Distance Pareto eigenvalue of a graph

Weekly e-seminar on "Graphs, Matrices and Applications"
Talk by

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Distance matrix of a connected graph

Graph

By a graph G we mean a finite set of vertices $V(G)$ and a set of edges $E(G)$ consisting of distinct pairs of vertices. Throughout the presentation we consider only simple and connected graphs of finite order.

Distance matrix

The *distance matrix* of a connected graph G of order n is defined to be $\mathcal{D}(G) = [d_{ij}]_n$, where d_{ij} is the distance between the vertices v_i and v_j in G. Thus, $\mathcal{D}(G)$ is a symmetric real matrix and have real eigenvalues.

Pareto eigenvalue

Definition

A real number λ is said to be a Pareto eigenvalue of $A \in \mathbb{M}_n$ if there exists a nonzero vector $\mathbf{x}(\geq 0) \in \mathbb{R}^n$ such that

$$A\mathbf{x} \geq \lambda\mathbf{x} \quad \text{and} \quad \lambda = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}},$$

also we call $\mathbf{x}$ to be a Pareto eigenvector of A associated with Pareto eigenvalue λ.

Distance Pareto eigenvalue

Distance Pareto eigenvalue of a connected graph G is a Pareto eigenvalue of the distance matrix of G.

Distance Pareto eigenvalue

Theorem

The distance Pareto eigenvalues of a connected graph G are given by

$$\Pi(G) = \{\rho(A) : A \in M\},$$

where M is the class of all principal sub-matrices of $\mathcal{D}(G)$ and $\rho(A)$ is the largest eigenvalue of A.

Corollary

The largest distance Pareto eigenvalue of a connected graph is the distance spectral radius of the graph.

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Theorem

For a connected graph of diameter d, the integers $0, 1, \ldots, d$ are always its distance Pareto eigenvalues.

Lemma

For any positive integer n, $\Pi(K_n) = \{0, 1, \ldots, n-1\}$.

Theorem

There are exactly $2(n-1)$ distance Pareto eigenvalues of S_n and they are

$$\mu_{2k} = 2(k-1),\ \mu_{2k-1} = k - 1 + \sqrt{k^2 - 3k + 3} \text{ where } k = 1, \ldots, n-1.$$

Here μ_k is the k^{th} smallest distance Pareto eigenvalue of the given graph.

Theorem

If G is a connected graph of order n and diameter d then $|\Pi(G)| \geq n + d - 1$, with equality if and only if $G = P_3$ or K_n.

Theorem

If G is a connected graph of order n and diameter d then $|\Pi(G)| \geq n + d - 1$, with equality if and only if $G = P_3$ or K_n.

Theorem

If G is a graph with n vertices, then

$$\rho_k(G) \geq n - k \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Equality holds if and only if $G = K_n$.

Here $\rho_k(G)$ is the k^{th} largest distance Pareto eigenvalue of G.

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Lemma

If G is a connected graph with at least two vertices, then

$$\rho_2(G) = \max\{\rho(A) : A \in P\},$$

where $P = \{(\mathcal{D}(G))(v) : v \in V(G)\}$

Theorem

If G is a graph of order n with a vertex of degree $n-1$, then

$$n - 2 \leq \rho_2(G) \leq 2(n-2),$$

the right hand equality holds if and only if $G = S_n$ and the left hand equality holds if and only if $G = K_n$.

Theorem

If G *is a connected graph of order* n *and diameter* 2 *then* $\rho_2(G) \leq 2(n-2)$, *with equality if and only if* $G = S_n$.

Theorem

If G *is a connected graph with* n *vertices and* $\omega(G)$[a] $\geq n-1$, *then*

$$n-2 \leq \rho_2(G) \leq \frac{n-3+\sqrt{n^2+10n-23}}{2},$$

with equality in the left hand side if and only if $G = K_n$ *and equality in the right hand side if and only if* $G = K_{n-1}^1$.

[a] $\omega(G)$ is the clique number i.e. the order of largest complete subgraph of G.

Theorem

For any non complete connected graph G with n vertices,

$$\rho_2(G) \geq \frac{n-2+\sqrt{n^2-4n+12}}{2},$$

with equality if and only if $G = K_n - e$.

Corollary

For any non complete connected graph G with n vertices,

$$\rho_2(G) \geq n-2+\frac{2}{n-1}$$

equality holds if and only if $G = P_3$.

Theorem

If G *is a connected graph of order* n *so that minimum transmission occur at a vertex* $v \in G$ *and* $\mathbf{x}$ *is the normalized distance Pareto eigenvector corresponding to* ρ_2*, then*

$$\rho_2(G) \geq \frac{2[W - Tr(v)]}{n-1},$$

with equality if and only if $x_u = \frac{1}{\sqrt{n-1}}$ *for* $u \neq v$.

Theorem

For any connected graph G *of order* n *other than* K_n *and* $K_n - e$

$$\rho_2(G) \geq \frac{n - 2 + \sqrt{n^2 - 4n + 20}}{2},$$

with equality if and only if $G = K_n - \{e_1, e_2\}$*, where* e_1 *and* e_2 *are not incident in* K_n.

Lemma

If $a \leq b$, *then* $\rho_2(K_{a,b}) = a + b - 3 + \sqrt{a^2 + b^2 + b - ab - 2a + 1}$.

Theorem

If G *is a connected bipartite graph of order* n, *then*

$$\rho_2(G) \geq n - 3 + \sqrt{n^2 + n + 1 + 3\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - n - 1 \right)}$$

equality is attained if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Theorem

If $\lambda_2(G)$ *is the second largest distance eigenvalue of a connected graph* G, *then* $\rho_2(G) > \lambda_2(G)$.

Theorem

If G *and* $G' = G - e$ *are connected graphs then* $\rho_2(G') \geq \rho_2(G)$.

Theorem

If $\lambda_2(G)$ *is the second largest distance eigenvalue of a connected graph* G*, then* $\rho_2(G) > \lambda_2(G)$.

Theorem

If G *and* $G' = G - e$ *are connected graphs then* $\rho_2(G') \geq \rho_2(G)$.

Corollary

Among all connected graphs of given order, second largest distance Pareto eigenvalue is maximum for some tree.

Lemma

Let G *be a tree,* G' *be any connected graph and* $G^v = G_v * G'_w$ *with* $v \in V(G), w \in V(G')$ *and* $\rho^u(G^v) = \rho(A_u)$ *where* $A_u = \mathcal{D}(G^v)(u)$. *If* $i, j, k \in V(G)$ *with* $i \sim j \sim k$ *then for any* $u \in V(G) \cup V(G')$

$$\rho^u(G^i) + \rho^u(G^k) \geq 2\rho^u(G^j).$$

Furthermore, either $\rho^u(G^i) > \rho^u(G^j)$ *or* $\rho^u(G^k) > \rho^u(G^j)$.

Lemma

Let G be a tree, G' be any connected graph and $G^v = G_v * G'_w$ with $v \in V(G), w \in V(G')$ and $\rho^u(G^v) = \rho(A_u)$ where $A_u = \mathcal{D}(G^v)(u)$. If $i, j, k \in V(G)$ with $i \sim j \sim k$ then for any $u \in V(G) \cup V(G')$

$$\rho^u(G^i) + \rho^u(G^k) \geq 2\rho^u(G^j).$$

Furthermore, either $\rho^u(G^i) > \rho^u(G^j)$ or $\rho^u(G^k) > \rho^u(G^j)$.

Theorem

Among all trees of given order, the second largest distance Pareto eigenvalue is maximized in the path graph.

Theorem

Among all trees of given order, the second largest distance Pareto eigenvalue is minimized in the star graph.

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Smallest five distance Pareto eigenvalues

Theorem

For any connected graph G with at least 3 vertices, $0, 1$ and 2 are the smallest three distance Pareto eigenvalues of G.

Theorem

The fourth smallest distance Pareto eigenvalue of a connected non complete graph is $1 + \sqrt{3}$.

Theorem

If G is a non complete graph with at least 4 vertices, then $\mu_5(G) \geq 3$. The equality holds if and only if $\omega(G) \geq 4$ or $diam(G) \geq 3$.

Theorem

If G is a connected graph of order $n \geq 4$, diameter 2 and $\omega(G) \leq 3$, then

$$\mu_5(G) = \begin{cases} \frac{1+\sqrt{33}}{2} & \text{if } C_5 \text{ or } S_4^+ \text{ is an induced subgraph of } G, \\ \frac{3+\sqrt{17}}{2} & \text{if neither } C_5 \text{ nor } S_4^+ \text{ is an induced subgraph of } G \text{ but } K_4 - e \text{ is,} \\ 4 & \text{otherwise.} \end{cases}$$

Corollary

If T is a tree with at least 4 vertices, then $\mu_5(T) = 4$ or 3 according as T is a star or not.

6th smallest distance Pareto eigenvalue

Theorem

If G is a connected graph with at least 5 vertices and $\mu_5(G) = 4$, then

$$\mu_6(G) = \begin{cases} 5 \text{ if } G = K_6, \\ 2 + \sqrt{7} \text{ otherwise.} \end{cases}$$

Theorem

If G is a connected graph with at least 5 vertices and $\mu_5(G) = \frac{3+\sqrt{17}}{2}$, then $\mu_6(G) = 4$.

Theorem

If G is a connected graph with at least 5 vertices and $\mu_5(G) = 3$, then

$$\mu_6(G) = \begin{cases} \frac{1+\sqrt{33}}{2} & \text{if } C_5 \text{ or } S_4^+ \text{ is an induced subgraph of } G, \\ \frac{3+\sqrt{17}}{2} & \text{if } C_5 \text{ and } S_4^+ \text{ are not induced subgraph of } G \text{ but } K_4 - e \text{ is}, \\ 4 & \text{if } C_5, S_4^+, K_4 - e \text{ are not induced subgraph of } G \text{ but at least one of } K_5, C_6, C_4, S_4, P_5 \text{ is}, \\ \rho(\mathcal{D}(P_4)) & \text{otherwise.} \end{cases}$$

Corollary

If T is a tree with $n \geq 5$ vertices, then

$$\mu_6(T) = \begin{cases} 2 + \sqrt{7} & \text{if } T = S_n, \\ 4 & \text{otherwise.} \end{cases}$$

Theorem

If G is a connected graph with at least 5 vertices and $\mu_5(G) = \frac{1+\sqrt{33}}{2}$, then

$$\mu_6(G) = \begin{cases} \frac{3+\sqrt{37}}{2} & \text{if } G = C_5, \\ \gamma & \text{if } G = C_3 * C_3, \\ \frac{3+\sqrt{17}}{2} & \text{if } K_4 - e \text{ is an induced subgraph of } G, \\ 4 & \text{otherwise.} \end{cases}$$

where γ is the largest root of $x^3 - x^2 - 11x - 7 = 0$

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THANK YOU

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Any questions, suggestions?