On the sum of first k largest Laplacian (signless) eigenvalues of graphs

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Laplacian eigenvalues of graphs and Brouwer's conjecture

Adjacency Matrix

The adjacency matrix of a graph G is the $n \times n$ matrix $A = A(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j, \\ 0, & \text{if } v_i \nsim v_j. \end{cases}$$

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Laplacian Matrix

The matrix $L(G) = D(G) - A(G)$ is known as Laplacian matrix and its spectrum is the Laplacian spectrum of the graph G. $L(G)$ can also be defined as $L(G) = (l_{ij})$, where

$$l_{ij} = \begin{cases} d_i, & \text{if } v_i = v_j, \\ -1, & \text{if } v_i \sim v_j, \\ 0, & \text{if } v_i \nsim v_j. \end{cases}$$

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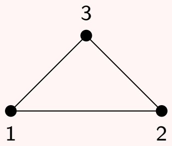

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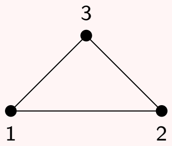

Degree based matrices

The diagonal matrix $D(G)$ is D(G)=Diag(2,2,2)

$$A(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, L(G) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, Q(G) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all square matrices of order n with entries from complex field $\mathbb{C}$. For $M \in \mathbb{M}_n(\mathbb{C})$, the square roots of the eigenvalues of MM^* or M^*M, where M^* is the complex conjugate of M are known as *singular values*. As MM^* is positive semi-definite, so singular values of M are non negative, denoted by $s_1(M) \geq s_2(M) \geq \cdots \geq s_n(M)$.

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Schatten p-norm

The *Schatten p-norm* of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ is the p-th root of the sum of the p-th powers of the singular values, that is

$$\|M\|_p = (s_1^p(M) + s_2^p(M) + \cdots + s_n^p(M))^{\frac{1}{p}},$$

where $n \geq p \geq 1$.

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where $n \geq p \geq 1$.

The Schatten 1-norm is the sum of all singular values and is known as *trace norm or nuclear norm* of M.

Ky Fan k-norm

The sum of first k singular values is the *Ky Fan k-norm*, that is for $p = 1$ and $n \geq k \geq 1$, we have

$$\|M\|_k = s_1(M) + s_2(M) + \cdots + s_k(M).$$

$\|M\|_1$ is the largest singular value of M and is called *spectral norm*.

The matrices $L(G)$ and $Q(G)$ are real symmetric and positive semi-definite, so their eigenvalues are ordered as $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 = q_n \leq q_{n-1} \leq \cdots \leq q_1$, where μ_1 and q_1 are their spectral radii, respectively.

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For $k = 1, 2, \dots, n$, let

$$S_k(G) = \sum_{i=1}^{k} \mu_i \qquad \text{and} \qquad S_k^+(G) = \sum_{i=1}^{k} q_i$$

be the sum of the k largest Laplacian eigenvalues and the sum of first k largest signless Laplacian eigenvalues of G. The parameter $S_k(G)$ is also of great importance in the well known theorem by Grone-Merris (1994) [12], a nice proof of which is due to Bai (2010) in [1], its statement is given in the following theorem.

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Grone-Merris-Bai Theorem

If G is any graph of order n and k, $1 \leq k \leq n$, is any positive integer, then $S_k(G) \leq \sum_{i=1}^{k} d_i^*(G)$, where $d_i^*(G) = |\{v \in V(G) : d_v \geq i\}|$, for $i = 1, 2, \ldots, n$.

Analogous to Grone-Merris-Bai Theorem, Brouwer [2] conjectured the upper bound for $S_k(G)$, which is known as Brouwer's conjecture, and is stated as follows.

Conjecture 1.1 (Brouwer's conjecture (2008))

If G is any graph with order n and size m, then

$$S_k(G) \leq m + \binom{k+1}{2}, \quad \text{for any} \quad k \in \{1, 2, \ldots, n\}.$$

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Next we obtained upper bounds for $S_k(G)$ of graph G.

Theorem 1.2

Let G be a connected graph of order $n \geq 4$ and size m having clique number $\omega \geq 2$. If $H = G \setminus K_\omega$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$, each of which is a c-cyclic graph and $p \geq 0$ trivial components, then

$$S_k(G) \leq \begin{cases} \omega(\omega-1) + n - p + 2r(c-1) + 2k, & \text{if } k \geq \omega - 1, \\ k(\omega+2) + n - p + 2r(c-1), & \text{if } k \leq \omega - 2. \end{cases} \quad (1)$$

Bounds for $S_k(G)$

Similar to Theorem 1.2, we have following result for complete bipartite graphs [9]

Theorem 1.3

Let G be a connected graph of order $n \geq 4$ and size m. Let K_{s_1,s_2}, $s_1 \leq s_2 \geq 2$, be the maximal complete bipartite subgraph of the graph G. If $H = G \setminus K_{s_1,s_2}$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$, each of which is a c-cyclic graph and $p \geq 0$ trivial components, then

$$S_k(G) \leq \begin{cases} 2s_1s_2 + n - p + 2r(c-1) + 2k, & \text{if } k \geq s_1 + s_2 - 1, \\ s_2 + ks_1 + n - p + 2r(c-1) + 2k, & \text{if } k \leq s_1 + s_2 - 2. \end{cases} \quad (2)$$

1

[1] Hilal A. Ganie, S. Pirzada, Bilal A. Rather and V. Trevisan, Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues if graphs, *Linear Algebra Appl.* **588** (2020) 1–18.

Truth of Brouwers conjecture

Theorem 1.4

Let G be a connected graph of order $n \geq 4$ and size m having clique number $\omega \geq 2$. If $H = G \setminus K_\omega$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$, each of which is a c-cyclic graph, then $S_k(G) \leq m + \dfrac{k(k+1)}{2}$, for all $k \in [1, \Delta_1]$ and $k \in [\beta_1, n]$, where $\Delta_1 = min\{\omega - 2, \gamma_1\}, \gamma_1 = \frac{2\omega+3-\sqrt{16\omega+8r(c-1)+9}}{2}$ and $\beta_1 = \frac{3+\sqrt{4\omega^2-4\omega+8r(c-1)+9}}{2}$.

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For graphs, having $K_{s,s}$, as the maximal complete bipartite subgraph. we have the following.

Theorem 1.5

Let G be a connected graph of order $n \geq 4$ and size m and let $K_{s,s}$, $s \geq 2$ be the maximal complete bipartite subgraph of graph G. If $H = G \setminus K_{s,s}$ is a graph having r non-trivial components $C_1, C_2, \ldots, C_r$, each of which is a c-cyclic graph and $p \geq 0$ trivial components, then for $s \geq \frac{5+\sqrt{8r(c-1)+34}}{2}$, Brouwer's conjecture holds for all k; and for $s < \frac{5+\sqrt{8r(c-1)+34}}{2}$, Brouwer's conjecture holds for all $k \in [x_1, n]$ and for all $k \in [1, y_1]$, where $x_1 = \frac{2s+3+\sqrt{20s-4s^2+8r(c-1)+9}}{2}$ and $y_1 = \frac{2s+3-\sqrt{20s-4s^2+8r(c-1)+9}}{2}$.

Chen [4] verified that Brouwer's conjecture is true for graphs in which the size m is restricted.

THEOREM 1.6

[4] Let G be a connected graph with $n \geq 4$ vertices and m edges having $p \geq 1$ pendent vertices.

(i) *If $p < \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-3)p}{2}$, then Brouwer's conjecture holds for $k \in [1, \frac{n-1}{2}]$.*

(ii) *If $p > \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-3)p}{2}$, then Brouwer's conjecture holds for $k \in [\frac{n-1}{2}, n]$.*

2

[2] Improved results on Brouwer's conjecture for sum of the Laplacian eigenvalues of a graph, *Linear Algebra Appl.* **557** (2018) 327–338.

Now, let the graph G have p vertices each of degree r. The following theorem verifies Brouwer's conjecture under certain restrictions on the size m of G.

Theorem 1.7

Let G be a connected graph with $n \geq 4$ vertices and m edges having $p \geq 1$ vertices of degree $r \geq 1$.

- **(i)** *If $m \geq \frac{(2n-r-1)r}{2}$, then Brouwer's conjecture holds for $k \in [1, r]$.*
- **(ii)** *If $p < \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-1-2r)p}{2}$, then Brouwer's conjecture holds for $k \in [r+1, \frac{n-1}{2}]$.*
- **(iii)** *If $p > \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-1-2r)p}{2}$, then Brouwer's conjecture holds for $k \in [\frac{n-1}{2}, n]$.*

[3]

[3] Hilal A. Ganie, S. Pirzada, Bilal A. Rather and R. U. Shaban, On Laplacian eigenvalues of graphs and Brouwer's conjecture, *J. Ramanujan Math. Soc.* **36(1)** (2021) 1–9.

Theorem 1.8

Let G be a connected graph with $n \geq 4$ vertices, m edges and having $p \geq 1$ and $q \geq 1 (q \neq p)$ vertices of degrees r and s $(s > r \geq 1)$, respectively.

(i) *If $m \geq \frac{(2n-r-1)r}{2}$, then Brouwer's conjecture holds for $k \in [1, r]$.*

(ii) *If $n > p + s + \frac{1}{2}$ and $m \geq \frac{s(2n-2p-s-1)}{2} + pr$; or $n < p + r + \frac{3}{2}$ and $m \geq \frac{(r+1)(2n-2p-r-2)}{2} + pr$, then Brouwer's conjecture holds for $k \in [r+1, s]$.*

(iii) *If $p + q < \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-2r-1)}{2}p - \frac{(n-2s-1)}{2}q$, then Brouwer's conjecture holds for all k, $s + 1 \leq k \leq (n-1)/2$.*

(iv) *If $p + q > \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-2r-1)}{2}p - \frac{(n-2s-1)}{2}q$, then Brouwer's conjecture holds for all k, $(n-1)/2 \leq k \leq n$.*

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In case G has $p \geq 1$ pendent vertices and $q \geq 1$ vertices of degree two, we have the following consequence of Theorem 1.8.

Corollary 1.9

Let G be a connected graph with $n \geq 4$ vertices and m edges having $p \geq 1$ pendent vertices and $q \geq 1$ vertices of degrees 2.

(i) *If $p + q < \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-3)}{2}p - \frac{(n-5)}{2}q$, then Brouwer's conjecture holds for all k, $3 \leq k \leq (n-1)/2$.*

(ii) *If $p + q > \frac{n}{2}$ and $m \geq \frac{(n-1)(3n-1)}{8} - \frac{(n-3)}{2}p - \frac{(n-5)}{2}q$, then Brouwer's conjecture holds for all k, $(n-1)/2 \leq k \leq n$.*

Conjecture 1.10 (Ashraf's conjecture (2013))

If G is any graph with order n and size m, then

$$S_k^+(G) \leq m + \binom{k+1}{2}, \quad \textit{for any} \quad k \in \{1, 2, \ldots, n\}.$$

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- Ashraf's conjecture is true for all graphs of order at most 10, besides it holds for $k \in \{1, n-1, n\}$.
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- Pirzada and Hilal (2015) obtained upper bounds for $S_k^+(G)$ in terms of various graph parameters, which are better than some previously known upper bounds on $S_k^+(G)$. They also showed that the conjecture is true for several classes of graphs.
- Chen (2018) verified the conjecture for $k = n-2$ and some new families of graphs

Laplacian energy conjecture of trees

Laplacian energy

The Laplacian energy $LE(G)$ [14] of a graph G as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \overline{d} \right|,$$

where $\overline{d}$ is the average of the Laplacian eigenvalues of G.

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where $\overline{d}$ is the average of the Laplacian eigenvalues of G.

Using the fact that $\sum_{1=i}^{n-1} \mu_i = 2m$, from [6], we have

$$LE(G) = 2\left(\sum_{i=1}^{\sigma} \mu_i - \sigma \overline{d} \right) = 2 \max_{1 \leq k \leq n} \left(\sum_{1=i}^{k} \mu_i - k\overline{d} \right), \tag{3}$$

where σ is the number of Laplacian eigenvalues greater than or equal to the average degree $\overline{d}$. By the definition of Laplacian energy, we see that $LE(G)$ is precisely the trace norm of the matrix $L(G) - \overline{d}I_n$, where I_n is the identity matrix.

Laplacian Energy Conjecture

Radenković and Gutman [25] (2007) studied the correlation between the energy and the Laplacian energy of trees and they computed the energy and Laplacian energy of all trees up to 14 vertices. They formulated the following conjecture.

Conjecture 2.1 (Laplacian energy conjecture)

If T is a tree of order n, then

$$LE(P_n) \leq LE(T) \leq LE(S_n).$$

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- Chang and Deng (2012) verified the conjecture 2.1 for trees of diameter 4 and 5 with perfect matching.
- Rahman, Ali and Rehman (2019) proved conjecture for some families of trees of diameter 4.

Proof of Conjecture for an arbitrary tree

Theorem 2.2

Let T be a tree of order $n \geq 4$ and let P_n be the path graph on n vertices. If T has s internal (non-pendent) vertices, then

$$LE(T) \geq LE(P_n),$$

provided that $s \leq \frac{1}{n-2}\left(\left(\frac{\pi-2}{\pi}\right)n^2 - 2n\right)$.

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Let T be a tree of order $n \geq 4$ having s internal (non-pendent) vertices. Then following holds,

- **(i)** If $s = 1$, then Conjecture 2.1 holds for all $n \geq 9$;
- **(ii)** If $s = 2$, then Conjecture 2.1 holds for all $n \geq 12$;
- **(iii)** If $s = 3$, then Conjecture 2.1 holds for all $n \geq 14$;
- **(iv)** If $s = 4$, then Conjecture 2.1 holds for all $n \geq 17$;
- **(v)** If $s = 5$, then Conjecture 2.1 holds for all $n \geq 20$;
- **(viii)** If $s \leq \frac{9n}{25} - 2$, then Conjecture 2.1 holds for all n.

Conjecture for trees of diameter 4

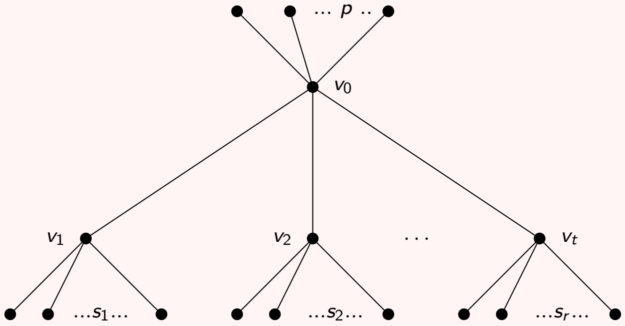

Figure: SNS tree.

Families not generated by Protoype tree

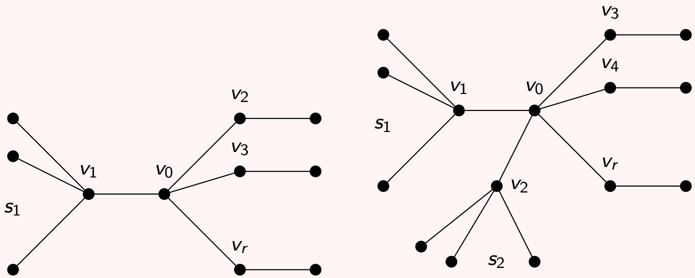

Figure: Trees $T^{'}$ and $T^{''}$

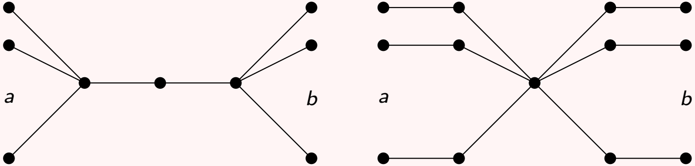

Figure: Double broom of diameter 4 and the tree $T(4; 2a, 2b)$

The following results are used in establishing conjecture for tress of diameter at most 4.

Lemma 2.3

[31] If P_n is a path on n vertices, then

$$LE(P_n) \leq 2 + \frac{4n}{\pi}.$$

Lemma 2.4

[11] Let $L(G)$ be the Laplacian matrix of G. Then $(d_1, d_2, \ldots, d_n) \preceq (\mu_1, \mu_2, \ldots, \mu_n)$, that is;

$$\sum_{i=1}^{k} \mu_i \geq 1 + \sum_{i=1}^{k} d_i \quad \text{for all} \quad k = 1, 2, \ldots, n-1.$$

The next lemma can be seen in [30].

Lemma 2.5

The number of Laplacian eigenvalues less than the average degree $2 - \frac{2}{n}$ of a tree T of order n is at least $\lfloor \frac{n}{2} \rfloor$.

Theorem 2.6

If T is a tree of diameter 3. Then conjecture 2.1 is true for all n.

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Let $\mathcal{T}_n(d)$ be the family of trees each of diameter d and order $n \geq 3$.

Theorem 2.7

Conjecture 2.1 is true for all trees of diameter 4, that is, for all trees of the family $\mathcal{T}_n(4)$.

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Theorem 2.7

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Now, we obtain a lower bound for the Laplacian energy of a tree T in terms of the sum of k_i largest Laplacian eigenvalues of T_i, where T_i, for $i = 1, 2$, are the components of T obtained by deleting any non-pendent edge.

Theorem 2.8

Let T be a tree of order $n \geq 8$ and let e be a non-pendent edge of T. Let $T - e = T_1 \cup T_2$ and let σ be the number of Laplacian eigenvalues of $T - e$ which are greater than or equal to the average degree $\overline{d}(T - e)$. Then

$$LE(T) \geq 2S_{k_1}(T_1) + 2S_{k_2}(T_2) - 4\sigma + \frac{4\sigma}{n},$$

where k_1, k_2 are respectively, the number of Laplacian eigenvalues of T_1, T_2 which are greater than or equal to $\overline{d}(T - e)$ with $k_1 + k_2 = \sigma$ and $S_k(T)$ is the sum of k largest Laplacian eigenvalues of T.

Assume Conjecture 2.1 holds for two components obtained by deleting a non-pendent edge of a tree T. Also, for both the components, let the number of Laplacian eigenvalues greater than or equal to the average degree be equal to their number of non-pendent vertices. Then Conjecture 2.1 holds for T, as can be seen as below.

Theorem 2.9

Let T_1 be a tree of order n_1 having r_1 non-pendent vertices and let T_2 be a tree of diameter at most 3 having order n_2 with $n_1 \geq n_2 \geq 6$. Let σ_1 be the number of Laplacian eigenvalues of T_1 greater than or equal to the average vertex degree $\overline{d}(T_1) = 2 - \frac{2}{n_1}$. Let $T = T_1 \cup T_2 \cup \{u, v\}$, where $u \in T_1$ and $v \in T_2$. If $\sigma_1 = r_1$, then Conjecture 2.1 holds for T, provided that $LE(T_1) \geq 2 + \frac{4n_1}{\pi}$.

Conclusion

In order to prove the conjecture in general, one must be aware of the distribution of Laplacian eigenvalues of trees around the average degree. The graph invariant σ plays a fundamental role in finding the lower bounds of $S_k(G)$, which in turn may help in proving the Laplacian energy conjecture. Thus to prove the Laplacian energy conjecture, we must study σ of the trees and use the gained information in verifying the Laplacian energy conjecture. Some tree transformation as in [30] can also help in verifying the Laplacian energy conjecture.

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THANK YOU