NORMALIZED LAPLACIAN SPECTRA OF CENTRAL VERTEX JOIN AND CENTRAL EDGE JOIN OF TWO REGULAR GRAPHS

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Contents

Introduction

- In graph spectra, we can find various relations between the spectrum and the structure of a graph.
- In 1931, E.Huckel discussed the idea of spectral graph theory in which the eigenvalues of graphs are used to represent the levels of energy of certain electrons.
- Let G be a bipartite graph. Then the eigenvalues of G are symmetric with respect to the origin.
- The multiplicity of the zero eigenvalue of the Laplacian matrix is equal to the number of connected components of the graph, and the multiplicity of the zero eigenvalue of the signless Laplacian matrix is equal to the number of bipartite connected components of the graph.
- Graph G is connected if and only if the second smallest Laplacian eigenvalue of G (called algebraic connectivity of G) is positive.

INTRODUCTION

- Eigenvalues$\mathscr{L}(G)$ lie in the interval [0, 2].
- Multiplicity of 0 is number of components.
- Multiplicity of 2 is number of bipartite components.
- G is bipartite if and only if for each $\lambda_i(\mathscr{L})$, the value 2- $\lambda_i(\mathscr{L})$ is also an eigenvalue of G.
- The stability of the molecules and other chemically important facts are closely related with the spectrum and the eigenvectors of the corresponding graphs.

Definition 0.1.

A graph G is said to be regular if all of its vertices have the same degree.

Definition 0.2.

The line graph of a graph G is the graph $l(G)$ with the edges of G as its vertices,and where two edges of G are adjacent in $l(G)$ if and only if they are incident in G.

Definition 0.3.

Let G be a simple graph of order p and size q . Then the adjacency matrix $A(G) = [a_{ij}]$ of the graph G is a square matrix of order p whose $(i, j)^{th}$ entry is defined by

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

- A is a real symmetric matrix.
- Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix with vertex degrees.
- Signless Laplacian matrix of G is $Q(G) = A(G) + D(G)$.
- The adjacency matrix of the complement of a graph G is $A(\bar{G}) = J_p - I_p - A(G)$.

Definition 0.4.

The normalised Laplacian matrix $\mathscr{L}(G)=(\mathscr{L}_{ij})$ of G is defined as

$$\mathscr{L}_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ and } \deg(v_i) \neq 0, \\ -\frac{1}{\sqrt{deg(v_i)deg(v_j)}}, & \text{if i} \neq \text{j and } v_i \text{ is adjacent to } v_j \ , \\ 0, & \text{otherwise.} \end{cases}$$

Let G be a graph without isolated vertices, then normalized Laplacian matrix of G is $\mathscr{L}(G) = D^{-\frac{1}{2}}(G)(D(G) - A(G))D^{-\frac{1}{2}}(G) = I_p - D^{-\frac{1}{2}}(G)A(G)D^{-\frac{1}{2}}(G)$

Example

The normalized Laplacian matrix of $K_{3,3}$ is

$$\mathscr{L}(K_{3,3}) = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 1 \end{bmatrix}$$

$$A(K_{3,3}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad L(K_{3,3}) = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}$$

$$Q(K_{3,3}) = \begin{bmatrix} 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 & 0 & 3 \end{bmatrix}$$

- Trace $(A(G))=0$.
- $L(G)$ and $Q(G)$ are real and symmetric and their eigenvalues are real and positive semi-definite , Trace$(L(G)) = 2q$ and Trace $Q(G) = 2q$, where q is the number of edges.
- $\mathscr{L}(G)$ is positive semi- definite and Trace$\mathscr{L}(G) = p$.
- Let G be an r-regular graph. Then $L(G) = r\mathscr{L}(G) = rI - A(G)$
- If G has exactly three distinct $\mathscr{L}$-eigenvalues, then the diameter of G is 2.

The characteristic polynomial of the $p \times p$ matrix M of G is defined as $\phi(M,x) = |xI_p - M|$, where I_p is the identity matrix of order p. The matrices $A(G), L(G)$ and $Q(G)$ are real and symmetric matrices, its eigenvalues are real. The eigenvalues of $A(G), L(G)$ and $Q(G)$ are denoted by $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_p, 0 = \mu_1 \leq \mu_2 \leq ... \leq \mu_p$, $\nu_1 \geq \nu_2 \geq ... \geq \nu_p$ and $0 = \tilde{\mu_1} \leq \tilde{\mu_2} \leq ... \leq \tilde{\mu_p}$ respectively.

- Let G be an r regular graph, then $\mu_i = r - \lambda_i$.
- Let G be an r regular graph, then $\tilde{\mu_i} = 1 - \frac{\lambda_i}{r}, 1 \leq i \leq p$.
- Two regular graphs are $\mathscr{L}$-cospectral if and only if they are cospectral.
- The number of walks of length k in G,from v_i to v_j, is the $(i,j)^{th}$ entry of A^k.
- A connected graph with diameter d has at least $d+1$ distinct eigenvalues.
- $\sum_{i=1}^{p} \lambda_i = 0, \sum_{i=1}^{p} \lambda_i^2 = 2q.$
- $\sum^{p} \nu_i = 2q,$

- $\sum_{i=1}^{p} \mu_i = 2q.$
- $\sum_{i=1}^{p} \tilde{\mu}_i = p.$
- $\sum_{i=1}^{p} \mu_i^2 = 2q + \sum_{i=1}^{p} d_i^2.$

DEFINITION 0.5.

Let G be a (p, q) graph . Then the incidence matrix of a graph G, $I(G)$ is the $p \times q$ matrix whose $(i, j)^{th}$ entry is 1 if v_i is incident to e_j and 0 otherwise.

- $I(G)I(G)^T = A(G) + D(G)$
- $I(G)I(G)^T = A(G) + rI_p$, G is an r-regular graph
- $I(G)^T I(G) = A(l(G)) + 2I_q$,

- Let G be a graph with p vertices. Then the characteristic polynomial of G is $\phi(G, \lambda) = \lambda^p + c_1\lambda^{p-1} + c_2\lambda^{p-2} + c_3\lambda^{p-3} + ... + c_p$. Then $c_k = \Sigma(-1)^{c_1(H)+c(H)}2^{c(H)}$,where $c_1(H)$ and $c(H)$ are the number of components in a subgraph H(with k vertices) of G which are edges and cycles respectively.
- $c_1 = 0$.
- $-c_2$ is the number of edges of G.
- $-c_3$ is twice the number of triangles in G.
- If c_{2k+1}=0,$k = 0, 1, ...,$ then G is bipartite.

- If $\lambda_1, \lambda_2, ..., \lambda_t$ are the distinct eigenvalues of G, then the spectrum of G is $Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{pmatrix}$, where m_j indicates the algebraic multiplicity of the eigenvalue λ_j, $1 \leq j \leq t$ of G.
- A graph G is bipartite if and only if the Laplacian spectrum and the signless Laplacian spectrum of G are equal.
- Two non-isomorphic graphs G and H are said to be cospetral($\mathscr{L}$-cospectral) if $A(G)$ ($\mathscr{L}(G)$) and $A(H)$ ($\mathscr{L}(H)$) have the same spectrum.
- The energy of the graph G is defined as $\varepsilon(G) = \sum_{i=1}^{p} |\lambda_i|$.
- The Laplacian energy of the graph G is defined as $LE(G) = \sum_{i=1}^{p} |\mu_i - \frac{2q}{p}|$.
- The signless Laplacian energy of the graph G is defined as $SLE(G) = \sum_{i=1}^{p} |\nu_i - \frac{2q}{p}|$.

- The Normalized Laplacian energy of the graph G is defined as $NLE(G) = \sum_{i=1}^{p} |\tilde{\mu}_i - 1|.$

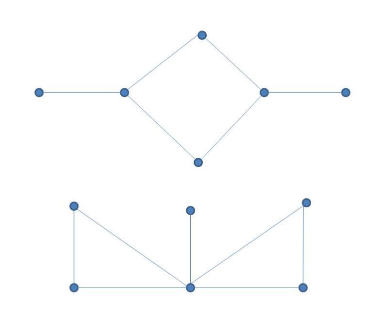

Figure 1: A-cospectral graphs

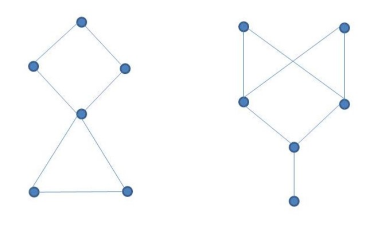

Figure 2: L-cospectral graphs

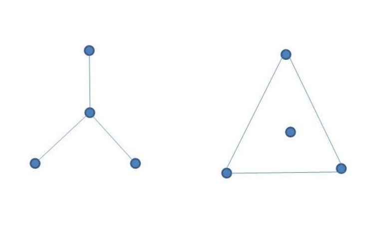

Figure 3: Q-cospectral graphs

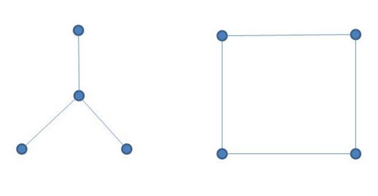

Figure 4: $\mathscr{L}$-cospectral graphs

- The Kemeny's constant $K(G)$ of a graph G is defined as $K(G) = \sum_{i=2}^{p} \frac{1}{\tilde{\mu}_i(G)}$
- The degree Kirchhoff index of G is defined as $Kf^*(G) = 2q\sum_{i=2}^{p} \frac{1}{\tilde{\mu}_i(G)} = 2qK(G)$
- Two graphs G_1 and G_2 are said to be equienergetic if $\varepsilon(G_1) = \varepsilon(G_2)$.

Definition 0.6.

The subdivision graph of a graph G is obtained by inserting new vertices between every edges of G. It is denoted by $S(G)$.
The adjacency matrix of $S(G)$ is

$$A(S(G)) = \begin{bmatrix} O_{p\times p} & I(G) \\ (I(G))^T & O_{q\times q} \end{bmatrix}.$$

Definition 0.7.

Let G be a simple graph with p vertices and q edges. The central graph of G, denoted by $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the nonadjacent vertices in G.

Let $\tilde{V}(G) = V(C(G)) - V(G)$, be the set of vertices in $C(G)$ corresponding to the edges of G.

The number of vertices and edges in $C(G)$ are $p + q$ and $q + \frac{p(p-1)}{2}$ respectively.

The adjacency matrix of $C(G)$ can be written as

$$A(C(G)) = \begin{bmatrix} A(\bar{G}) & I(G) \\ I(G)^T & O_{m\times m} \end{bmatrix}.$$

Example 0.8.

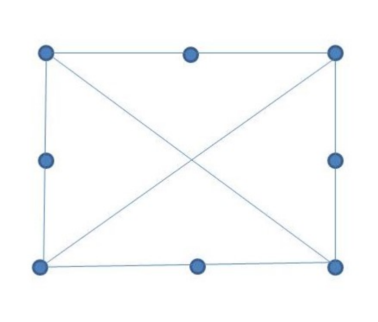

Figure 5: $C(K_{2,2})$

$\overline{A(K(2,2))}$

$$A(C(K_{2,2)})=\begin{bmatrix}0&0&1&0&1&0&0&1\\0&0&0&1&1&1&0&0\\1&0&0&0&0&1&1&0\\0&1&0&0&0&0&1&1\\1&1&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&1&1&0&0&0&0\\1&0&0&1&0&0&0&0\end{bmatrix}$$

$I(K(2,2))$

$O_{4\times 4}$

$[I(K(2,2))]^T$

Definition 0.9.

The Kronecker product of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and the vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if (x_1, y_1) and (x_2, y_2) are edges in G_1 and G_2 respectively.

Definition 0.10.

Let A and B be two matrices of same order. Then the Hadamard product $A \circ B$ of A and B is a matrix with same order and entries are given by $(A \circ B)_{ij} = (A)_{ij}.(B)_{ij}$ (that is entry wise multiplication).

Definition 0.11.

The join of two graphs G_1 and G_2, $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2.

Proposition 0.12.

If G_1 is an r_1-regular graph with n_1 vertices and G_2 is an r_2-regular graph with n_2 vertices, then the characteristic polynomial of $G_1 \vee G_2$ is given by

$$\phi(G_1 \vee G_2, x) = \frac{\phi(G_1, x)\phi(G_2, x)}{(x - r_1)(x - r_2)}[(x - r_1)(x - r_2) - n_1 n_2].$$

Lemma 0.13.

Let U, V, W and X be matrices with U invertible. Let

$$S = \begin{pmatrix} U & V \\ W & X \end{pmatrix}.$$

Then $det(S) = det(U)det(X - WU^{-1}V)$.
If X is invertible, then $det(S) = det(X)det(U - VX^{-1}W)$.
If U and W are commutes, then $det(S) = det(UX - WV)$.

Lemma 0.14.

Let G be a connected r-regular graph on p vertices with an adjacency matrix A having t distinct eigenvalues $r = \lambda_1, \lambda_2, ..., \lambda_t$. Then there exists a polynomial

$$P(x) = p\frac{(x-\lambda_2)(x-\lambda_3)...(x-\lambda_t)}{(r-\lambda_2)(r-\lambda_3)...(r-\lambda_t)}.$$

such that $P(A) = J_p, P(r) = p$ and $P(\lambda_i) = 0$ for $\lambda_i \neq r$.

Definition 0.15.

The M-coronal $\chi_M(x)$ of $n \times n$ matrix M is defined as the sum of the entries of the matrix $(xI_n - M)^{-1}$ (if exists), that is,

$$\chi_M(x) = J_{n\times 1}^T (xI_n - M)^{-1} J_{n\times 1}.$$

$\chi_M(x,\alpha) = J_{n\times 1}^T (xI_n - M \circ (\alpha J_n + (1-\alpha)I_n))^{-1} J_{n\times 1}$, if $\alpha = 1$, then $\chi_M(x,1)$ is the usual M-coronal $\chi_M(x)$.

Lemma 0.16.

Let G be an r-regular graph on p vertices, then $\chi_{\mathscr{L}(G)}(x,\alpha) = \frac{p}{x+\alpha-1}$.

For any real number k

$\chi_{k\mathscr{L}(G)}(x) = \chi_{\mathscr{L}(G)}(x)$

Lemma 0.17.

Let A be an $n \times n$ real matrix. Then $det(A + \alpha J_n) = det(A) + \alpha J_{n\times 1}^T adj(A) J_{n\times 1}$, where α is a real number and $adj(A)$ is the adjoint of A .

Corollary 0.18.

Let A be an $n \times n$ real matrix. Then

$$det(xI_n - A - \alpha J_n) = (1 - \alpha\chi_A(x))det(xI_n - A).$$

Lemma 0.19.

For any real numbers $a, b > 0, (aI_n - bJ_n)^{-1} = \frac{1}{a}I_n + \frac{b}{a(a-nb)}J_n$.

Proposition 0.20.

Let $A, B \in R^{n \times n}$. Let λ be an eigenvalue of matrix A with eigenvector x and μ be an eigenvalue of matrix B with eigenvector y, then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with eigenvector $x \otimes y$.

Theorem 0.21.

Let G be an r-regular graph with n vertices. Then the normalized Laplacian characteristic polynomial of central graph of G is

$$f(\mathscr{L}(C(G)),x) = (x-1)^{m-n}\prod_{i=1}^{n}\left((x-1)\left((x-1)-\frac{P(\lambda_i)-1-\lambda_i}{n-1}\right)-\frac{(\lambda_i+r)}{2(n-1)}\right).$$

Proof

Corollary 0.22.

Let G be an r-regular graph on n vertices. Then the normalized Laplacian spectrum of $C(G)$ consists of

1. 1 *repeated* $m-n$ *times* .
2. *Two roots of the equation* $(x-1)\left((x-1)+\frac{n-1-r}{n-1}\right)-\frac{r}{(n-1)}=0.$
3. *Two roots of the equation* $(x-1)\left((x-1)+\frac{-1-\lambda_i}{n-1}\right)-\frac{(\lambda_i+r)}{2(n-1)}=0$ for $i=2,...,n.$

Example 0.23.

Let $G = K_{p,p}$.Then the adjacency eigenvalues of G are 0 (multiplicity $2p-2$) and $\pm p$. Therefore, the normalised eigenvalues of $C(G)$ are $0, \frac{3p-1}{2p-1}, 1$ (repeated $p^2 - 2p$ times), roots of the equation $(2p-1)x^2 + (1-3p)x + p = 0$ and roots of the equation $(4p-2)x^2 - (8p-2)x + 3p = 0$ (each root repeated $2p-2$ times).

Definition 0.24.

Let G_1 and G_2 be any two graphs on n_1, n_2 vertices and m_1, m_2 edges respectively. The central vertex join of G_1 and G_2 is the graph $G_1 \dot{\vee} G_2$, is obtained from $C(G_1)$ and G_2 by joining each vertex of G_1 with every vertex of G_2.
Note that the central vertex join $G_1 \dot{\vee} G_2$ has $m_1 + n_1 + n_2$ vertices and $m_1 + m_2 + n_1 n_2 + \frac{n_1(n_1-1)}{2}$ edges.

Definition 0.25.

Let G_i be any two graphs with n_i vertices and m_i edges respectively for $i = 1, 2$. Then the central edge join of two graphs G_1 and G_2 is the graph $G_1 \veebar G_2$ is obtained from $C(G_1)$ and G_2 by joining each vertex corresponding to edges of G_1 with every vertex of G_2.
Note that the central edge join $G_1 \veebar G_2$ has $m_1 + n_1 + n_2$ vertices and $m_1 + m_2 + m_1 n_2 + \frac{n_1(n_1-1)}{2}$ edges.

Example 0.26.

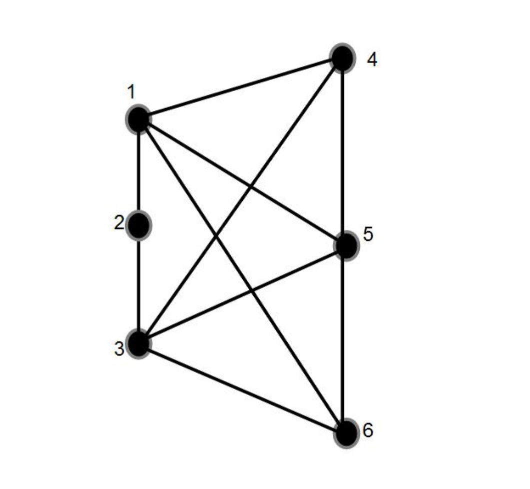

Figure 6: $P_2\dot{\vee}P_3$

Example 0.27.

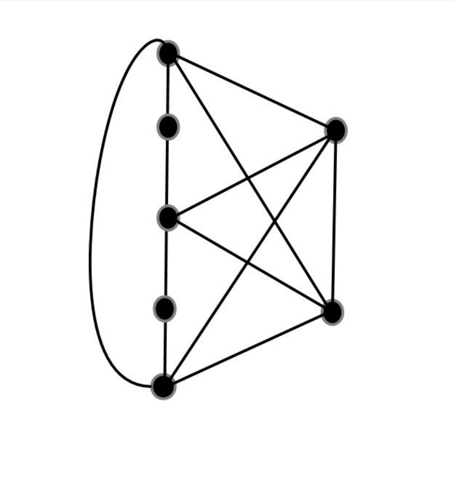

Figure 7: $P_3 \dot{\vee} P_2$

Example 0.28.

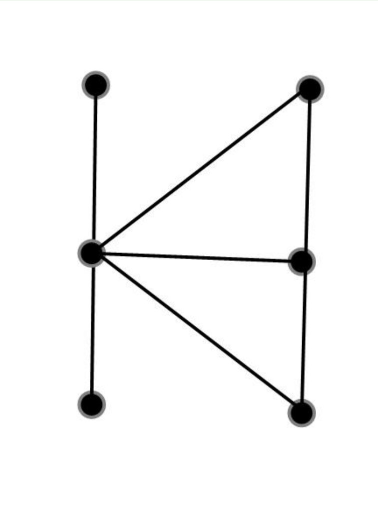

Figure 8: $P_2 \veebar P_3$

Example 0.29.

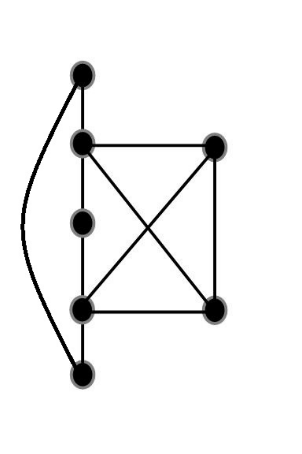

Figure 9: $P_3 \veebar P_2$

$$A(P_2 \dot{\vee} P_3) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad A(P_3 \dot{\vee} P_2) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A(P_2 \veebar P_3) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad A(P_3 \veebar P_2) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Theorem 0.30.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for i=1,2. Then the normalized Laplacian characteristic polynomial of $G_1\dot{\vee}G_2$ is given by

$$
\begin{aligned}
&f(\mathscr{L}(G_1\dot{\vee}G_2),x)\\
&=(x-1)^{m_1-n_1}\prod_{i=2}^{n_2}\left((x-\frac{n_1}{r_2+n_1})I_{n_2}-\frac{r_2}{r_2+n_1}\tilde{\mu}_i(G_2)\right)\\
&\quad\times\Bigg[\left(x-\frac{n_1}{r_2+n_1}\right)\left((x-1)^2-\frac{r_1}{(n_1+n_2-1)}-\frac{(1+r_1)(x-1)}{n_1+n_2-1}\right)\\
&\qquad\qquad-\frac{n_1n_2(x-1)-n_1(x-1)(xn_1+xr_2-n_1)}{(n_1+n_2-1)(n_1+r_2)}\Bigg]\\
&\quad\times\prod_{i=2}^{n_1}\left((x-1)^2-\frac{r_1}{(n_1+n_2-1)}-\frac{(1+r_1)(x-1)}{n_1+n_2-1}+\frac{r_1\tilde{\mu}_i(G_1)}{2(n_1+n_2-1)}+\frac{r_1(x-1)\tilde{\mu}_i(\cdots}{n_1+n_2-\cdots}\right.
\end{aligned}
$$

Corollary 0.31.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for i=1,2. Then the normalized Laplacian spectrum of $G_1\dot{\vee}G_2$ consists of

- 1 *repeated* $m_1 - n_1$ *times.*
- *Roots of the equation,*
$$(x - \frac{n_1}{r_2+n_1}) - \frac{r_2}{r_2+n_1}\tilde{\mu}_i(G_2) = 0 \text{ for } i = 2, ..., n_2.$$
- *Three roots of the equation,*
$$\left(x - \frac{n_1}{r_2+n_1}\right)\left((x-1)^2 - \frac{r_1}{n_1+n_2-1} - \frac{(1+r_1)(x-1)}{n_1+n_2-1}\right) - \left(\frac{n_1n_2(x-1) - n_1(x-1)(xn_1+xr_2-n_1)}{(n_1+n_2-1)(n_1+r_2)}\right) = 0.$$
- *Two roots of the equation,*
$$(x-1)^2 - \frac{r_1}{(n_1+n_2-1)} - \frac{(1+r_1)(x-1)}{n_1+n_2-1} + \frac{r_1\tilde{\mu}_i(G_1)}{2(n_1+n_2-1)} + \frac{r_1(x-1)\tilde{\mu}_i(G_1)}{n_1+n_2-1} = 0 \text{ for } i =$$

Example 0.32.

Let $G_1 = K_{p,p}$ and $G_2 = K_2$.
Normalised Laplacian eigenvalues of G_1 are 0,1(repeated $2p-2$) and 2.
Normalised Laplacian eigenvalues of G_2 are 0 and 2.
Normalised Laplacian eigenvalues of $G_1 \dot{\vee} G_2$ are $0, \frac{2p+2}{2p+1}$,1 (repeated $p^2 - 2p$ times),
Roots of the equation $(4p^2 + 4p + 1)x^2 - (10p^2 + 11p + 3)x + (6p^2 + 6p + 2) = 0$,
Roots of the equation $(4p + 2)x^2 - (6p + 6)x + 2p + 4 = 0$ and roots of the equation $(4p + 2)x^2 - (8p + 6)x + 3p + 4 = 0$(each root repeated $2p - 2$ times).

Theorem 0.33.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for i=1,2. Then the normalized Laplacian characteristic polynomial of $G_1 \veebar G_2$ is given by

$$f(\mathscr{L}(G_1 \veebar G_2), x) = (x-1)^{m_1-n_1-1} \prod_{i=2}^{n_2} \left((x - \frac{m_1}{r_2+m_1}) I_{n_2} - \frac{r_2}{r_2+m_1} \tilde{\mu}_i(G_2) \right)$$

$$\prod_{i=2}^{n_1} \left[(x-1)^2 + \frac{(-1-r_1+r_1\tilde{\mu}_i(G_1))(x-1)}{n_1-1} - \frac{(2r_1 - r_1\tilde{\mu}_i(G_1))}{(n_1-1)(n_2+2)} \right]$$

$$\times \left[\left((x-1)^2 + \frac{(n_1-1-r_1)(x-1)}{n_1-1} - \frac{2r_1}{(n_1-1)(n_2+2)} \right) \right.$$

$$\left. \left((x - \frac{m_1}{r_2+m_1})(x-1) - \frac{n_2 m_1}{(n_2+2)(r_2+m_1)} \right) - \frac{n_2 r_1^2 n_1}{(n_1-1)(n_2+2)^2(r_2+m_1)} \right].$$

Proof

Corollary 0.34.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for i=1,2. Then the normalized Laplacian spectrum of $G_1 \underline{\vee} G_2$ consists of

- 1 *repeated* $m_1 - n_1 - 1$ *times.*
- *Roots of the equation,*
$$(x - \frac{m_1}{r_2 + m_1}) - \frac{r_2}{r_2 + m_1}\tilde{\mu}_i(G_2) = 0 \text{ for } i = 2, ..., n_2.$$
- *Four roots of the equation,*
$$\left((x-1)^2 + \frac{(n_1 - 1 - r_1)(x-1)}{n_1 - 1} - \frac{2r_1}{(n_1 - 1)(n_2 + 2)}\right)$$
$$\left((x - \frac{m_1}{r_2 + m_1})(x-1) - \frac{n_2 m_1}{(n_2 + 2)(r_2 + m_1)}\right) - \frac{n_2 r_1^2 n_1}{(n_1 - 1)(n_2 + 2)^2 (r_2 + m_1)} = 0$$
- *Two roots of the equation,*
$$(x-1)^2 + \frac{(-1 - r_1 + r_1\tilde{\mu}_i(G_1))\,(x-1)}{n_1 - 1} - \frac{(2r_1 - r_1\tilde{\mu}_i(G_1))}{(n_1 - 1)(n_2 + 2)} = 0 \text{ for } i = 2, ..., n_1.$$

Example 0.35.

Let $G_1 = K_{p,p}$ and $G_2 = K_2$.
Normalised Laplacian eigenvalues of G_1 are 0,1(repeated $2p-2$) and 2.
Normalised Laplacian eigenvalues of G_2 are 0 and 2.
Normalised Laplacian eigenvalues of $G_1 \underline{\vee} G_2$ are 0, $\frac{p^2+2}{p^2+1}$,1 (repeated p^2-2p-1 times), roots of the equation $(32p^3-16p^2+32p-16)x^3+(-112p^3+48p^2-80p+32)x^2+(120p^3-40p^2+56p-16)x+8p^2-40p^3-8p=0$, roots of the equation $(2p-1)x^2-(3p-1)x+p=0$ and roots of the equation $(8p-4)x^2-(16p-4)x+7p=0$(each root repeated $2p-2$ times).

Theorem 0.36.

Let G_1 and G_2 (not necessarily distinct) be $\mathscr{L}$-cospectral regular graphs, H_1 and H_2 (not necessarily distinct) are another $\mathscr{L}$-cospectral regular graphs. Then $\mathscr{L}(G_1\dot{\vee}H_1)$ and $\mathscr{L}(G_2\dot{\vee}H_2)$ (respectively, $\mathscr{L}(G_1 \veebar H_1)$ and $\mathscr{L}(G_2 \veebar H_2)$) are $\mathscr{L}$-cospectral non-regular graphs. Proof

Example 0.37.

Consider two regular $\mathscr{L}$-cospectral graphs H and$F$(see Figs.6,7).
Graphs $C(H)$ and $C(F)$ are shown in Figs.8,9.
Graphs $(H\dot{\vee}K_2)$ and $(F\dot{\vee}K_2)$ are non-isomorphic.
Then, $(H\dot{\vee}K_2)$ and $(F\dot{\vee}K_2)$ are non-regular $\mathscr{L}$-cospectral graphs.

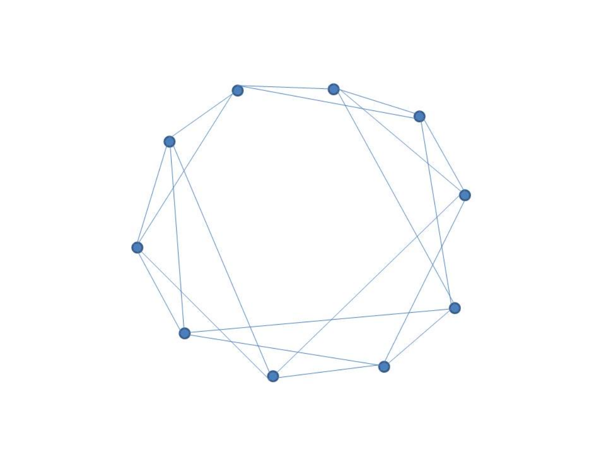

Figure 10: H

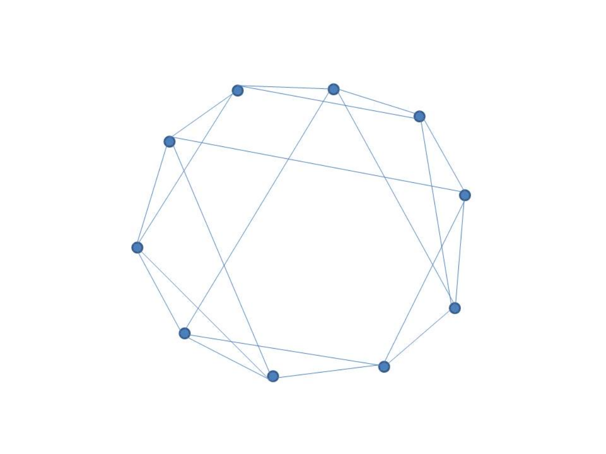

Figure 11: F

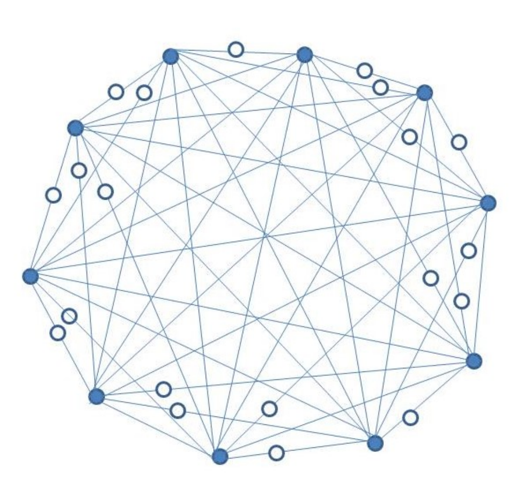

Figure 12: $C(H)$

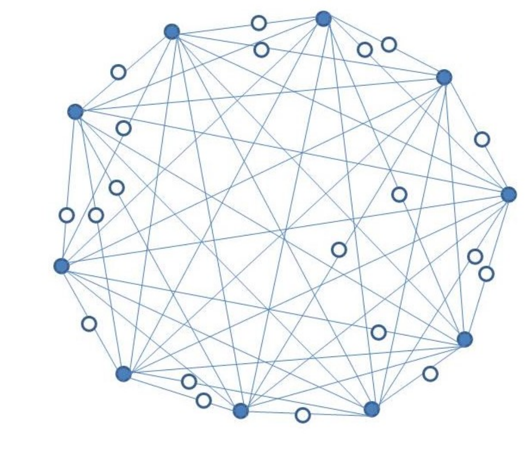

Figure 13: $C(F)$

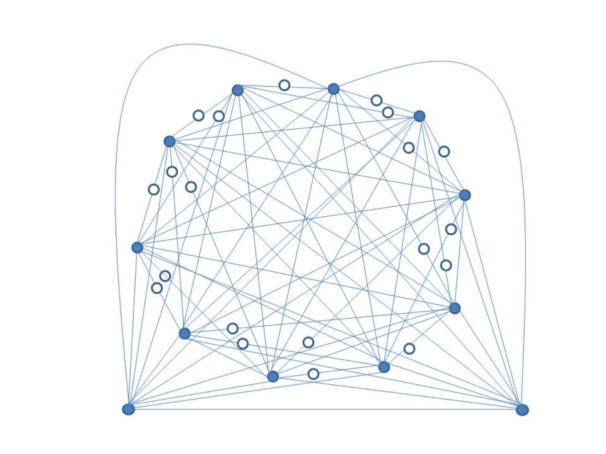

Figure 14: $H\dot{\vee}K_2$

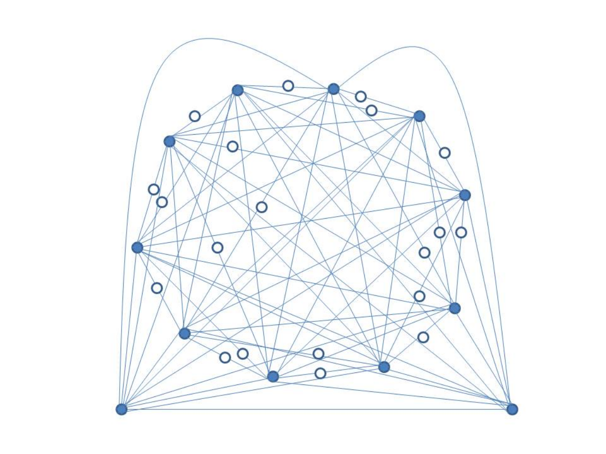

Figure 15: $F\dot{\vee}K_2$

Theorem 0.38.

Let G be an r-regular graph of order n and size m. Then

$$K(C(G)) = \left[m - n + \frac{n-1}{n-1+r} + \sum_{i=2}^{n} \frac{2(2n+\lambda_i - 1)}{2n+\lambda_i - r}\right].$$

Proof

Theorem 0.39.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for i=1,2. Then

$$K(G_1\dot{\vee}G_2) = \Bigg[m_1 - n_1 + \frac{n_1(2n_1+3n_2+r_1+r_2)+2n_2r_2-2n_1-r_2-r_1r_2}{n_1(n_1+2n_2-2)+n_2r_2} + \sum_{i=2}^{n_2}\frac{r_2+n_1}{n_1+r_2\tilde{\mu}_i(G_2)} + \sum_{i=2}^{n_1}\frac{2(-r_1\tilde{\mu}_i(G_1)+2n_1+2n_2+r_1-1)}{2n_1+2n_2-r_1\tilde{\mu}_i(G_1)}\Bigg].$$

Proof

Theorem 0.40.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for i=1,2. Then

$$K(G_1 \underline{\vee} G_2) = \Big[m_1 - n_1 - 1 + \frac{r_2(4n_1n_2 - n_2^2 + 6n_2r_1 + 4n_1 - 4n_2 + 4r_1 - 4)}{m_1(2n_2^2r_1 + 2n_1n_2 + 6n_2r_1 + 4n_1 + 2n_2) + 2n_2r_1 + n_2^2r}$$

$$\frac{n_2^2(n_1r_2 + 2n_1m_1 + 2r_1r_2 + 3m_1r_1) + m_1(10n_1n_2 - 2n_2^2 + 10n_2r_1 + 12n_1 - 10n_2 + 8r}{m_1(2n_2^2r_1 + 2n_1n_2 + 6n_2r_1 + 4n_1 + 2n_2) + 2n_2r_1 + n_2^2r_1r_2}$$

$$\sum_{i=2}^{n_2} \frac{r_2 + m_1}{m_1 + r_2\tilde{\mu}_i(G_2)} + \sum_{i=2}^{n_1} \frac{-n_2r_1\tilde{\mu}_i(G_1) - 2r_1\tilde{\mu}_i(G_1) + 2n_1n_2 + n_2r_1 + 4n_1 - n_2 + 2r_1}{n_1n_2 + n_2r_1 + 2n_1 - n_2r_1\tilde{\mu}_i(G_1) - 3r_1\tilde{\mu}_i(G_1)}$$

Proof

References

[1] R.B Bapat. Graphs and Matrices, Hindustan Book Agency,2010.

[2] D.M. Cvetković, M. Doob, and H. Sachs. Spectra of graphs: Theory and application. Academic press, 1980.

[3] D Cvetkovic, Peter Rowlinson, and S Simic.An Introduction to the Theory of Graph Spectra Cambridge University Press, 2001.

[4] A.Das and P.Panigrahi, Spectra of R-vertex join and R-edge join of two graphs. Discussiones Mathematicae-General Algebra and Applications. 38(1):19-32,2018.

[5] Xiaogang Liu and Zuhe Zhang. Spectra of subdivision-vertex join and subdivision-edge join of two graphs.Bulletin of the Malaysian Mathematical Sciences Society, 42(1):15–31, 2019.

[6] J Vernold Vivin,MM Akbar Ali and K Thilagavathi. On harmonious coloring of central graphs.Advances and applications in discrete mathematics, 2(1):17-33, 2008.

[7] Cam McLeman and Erin McNicholas. Spectra of coronae Linear algebra and its applications ,435(5):998-1007, 2011.

References

[8] TK Jahfar and AV Chithra. Central vertex join and central edge join of two graphs. AIMS Mathematics, 5(6):7214-7233, 2020.

[9] Haiyan Chen and Fuji Zhang. Resistance distance and normalized Laplacian spectrum. Discrete applied mathematics, 155(5):654-661,2007.

Thank You

Proof

Let $I(G)$ be the incidence matrix of G and $A(\bar{G})$ be the adjacency matrix of complement of graph G. Then the adjacency matrix of $C(G)$ is

$$A(C(G)) = \begin{pmatrix} A(\bar{G}) & I(G) \\ I(G)^T & O_{m\times m} \end{pmatrix}.$$

The degree matrix of $C(G)$ is

$$D(C(G)) = \begin{pmatrix} (n-1)I_n & O_{n\times m} \\ O_{m\times n} & 2I_m \end{pmatrix}.$$

Then the normalized Laplacian matrix of $C(G)$ is

$$\mathscr{L}(C(G)) = I_{m+n} - D^{-\frac{1}{2}}(C(G))A(C(G))D^{-\frac{1}{2}}(C(G)) = \begin{pmatrix} I_n - \frac{A(\bar{G})}{n-1} & -\frac{I(G)}{\sqrt{2(n-1)}} \\ -\frac{I(G)^T}{\sqrt{2(n-1)}} & I_m \end{pmatrix}.$$

Using Lemmas **??** and **??**, we have the normalized Laplacian characteristic polynomial of $C(G)$ is

$$
\begin{aligned}
f(\mathscr{L}(C(G)),x) =& det \begin{pmatrix} (x-1)I_n + \frac{J_n - I_n - A(G)}{n-1} & \frac{I(G)}{\sqrt{2(n-1)}} \\ \frac{I(G)^T}{\sqrt{2(n-1)}} & (x-1)I_m \end{pmatrix} \\
=&(x-1)^m det\left((x-1)I_n + \frac{J_n - I_n - A(G)}{n-1} - \frac{I(G)I(G)^T}{2(x-1)(n-1)}\right) \\
=&(x-1)^{m-n} det\left((x-1)\left((x-1)I_n + \frac{J_n - I_n - A(G)}{n-1}\right) - \frac{I(G)I(G)^T}{2(n-1)}\right) \\
=&(x-1)^{m-n} det\left((x-1)\left((x-1)I_n + \frac{J_n - I_n - A(G)}{n-1}\right) - \frac{(A(G)+rI_n)}{2(n-1)}\right) \\
f(\mathscr{L}(C(G)),x) =&(x-1)^{m-n} \prod_{i=1}^{n} \left((x-1)\left((x-1) + \frac{P(\lambda_i) - 1 - \lambda_i}{n-1}\right) - \frac{(\lambda_i + r)}{2(n-1)}\right).
\end{aligned}
$$

Back

Let $I(G_1)$ be the incidence matrix of G_1. Then by a proper labeling of vertices, the adjacency matrix of $G_1\dot{\vee}G_2$ can be written as

$$A(G_1\dot{\vee}G_2) = \begin{pmatrix} A(\bar{G_1}) & I(G_1) & J_{n_1\times n_2} \\ I(G_1)^T & O_{m_1\times m_1} & O_{m_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2) \end{pmatrix}.$$

The degree matrix of $G_1\dot{\vee}G_2$ is

$$D(G_1\dot{\vee}G_2) = \begin{pmatrix} (n_1+n_2-1)I_{n_1} & O_{n_1\times m_1} & O_{n_1\times n_2} \\ O_{m_1\times n_1} & 2I_{m_1} & O_{m_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & (r_2+n_1)I_{n_2} \end{pmatrix}.$$

$$D^{-\frac{1}{2}}(G_1\dot{\vee}G_2) = \begin{pmatrix} \frac{I_{n_1}}{\sqrt{(n_1+n_2-1)}} & O_{n_1\times m_1} & O_{n_1\times n_2} \\ O_{m_1\times n_1} & \frac{I_{m_1}}{\sqrt{2}} & O_{m_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & \frac{I_{n_2}}{\sqrt{(r_2+n_1)}} \end{pmatrix}.$$

Then the normalized Laplacian matrix of $G_1\dot{\vee}G_2$ is

$$\mathscr{L}(G_1\dot{\vee}G_2) = \begin{pmatrix} I_{n_1} - \frac{A(\bar{G_1})}{n_1+n_2-1} & -\frac{I(G_1)}{\sqrt{2(n_1+n_2-1)}} & -\frac{J_{n_1\times n_2}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} \\ -\frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} & I_{m_1} & O_{m_1\times n_2} \\ -\frac{J_{n_2\times n_1}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} & O_{n_2\times m_1} & I_{n_2} - \frac{A(G_2)}{(n_1+r_2)} \end{pmatrix}$$

$$\mathscr{L}(G_1\dot{\vee}G_2)=\begin{pmatrix} I_{n_1}-\frac{A(\bar{G_1})}{n_1+n_2-1} & -\frac{I(G_1)}{\sqrt{2(n_1+n_2-1)}} & -\frac{J_{n_1\times n_2}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} \\ -\frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} & I_{m_1} & O_{m_1\times n_2} \\ -\frac{J_{n_2\times n_1}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} & O_{n_2\times m_1} & \mathscr{L}(G_2)\circ B \end{pmatrix},$$

where $B=\frac{r_2}{r_2+n_1}J_{n_2}+\frac{n_1}{r_2+n_1}I_{n_2}$.
Using Definition **??**, Lemmas **??**,**??**, **??** and Corollary **??**, we have the normalized Laplacian characteristic polynomial of $G_1\dot{\vee}G_2$ is

$$f(\mathscr{L}(G_1\dot{\vee}G_2),x) = det\begin{pmatrix} (x-1)I_{n_1}+\frac{A(\bar{G_1})}{n_1+n_2-1} & \frac{I(G_1)}{\sqrt{2(n_1+n_2-1)}} & \frac{J_{n_1\times n_2}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} \\ \frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} & (x-1)I_{m_1} & O_{m_1\times n_2} \\ \frac{J_{n_2\times n_1}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} & O_{n_2\times m_1} & xI_{n_2}-(\mathscr{L}(G_2)\circ B) \end{pmatrix}$$

$$= \det\Big(xI_{n_2}-(\mathscr{L}(G_2)\circ B)\Big)\det S,$$

$$\text{where S} = \begin{pmatrix} (x-1)I_{n_1}+\frac{A(\bar{G_1})}{n_1+n_2-1} & \frac{I(G_1)}{\sqrt{2(n_1+n_2-1)}} \\ \frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} & (x-1)I_{m_1} \end{pmatrix} - \Bigg[\begin{pmatrix} \frac{J_{n_1\times n_2}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} \\ O_{m_1\times n_2} \end{pmatrix}(xI_{n_2}-($$

$$\times\begin{pmatrix} \frac{J_{n_2\times n_1}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} & O_{n_2\times m_1} \end{pmatrix}\Bigg]$$

$$= \begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{G_1})}{n_1+n_2-1} - \frac{J_{n_1 \times n_2}}{\sqrt{(n_1+n_2-1)(n_1+r_2)}} \left(xI_{n_2} - (\mathscr{L}(G_2) \circ B)\right)^{-1} \frac{J_{n_2 \times n_1}}{\sqrt{(n_1+n_2-1)(n_1+ \\ \frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} \end{pmatrix}$$

$$= \begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{G_1})}{n_1+n_2-1} - \frac{\chi_{\mathscr{L}(G_2)}(x, \frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)} J_{n_1} & \frac{I(G_1)}{\sqrt{2(n_1+n_2-1)}} \\ \frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} & (x-1)I_{m_1} \end{pmatrix}.$$

Therefore,

$$det(S) = det\begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{G_1})}{n_1+n_2-1} - \frac{\chi_{\mathscr{L}(G_2)}(x, \frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)}J_{n_1} & \frac{I(G_1)}{\sqrt{2(n_1+n_2-1)}} \\ \frac{I(G_1)^T}{\sqrt{2(n_1+n_2-1)}} & (x-1)I_{m_1} \end{pmatrix}$$

$$=(x-1)^{m_1} det\left((x-1)I_{n_1} + \frac{A(\bar{G_1})}{n_1+n_2-1} - \frac{\chi_{\mathscr{L}(G_2)}(x, \frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)}J_{n_1} - \frac{I(G_1)}{2(n_1+n_2}\right.$$

$$=(x-1)^{m_1} det\Big((x-1)I_{n_1} + \frac{A(\bar{G_1})}{n_1+n_2-1} - \frac{\chi_{\mathscr{L}(G_2)}(x, \frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)}J_{n_1}$$

$$- \frac{r_1}{(n_1+n_2-1)(x-1)}I_{n_1} + \frac{r_1\mathscr{L}(G_1)}{2(n_1+n_2-1)(x-1)}\Big)$$

$$=(x-1)^{m_1} det\Bigg((x-1)I_{n_1}+\frac{J_{n_1}-I_{n_1}-A(G_1)}{n_1+n_2-1}-\frac{\chi_{\mathscr{L}(G_2)}(x,\frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)}J_{n_1}$$

$$-\frac{r_1}{(n_1+n_2-1)(x-1)}I_{n_1}+\frac{r_1\mathscr{L}(G_1)}{2(n_1+n_2-1)(x-1)}\Bigg)$$

$$=(x-1)^{m_1} det\Bigg((x-1)I_{n_1}+\frac{J_{n_1}-I_{n_1}-r_1I_{n_1}+r_1\mathscr{L}(G_1)}{n_1+n_2-1}-\frac{\chi_{\mathscr{L}(G_2)}(x,\frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)}J$$

$$-\frac{r_1}{(n_1+n_2-1)(x-1)}I_{n_1}+\frac{r_1\mathscr{L}(G_1)}{2(n_1+n_2-1)(x-1)}\Bigg)$$

$$=(x-1)^{m_1}\Bigg(1-\Bigg(\frac{\chi_{\mathscr{L}(G_2)}(x,\frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)}-\frac{1}{n_1+n_2-1}\Bigg)$$

$$\chi_{k\mathscr{L}(G_1)}\left(x-1-\frac{r_1}{(n_1+n_2-1)(x-1)}-\frac{1+r_1}{n_1+n_2-1}\right)\Bigg)$$

$$\times det\Bigg(\Bigg(x-1-\frac{r_1}{(n_1+n_2-1)(x-1)}-\frac{1+r_1}{n_1+n_2-1}\Bigg)I_{n_1}+\frac{r_1\mathscr{L}(G_1)}{2(n_1+n_2-1)(x-1)}+$$

It is easy to see that

$$det\left(xI_{n_2} - (\mathscr{L}(G_2) \circ B)\right) = det\left((x - \frac{n_1}{r_2 + n_1})I_{n_2} - \frac{r_2}{r_2 + n_1}\mathscr{L}(G_2)\right).$$

$$\chi_{\mathscr{L}(G_2)}\left(x, \frac{r_2}{r_2 + n_1}\right) = \frac{n_2}{x - 1 + \frac{r_2}{r_2+n_1}}.$$

$$\chi_{k\mathscr{L}(G_1)}\left(x - 1 - \frac{r_1}{(n_1 + n_2 - 1)(x - 1)} - \frac{1 + r_1}{n_1 + n_2 - 1}\right) = \frac{n_1}{x - 1 - \frac{r_1}{(n_1+n_2-1)(x-1)} - \frac{}{n}}$$

Therefore,

$$det(S) = (x-1)^{m_1-n_1}\left(1-\left(\frac{\chi_{\mathscr{L}(G_2)}(x, \frac{r_2}{r_2+n_1})}{(n_1+n_2-1)(n_1+r_2)} - \frac{1}{n_1+n_2-1}\right)\right.$$

$$\left.\times\left(\frac{n_1}{x-1-\frac{r_1}{(n_1+n_2-1)(x-1)}-\frac{1+r_1}{n_1+n_2-1}}\right)\right)$$

$$\times\, det\left(\left((x-1)^2-\frac{r_1}{(n_1+n_2-1)}-\frac{(1+r_1)(x-1)}{n_1+n_2-1}\right)I_{n_1}+\frac{r_1\mathscr{L}(G_1)}{2(n_1+n_2-1)}+\frac{r_1(x}{n_1}\right.$$

$$f(\mathscr{L}(G_1\dot{\vee}G_2),x) = \det\left(xI_{n_2}-(\mathscr{L}(G_2)\circ B)\right)\det S$$

$$= det\left((x-\frac{n_1}{r_2+n_1})I_{n_2}-\frac{r_2}{r_2+n_1}\mathscr{L}(G_2)\right)$$

$$\times\,(x-1)^{m_1-n_1}\left[1-\left(\frac{\frac{n_2}{x-1+\frac{r_2}{r_2+n_1}}}{(n_1+n_2-1)(n_1+r_2)}-\frac{1}{n_1+n_2-1}\right)\right.$$

$$\left(\quad n_1 \quad\right)$$

$$
\begin{aligned}
=&(x-1)^{m_1-n_1}\prod_{i=2}^{n_2}\left((x-\frac{n_1}{r_2+n_1})I_{n_2}-\frac{r_2}{r_2+n_1}\tilde{\mu}_i(G_2)\right)\\
&\times\Bigg[\left(x-\frac{n_1}{r_2+n_1}\right)\left((x-1)^2-\frac{r_1}{(n_1+n_2-1)}-\frac{(1+r_1)(x-1)}{n_1+n_2-1}\right)\\
&\qquad-\frac{n_1n_2(x-1)-n_1(x-1)(xn_1+xr_2-n_1)}{(n_1+n_2-1)(n_1+r_2)}\Bigg]\\
&\prod_{i=2}^{n_1}\left((x-1)^2-\frac{r_1}{(n_1+n_2-1)}-\frac{(1+r_1)(x-1)}{n_1+n_2-1}+\frac{r_1\tilde{\mu}_i(G_1)}{2(n_1+n_2-1)}+\frac{r_1(x-1)\tilde{\mu}_i(G}{n_1+n_2-1}\right.
\end{aligned}
$$

Back

Let $I(H_1)$ be the incidence matrix of H_1. Then by a proper labeling of vertices, adjacency matrix of $H_1 \veebar H_2$ can be written as

$$A(H_1 \veebar H_2) = \begin{pmatrix} A(\bar{H}_1) & I(H_1) & O_{n_1 \times n_2} \\ I(H_1)^T & O_{m_1 \times m_1} & J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & J_{n_2 \times m_1} & A(H_2) \end{pmatrix}.$$

The degree matrix of $H_1 \veebar H_2$ is

$$D(H_1 \veebar H_2) = \begin{pmatrix} (n_1 - 1)I_{n_1} & O_{n_1 \times m_1} & O_{n_1 \times n_2} \\ O_{m_1 \times n_1} & (n_2 + 2)I_{m_1} & O_{m_1 \times n_2} \\ O_{n_2 \times n_1} & O_{n_2 \times m_1} & (r_2 + m_1)I_{n_2} \end{pmatrix}.$$

$$D^{-\frac{1}{2}}(H_1 \veebar H_2) = \begin{pmatrix} \frac{I_{n_1}}{\sqrt{(n_1-1)}} & O_{n_1\times m_1} & O_{n_1\times n_2} \\ O_{m_1\times n_1} & \frac{I_{m_1}}{\sqrt{n_2+2}} & O_{m_1\times n_2} \\ O_{n_2\times n_1} & O_{n_2\times m_1} & \frac{I_{n_2}}{\sqrt{(r_2+m_1)}} \end{pmatrix}.$$

Then the NL matrix of $H_1 \veebar H_2$ is

$$\mathscr{L}(H_1 \veebar H_2) = \begin{pmatrix} I_{n_1} - \frac{A(\bar{H}_1)}{n_1-1} & -\frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} & O_{n_1\times n_2} \\ -\frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & I_{m_1} & -\frac{J_{m_1\times n_2}}{\sqrt{(n_2+2)(r_2+m_1)}} \\ O_{n_2\times m_1} & -\frac{J_{n_2\times m_1}}{\sqrt{(n_2+2)(r_2+m_1)}} & I_{n_2} - \frac{A(H_2)}{(m_1+r_2)} \end{pmatrix}$$

$$= \begin{pmatrix} I_{n_1} - \frac{A(\bar{H}_1)}{n_1-1} & -\frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} & O_{n_1\times n_2} \\ -\frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & I_{m_1} & -\frac{J_{m_1\times n_2}}{\sqrt{(n_2+2)(r_2+m_1)}} \\ O_{n_2\times m_1} & -\frac{J_{n_2\times m_1}}{\sqrt{(n_2+2)(r_2+m_1)}} & \mathscr{L}(H_2)\circ C \end{pmatrix}$$

where $C = \frac{r_2}{r_2+m_1} J_{n_2} + \frac{m_1}{r_2+m_1} I_{n_2}$.
Using Definition **??** and Lemmas **??**,**??**, **??**, we have the NL characteristic

polynomial of $H_1 \veebar H_2$ is

$$f(\mathscr{L}(H_1 \veebar H_2), x) = det \begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{H}_1)}{n_1-1} & \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} & O_{n_1 \times n_2} \\ \frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & (x-1)I_{m_1} & \frac{J_{m_1 \times n_2}}{\sqrt{(n_2+2)(r_2+m_1)}} \\ O_{n_2 \times m_1} & \frac{J_{n_2 \times m_1}}{\sqrt{(n_2+2)(r_2+m_1)}} & xI_{n_2} - (\mathscr{L}(H_2) \circ C) \end{pmatrix}$$

$$= \det\left(xI_{n_2} - (\mathscr{L}(H_2) \circ C)\right) \det S,$$

$$\text{where S} = \begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{H}_1)}{n_1-1} & \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} \\ \frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & (x-1)I_{m_1} \end{pmatrix} - \Bigg[\begin{pmatrix} O_{n_1 \times n_2} \\ \frac{J_{m_1 \times n_2}}{\sqrt{(n_2+2)(r_2+m_1)}} \end{pmatrix}$$

$$\times (xI_{n_2} - (\mathscr{L}(H_2) \circ C))^{-1} \Big(O_{n_2 \times m_1}$$

$$S = \begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{H}_1)}{n_1-1} & & \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} \\ \frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & (x-1)I_{m_1} - \frac{J_{m_1\times n_2}}{\sqrt{(n_2+2)(r_2+m_1)}}(xI_{n_2} - (\mathscr{L}(H_2)\circ C))^{-1} & \ldots \end{pmatrix}$$

$$= \begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{H}_1)}{n_1-1} & \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} \\ \frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & (x-1)I_{m_1} - \frac{\chi_{\mathscr{L}(H_2)}(x, \frac{r_2}{r_2+m_1})}{(n_2+2)(r_2+m_1)}J_{m_1} \end{pmatrix}.$$

By using Lemma **??**, Corollary **??** and equation (**??**) we have

$$det(S) = det\begin{pmatrix} (x-1)I_{n_1} + \frac{A(\bar{H}_1)}{n_1-1} & \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}} \\ \frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}} & (x-1)I_{m_1} - \frac{\chi_{\mathscr{L}(H_2)}(x,\frac{r_2}{r_2+m_1})}{(n_2+2)(r_2+m_1)}J_{m_1} \end{pmatrix}$$

$$= det\left((x-1)I_{m_1} - \frac{\chi_{\mathscr{L}(H_2)}(x,\frac{r_2}{r_2+m_1})}{(n_2+2)(r_2+m_1)}J_{m_1}\right) det\Bigg[(x-1)I_{n_1} + \frac{A(\bar{H}_1)}{n_1-1}$$

$$- \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}}\left((x-1)I_{m_1} - \frac{\chi_{\mathscr{L}(H_2)}(x,\frac{r_2}{r_2+m_1})}{(n_2+2)(r_2+m_1)}J_{m_1}\right)^{-1}\frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}}$$

$$= (x-1)^{m_1}\left(1 - \frac{\chi_{\mathscr{L}(H_2)}(x,\frac{r_2}{r_2+m_1})}{(n_2+2)(r_2+m_1)}.\frac{m_1}{x-1}\right) det\Bigg[(x-1)I_{n_1} + \frac{J_{n_1} - I_{n_1} - A(H_1)}{n_1-1}$$

$$- \frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}}\left(\frac{1}{(x-1)}I_{m_1} + \frac{\beta}{(x-1)(x-1-m_1\beta)}J_{m_1}\right)\frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}}$$

where $\beta = \frac{\chi_{\mathscr{L}(H_2)}(x, \frac{r_2}{r_2+m_1})}{(n_2+2)(r_2+m_1)}$

$$
\begin{aligned}
detS =&(x-1)^{m_1}\left(1-\beta\frac{m_1}{x-1}\right)det\Bigg[(x-1)I_{n_1}+\frac{J_{n_1}-I_{n_1}-A(H_1)}{n_1-1}-\frac{I(H_1)I(}{(x-1)(n_1-}\\
&-\frac{I(H_1)}{\sqrt{(n_1-1)(n_2+2)}}\left(\frac{\beta}{(x-1)(x-1-m_1\beta)}J_{m_1}\right)\frac{I(H_1)^T}{\sqrt{(n_1-1)(n_2+2)}}\Bigg]\\
=&(x-1)^{m_1}\left(1-\beta\frac{m_1}{x-1}\right)det\Bigg[(x-1)I_{n_1}+\frac{J_{n_1}-I_{n_1}-A(H_1)}{n_1-1}\\
&-\frac{I(H_1)I(H_1)^T}{(x-1)(n_1-1)(n_2+2)}-\frac{\beta}{(x-1)(x-1-m_1\beta)}\frac{I(H_1)J_{m_1}I(H_1)^T}{(n_1-1)(n_2+2)}\Bigg]\\
=&(x-1)^{m_1}\left(1-\beta\frac{m_1}{x-1}\right)det\Bigg[(x-1)I_{n_1}+\frac{J_{n_1}-I_{n_1}-A(H_1)}{n_1-1}\\
&-\frac{I(H_1)I(H_1)^T}{(x-1)(n_1-1)(n_2+2)}-\frac{\beta r_1^2 J_{n_1}}{(x-1)(x-1-m_1\beta)(n_1-1)(n_2+2)}\Bigg]\\
=&(x-1)^{m_1}\left(1-\beta\frac{m_1}{x-1}\right)det\Bigg[(x-1)I_{n_1}+\frac{J_{n_1}-I_{n_1}-A(H_1)}{n_1-1}-\frac{I(H_1)I(}{(x-1)(n_1-}
\end{aligned}
$$