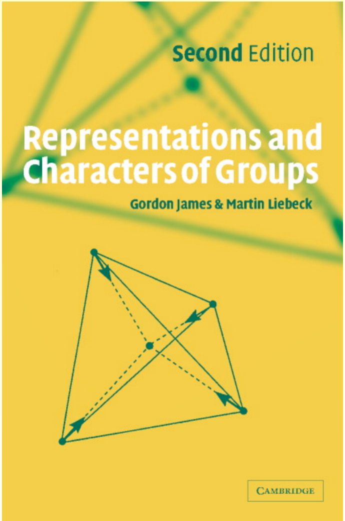
When f is the sgn function defined as $f(\pi) = \text{sgn}(\pi)$ for all $\pi \in \mathfrak{S}_n$, then, clearly $\det_{\text{sgn}}(A)$ is the usual determinant of A. i.e. we have $\det_{\text{sgn}}(A) = \det A$. If f is the id or all ones function defined as $f(\pi) = 1$ for all $\pi \in \mathfrak{S}_n$, then $\det_{\text{id}}(A) = \text{perm}(A)$, where $\text{perm}(A)$ is the permanent of A.

sgn, id are characters of irreducible representations and $\mathfrak{S}_n$ has $p(n)$ characters. We index ireducible characters by $\lambda \vdash n$ and hence have $\chi_\lambda : \mathfrak{S}_n \mapsto \mathbb{Z}$.

Character table of $\mathfrak{S}_4$

	e	$(1,2)$	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4)$
χ_{Triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1
χ	3	1	-1	0	-1
χ_4	3	-1	-1	0	1
χ_5	2	0	2	-1	0


Algebra, Logic and Applications Series Volume 8
Multilinear Algebra
Russell Merris
Gordon and Breach Science Publishers


Second Edition
Representations and Characters of Groups
Gordon James & Martin Liebeck
CAMBRIDGE

In this talk, we will consider matrices that arise from graphs. We would like propertices of the matrices to be invariant of the vertex labelling of the graphs.

Theorem 3

Let $\sigma \in \mathfrak{S}_n$ *and let* P_σ *be its corresponding permutation matrix. Let* $f : \mathfrak{S}_n \to \mathbb{Z}$ *be a function such that* $f(\tau) = f(\sigma^{-1}\tau\sigma)$ *for all* $\sigma, \tau \in \mathfrak{S}_n$*. Then,*

$\det_f(A) = \det_f(P_\sigma^{-1} A P_\sigma)$.

Recall $\det_f(A) = \sum_{\tau \in \mathfrak{S}_n} f(\tau) \prod_{i=1}^n a_{i,\tau(i)}$

$$\begin{aligned}
\det_f(P_\sigma^{-1} A P_\sigma) &= \sum_{\tau \in \mathfrak{S}_n} f(\tau) \prod_{i=1}^{n} a_{\sigma(i),\sigma(\tau(i))} \\
&= \sum_{\tau \in \mathfrak{S}_n} f(\tau) \prod_{j=1}^{n} a_{j,\sigma(\tau(\sigma^{-1}(j)))} \\
&= \sum_{\tau \in \mathfrak{S}_n} f(\sigma^{-1}\tau\sigma) \prod_{j=1}^{n} a_{j,\tau(j)} \\
&= \det_f(A)
\end{aligned}$$

Immanantal Polynomials

Similar to the charateristic polynomial $\det(xI - M)$, one can consider $\det_\lambda(xI - M)$, the immanantal polynomial of M indexed by λ.

q-analogue of the laplacian of a graph

$L = D - A$ is the laplacian of G. For a variable q, define $\mathcal{L}_q = I + q^2(D - I) - qA$. This is as amazing a matrix as the laplacian L with connections to zeta functions and to inverses of q-distance matrices.

When $q = 1$, it is easy to see that $\mathcal{L}_q = L$. For trees, we are interested in seeing if results on L are also true for $\mathcal{L}_q$ at least for some $q \in \mathbb{R}$.

Theorem 4 (Bapat, Lal and Pati)

Let $\mathcal{L}_q$ be the q-laplacian of a tree T. Then, $\det \mathcal{L}_q = 1 - q^2$ *and $\mathcal{L}_q$ is positive semidefinite iff $q \in [-1, 1]$.*

We recap the three ingredients needed to state results.

1. The poset GTS_n on trees with n vertices.
2. q-laplacian $\mathcal{L}_q$ of trees.
3. Immanantal polynomials indexed by $\lambda \vdash n$.

For an $n \times n$ matrix A and $\lambda \vdash n$, recall $f_\lambda^A(x) = d_\lambda(xI - A)$. The polynomial $f_\lambda^A(x)$ is called the immanantal polynomial of A corresponding to $\lambda \vdash n$.

Thus, in this notation, $f_{1^n}^A(x)$ is the characteristic polynomial of A. Let T be a tree on n vertices with Laplacian matrix L_T and define

$$f_\lambda^{L_T}(x) = d_\lambda(xI - L_T) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{L_T} x^{n-r} \tag{1}$$

where the $c_{\lambda,r}^{L_T}$'s are coefficients of the Laplacian immanantal polynomial of T in absolute value.

Known results

Immanantal polynomials were studied by Merris who showed that the Laplacian immanantal polynomial corresponding to the partition $\lambda = 2, 1^{n-2}$ (also called the second immanantal polynomial) of a tree T has connections with the centroid of T.

When $\lambda = 1^n$, Gutman and Pavlovic conjectured the following inequality which was proved by Gutman and Zhou and independently by Mohar.

Known results - II

Theorem 5 (Gutman and Zhou, Mohar)

Let T be any tree on n vertices and let S_n and P_n be the star and the path trees on n vertices respectively. Then, for $0 \leq r \leq n$, we have

$$c_{1^n,r}^{L_{S_n}} \leq c_{1^n,r}^{L_T} \leq c_{1^n,r}^{L_{P_n}}.$$

Thus, in absolute value, any tree T has coefficients of its Laplacian characteristic polynomial sandwiched between the corresponding coefficients of the star and the path trees.

Known results - III

Chan, Lam and Yeo have shown the following in their preprint.

Theorem 6 (Chan, Lam and Yeo)

Let T be any tree on n vertices with Laplacian L_T and let S_n and P_n be the star and the path trees on n vertices respectively. Then, for all $\lambda \vdash n$ and $0 \leq r \leq n$,

$$c_{\lambda,r}^{L_{S_n}} \leq c_{\lambda,r}^{L_T} \leq c_{\lambda,r}^{L_{P_n}}. \tag{2}$$

We will work with $\mathcal{L}_T^q$, the q-analogue of T's Laplacian.

As done in (1), define

$$f_{\lambda}^{\mathcal{L}_T^q}(x) = d_{\lambda}(xI - \mathcal{L}_T^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_T^q}(q) x^{n-r}. \tag{3}$$

Thus, each $c_{\lambda,r}^{\mathcal{L}_T^q}(q)$ is a polynomial in the variable q. How do we compare two polynomials and define one to be bigger than another?

As done in (1), define

$$f_{\lambda}^{\mathcal{L}_T^q}(x) = d_{\lambda}(xI - \mathcal{L}_T^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_T^q}(q) x^{n-r}. \tag{3}$$

Thus, each $c_{\lambda,r}^{\mathcal{L}_T^q}(q)$ is a polynomial in the variable q. How do we compare two polynomials and define one to be bigger than another?

We want the difference $c_{\lambda,r}^{\mathcal{L}_T^q}(q) - c_{\lambda,r}^{\mathcal{L}_{S_n}^q}(q) \in \mathbb{R}^+[q]$. That is, the difference polynomial has non-negative coefficients. This is an established way to get q-analogue of inequalities. Similarly, we want $c_{\lambda,r}^{\mathcal{L}_{P_n}^q}(q) - c_{\lambda,r}^{\mathcal{L}_T^q}(q) \in \mathbb{R}^+[q]$.

Recall (2).

$$c_{\lambda,r}^{L_{S_n}} \leq c_{\lambda,r}^{L_T} \leq c_{\lambda,r}^{L_{P_n}}.$$

These results show that the max and the min values of the absolute value of the coefficients of the laplacian immanantal polynomials are attained for the path tree and the star tree.

We wanted to get the GTS_n poset into the picture. We show the following.

Our result

Theorem 7 (Mukesh and SS, 2019)

Let T_1 and T_2 be trees with n vertices and let T_2 cover T_1 in GTS_n. Let $\mathcal{L}_{T_1}^q$ and $\mathcal{L}_{T_2}^q$ be the q-Laplacians of T_1 and T_2 respectively. For $\lambda \vdash n$, let

$$
\begin{aligned}
f_\lambda^{\mathcal{L}_{T_1}^q}(x) &= d_\lambda(xI - \mathcal{L}_{T_1}^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) x^{n-r} \text{ and} \\
f_\lambda^{\mathcal{L}_{T_2}^q}(x) &= d_\lambda(xI - \mathcal{L}_{T_2}^q) = \sum_{r=0}^{n} (-1)^r c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) x^{n-r}.
\end{aligned}
$$

Then, for all $\lambda \vdash n$ and for all $0 \le r \le n$, we assert that $c_{\lambda,r}^{\mathcal{L}_{T_1}^q}(q) - c_{\lambda,r}^{\mathcal{L}_{T_2}^q}(q) \in \mathbb{R}^+[q^2]$.

The proof involves a dual notion of "vertex orientations" and putting a "statistic" on each such orientation. This idea was motivated by Chan and Lam's proof where they show a "positive proof" that the immanant $d_\lambda(L_T) \geq 0$.

Thus for all $q \in \mathbb{R}$, we get monotonicity results. In particular, we get the following.

Corollary 8

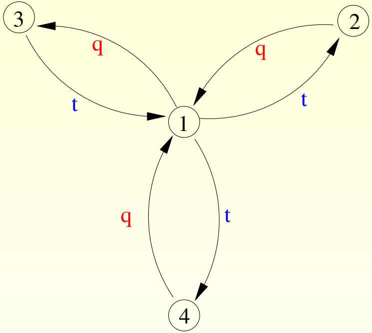
Setting $q = 1$ in $\mathcal{L}_T^q$, we infer that for all r, the coefficient of x^{n-r} in the immanantal polynomial of the Laplacian L_T of T decreases in absolute value as we go up GTS_n. Using Lemma 1, we thus get a more refined and hence stronger result than Gutman & Zhou and Mohar.

Corollary 9

Let T_1, T_2 be trees on n vertices with respective q-Laplacians $\mathcal{L}^q_{T_1}, \mathcal{L}^q_{T_2}$. Let $T_2 \geq_{\mathsf{GTS}_n} T_1$ and let $d_\lambda(\mathcal{L}^q_{T_i})$ denote the immanant of $\mathcal{L}^q_{T_i}$ for $1 \leq i \leq 2$ corresponding to the partition $\lambda \vdash n$. By comparing the constant term of the immanantal polynomial, for all $\lambda \vdash n$, we infer $d_\lambda(\mathcal{L}^q_{T_2}) \leq d_\lambda(\mathcal{L}^q_{T_1})$. This refines the inequalities given by Chan, Lam and Yeo.

What else?

In earlier work with Bapat, we had defined a q,t-laplacian $\mathcal{L}_{q,t}^{T}$ of a tree T as follows:



$$\mathcal{L}_{q,t}^{T} = \begin{pmatrix} 1+2qt & -t & -q & -t \\ -q & 1 & 0 & 0 \\ -t & 0 & 1 & 0 \\ -q & 0 & 0 & 1 \end{pmatrix}$$

Theorem 10

When $q, t \in \mathbb{R}$ with $qt \geq 0$, our main result goes through. Further, when $q, t \in \mathbb{C}$ too with $qt \geq 0$, our main result goes through.

Corollary 11

Set $q = z$ and $t = \overline{z}$ in $\mathcal{L}_{q,t}^T$. For example, set $q = \iota$ and $t = -\iota$ (where $\iota = \sqrt{-1}$) in $\mathcal{L}_{q,t}^T$.

This specialisation when $q = \iota$ and $t = -\iota$ gives rise to a class of Adjacency and Laplacian matrices.

Such adjacency matrices exist in the literature by the name Hermitian adjacency matrices. This was defined first by Bapat, Pati and Kalita and later by Liu and Li and by Guo and Mohar.

Such laplacian matrices also exist in the literature by the name Hermitian laplacian matrices. This was defined by Yu and Qu.

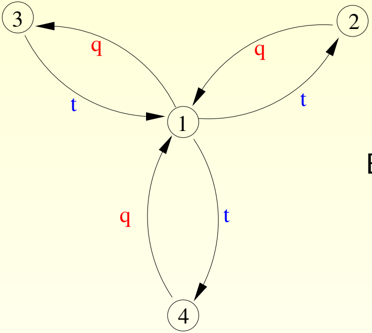
These are recent advancements and our result shows results about their immanantal polynomial and the GTS_n poset.

Corollary 12

Let T_1, T_2 be trees on n vertices with $T_2 \geq_{\mathsf{GTS}_n} T_1$. Then, in absolute value, the coefficients of the immanantal polynomials of the Hermitian Laplacian of T_1 are larger than the corresponding coefficient of the immanantal polynomials of the Hermitian Laplacian of T_2.

Bapat, Lal and Pati introduced the exponential distance matrix ED_T of a tree T. We show that when $q \neq \pm 1$, the coefficients of the characteristic polynomial of ED_T, in absolute value decrease when we go up GTS_n.

In ED_q^T, the exponential distance matrix of T, the (i,j)-th entry is $q^{d_{i,j}}$. There is a bivariate version $\mathrm{ED}_{q,t}^T$ where each edge has a separate direction incorporated.



$$\mathrm{ED}_{q,t}^T = \begin{pmatrix} 1 & t & q & t \\ q & 1 & q^2 & qt \\ t & t^2 & 1 & t^2 \\ q & qt & q^2 & 1 \end{pmatrix}$$

Clearly $\mathrm{ED}_q^T = \mathrm{ED}_{q,q}^T$.

Theorem 13 (Bapat, Lal and Pati)

Let T be a tree on n vertices. Then, $(\mathrm{ED}_q^T)^{-1} = \frac{1}{1-q^2}\mathcal{L}_q^T$.

Theorem 14 (Bapat, SS)

Let T be a tree on n vertices. Then, $(\mathrm{ED}_{q,t}^T)^{-1} = \frac{1}{1-qt}\mathcal{L}_{q,t}^T$.

Using Jacobi's theorem on minors of the inverse of a matrix, we can transfer results about the coefficients of the characteristic polynomial of ED_q^T and couple them with the GTS poset.

The following remarkable relation on immanants indexed by partitions with two columns is known.

Theorem 15 (Merris, Watkins)

Let A be an invertible $n \times n$ matrix. Then $\lambda \vdash n$ is a two column partition if and only if

$$d_\lambda(A)\det(A^{-1}) = d_\lambda(A^{-1})\det(A).$$

Using this Theorem of Merris and Watkins, we get results on coefficients of immanantal polynomials indexed by λ for two column partitions as one goes up the poset GTS.

Questions/Comments?