Toppleable permutations, excedances and acyclic orientations

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(joint with D. Hathcock and P. Tetali, `arXiv:2010.11236`,
and with B. Bényi, `arXiv:2104.13654`)

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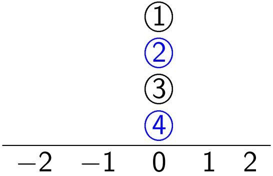
Outline

1. A dynamical system on labelled chips
2. A new dynamical system on permutations
3. Main ideas in the proofs
4. Extension to collapsed permutations

A sorting algorithm

- Defined by Hopkins, McConville and Propp, (*Elec. J. Comb.*, 2017).
- Start with chips labelled $1, \ldots, n$ initially at the origin in $\mathbb{Z}$.
- At each time step, do the following:
  1. If no position has two or more chips, stop. Else, go to step 2.
  2. Choose a position i uniformly at random among positions occupied by more than one chip.
  3. Pick two chips uniformly from those at site i.
  4. If the two chips are α, β with $\alpha < \beta$, then move α to position $i - 1$ and β to $i + 1$.
  5. Go to step 1.

Example: $n = 4$



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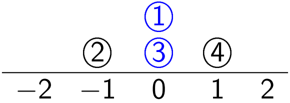

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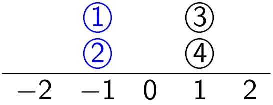

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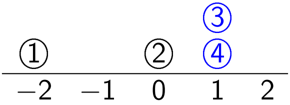

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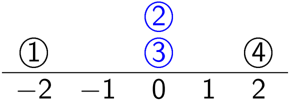


Example: $n = 4$

①	②		③	④
-2	-1	0	1	2

The main result

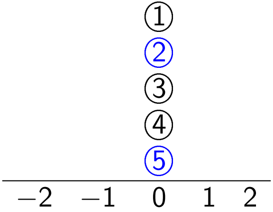
Theorem (Hopkins, McConville and Propp, *Elec. J. Comb.*, 2017)

When n is even, the chips end up at positions

$$-\frac{n}{2}, \ldots, -1, 1, \ldots, \frac{n}{2}$$

and are always sorted.

Example: $n = 5$



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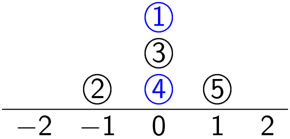

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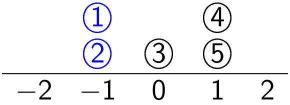

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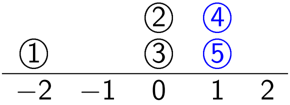

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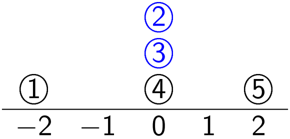


Example: $n = 5$

$$\begin{array}{ccccc} ① & ② & ④ & ③ & ⑤ \\ \hline -2 & -1 & 0 & 1 & 2 \end{array}$$

Open problem

When n is odd, it is easy to see that the chips end up at positions

$$-\frac{n-1}{2}, \ldots, \frac{n-1}{2}.$$

Conjecture (Hopkins, McConville and Propp, *Elec. J. Comb.*, 2017)

When n is odd, the chips get sorted with probability tending to $1/3$ as $n \to \infty$.

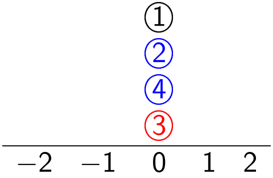
Further work

- *Root system chip firing*:
  1. Galashin, Hopkins, McConville and Postnikov (*SLC* 2018),
  2. Galashin, Hopkins, McConville and Postnikov (*Math. Z.* 2019),
  3. Hopkins and Postnikov (*Alg. Comb.* 2019).
- Progress towards proving the $1/3$-conjecture:
  1. Klivans and Liscio (*SLC* 2020),
  2. Felzenszwalb and Klivans (*JCTA* 2021).
  3. Klivans and Liscio (`arXiv:2006.12324`).

Modification of the process

- Suppose n is even and fix $r \in [n]$.
- Assume that the chip labelled r is infinitely heavy, and cannot be moved.
- Then one ends up in a configuration which has 2 chips at the origin (one of which is r) and 1 chip each at positions

$$-\frac{n}{2}+1, \ldots, -1, 1, \ldots, \frac{n}{2}-1.$$



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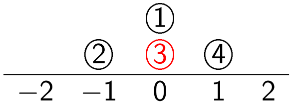

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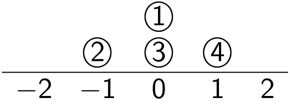

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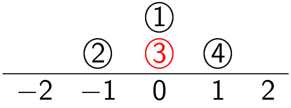
Motivation

- Consider the last stage where r is still infinitely heavy. E.g.



- That configuration can be considered as a permutation $\pi \in S_{n-1}$ plus an extra label, r.
- In the above example, $\pi = 213$, $r = 3$.
- According to HMP, all pairs (π, r) that arise this way end up sorted.
- It is natural to ask what are all the pairs which end up being sorted.

Notation

- Let $\pi = (\pi_1, \dots, \pi_n) \in S_n$ and $r \in [n+1]$.
- Let $L_n = \{-\lfloor (n+1)/2 \rfloor, \dots, -1, 0, 1, \dots, \lfloor n/2 \rfloor + 1\}$.

1. Place the elements $\pi_1, \dots, \pi_n$ in positions

$$-\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, -1, 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

2. Increase the labels in π greater than or equal to r by 1.
3. Add r to the origin.

- We will call this initial condition $\pi^{(r)}$.
- Eg with $r = 2$: $\rho = 3142 \in S_4$, $\sigma = 25134 \in S_5$.

$$\rho^{(2)} = \begin{array}{cccccc} & & 1 & & & \\ & 4 & 2 & 5 & 3 & \\ \hline -2 & -1 & 0 & 1 & 2 & 3 \end{array}, \quad \sigma^{(2)} = \begin{array}{ccccccc} & & & 1 & & & \\ & 3 & 6 & 2 & 4 & 5 & \\ \hline -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}.$$

Definitions

- For $\pi \in S_n$ and $r \in [n+1]$, we consider the toppling dynamics.
- The toppling dynamical system on L_n induces a map $T : S_n \times [n+1] \to S_{n+1}$.
- Let id be the identity (namely sorted) permutation.

Definition

We say that a permutation π is r-toppleable if $T(\pi, r) = \mathrm{id}$, and we say that π is toppleable if π is r-toppleable for all $r \in [n+1]$.

Basic properties

Proposition

Fix $\pi \in S_n$ and $r \in [n+1]$. The toppling dynamical system on L_n with initial condition $\pi^{(r)}$ satisfies the following properties.

1. The final configuration is deterministic.
2. At every time step, the configuration lives in L_n.
3. In the final configuration, there is precisely one chip at every position in L_n, except the origin (resp. position 1) when n is odd (resp. even).

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Main idea: No position contains more than 2 chips at any time.

Symmetry for n odd

Proposition (Symmetry)

- Suppose $n \geq 3$ is odd, $r \in [n+1]$, $\pi = (\pi_1, \ldots, \pi_n) \in S_n$.
- Let $\hat{\pi} = (n+1-\pi_n, \ldots, n+1-\pi_1)$.
- Then the toppling dynamics on $\pi^{(r)}$ is isomorphic to that on $\hat{\pi}^{(n+2-r)}$ via the map which reflects configurations about the origin and interchanges chip i with $n+2-i$.
- Since $\widehat{\mathrm{id}} = \mathrm{id}$, π is r-toppleable if and only if $\hat{\pi}$ is $(n+2-r)$-toppleable.

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Main idea: Isomorphism for any single toppling move.

Number of toppleable permutations

- Let $t_r(n)$ be the number of r-toppleable permutations.
- Let $t(n)$ be the number of toppleable permutations in S_n.
- For $n = 3$, there are four 1-toppleable permutations, namely 123, 213, 132 and 231, . . .

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- and four 4-toppleable permutations, namely 123, 213, 132 and 312.
- Therefore, $t_1(3) = t_4(3) = 4$.
- The common permutations among these turn out also to be 2- and 3-toppleable.
- Hence $t(3) = t_2(3) = t_3(3) = 3$.

Data

$n \setminus r$	1	2	3	4	5	6	7	8	9
3	4	3	3	4					
4	14	10	7	7	8				
5	46	38	31	31	38	46			
6	230	184	146	115	115	130	146		
7	1066	920	790	675	675	790	920	1066	
8	6902	5836	4916	4126	3451	3451	3842	4264	4718

The number of r-toppleable permutations, $t_r(n)$, for $3 \leq n \leq 8$.
Note the symmetry for odd n.

Statement of Main Result 3

Statement of monotonicity theorem

Background for the main result: excedance sets

- An excedance of a permutation π is any position i such that $\pi_i > i$.
- The positions at which there are excedances for π is called the excedance set of π.
- Ehrenborg and Steingrímsson (Adv. Appl. Math., 2000) initiated the study of permutations whose excedance set is $\{1, \ldots, k\}$ for $0 \leq k \leq n-1$.
- They gave a formula for the number $a_{n,k}$ of such permutations in S_n.
- One surprising result they found is that $a_{n,k} = a_{n,n-1-k}$.
- A related result of Clark and Ehrenborg (Europ. J of C, 2010) is

$$\sum_{r,s\geq 0} a_{r+s,s} \frac{x^r}{r!} \frac{y^s}{s!} = \frac{e^{-x-y}}{(e^{-x}+e^{-y}-1)^2}.$$

Main result 1

Theorem (A., Hathcock and Tetali, 2020+)

For all n,

$$t(n) = t_{\lfloor n/2 \rfloor + 1}(n) = t_{\lfloor n/2 \rfloor + 2}(n).$$

Furthermore,

$$t(n) = a\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right).$$

Using the exponential generating function, de Andrade, Lundberg and Nagle (Europ. J. of C, 2015) obtained the asymptotic formula,

$$t(n) = \frac{1}{2\log 2\sqrt{1-\log 2} + o(1)} \frac{n!}{(2\log 2)^n}.$$

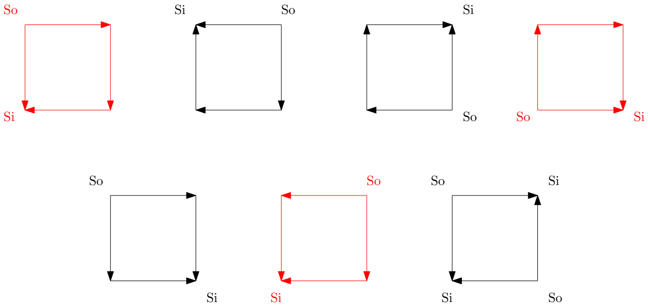
Acyclic orientations and chromatic polynomials

- Let G be a simple (no loops or multiple edges) undirected graph.
- An orientation of G is an assignment of arrows to the edges of G.
- An acyclic orientation (AO) is an orientation in which there is no directed cycle.
- A proper colouring of G is an assignment of colours to vertices such that no two adjacent vertices get the same colour.
- The chromatic polynomial of G, denoted $\chi_G(q)$, is the number of proper colourings of G with q colours.

Theorem (Stanley, *Disc. Math.*, 1973)

The number of acyclic orientations of G (up to sign) is $\chi_G(-1)$.

Example: C_4, the 4-cycle



There are 14 acyclic orientations for C_4. Seven are shown here. The other seven are obtained by reversing each of the arrows. The chromatic polynomial is $\chi_{C_4}(q) = q^4 - 4q^3 + 6q^2 - 3q$.

Acyclic orientations with unique sink

Definition

An acyclic orientation with a unique sink (AUSO) is an acyclic orientation with exactly one sink.

Theorem (Greene and Zaslavsky, *Trans. of the AMS*, 1983)

The number of AUSOs of G (up to sign) is independent of the sink and equal to (up to sign) the linear coefficient of $\chi_G(-1)$.

C_4 has 3 AUSOs, shown in red on the previous page.

Main result 2

Recall that $K_{m,n}$ is the complete bipartite graph with parts of size m and n.
For example, $C_4 \cong K_{2,2}$.

Theorem (A., Hathcock and Tetali, 2020+)

For all n, $t(n)$ is equal to the number of acyclic orientations with a fixed unique sink of $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor + 1}$.

This proof is bijective.

Poly-Bernoulli numbers

- The well-known polylogarithm function is given by

$$\mathrm{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k}.$$

- Recall that a position k is an ascent in a permutation if $\pi_k < \pi_{k+1}$.
- The Eulerian number $\left\langle {m \atop j} \right\rangle$ is the number of permutations in S_n with j ascents.
- For a non-negative integer m,

$$\mathrm{Li}_{-m}(z) = \frac{\sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle z^{m-j}}{(1-z)^{m+1}}.$$

Poly-Bernoulli numbers

- Poly-Bernoulli numbers of type B were defined by Kaneko (1997) via the exponenital generating function,

$$\sum_{n=0}^{\infty} B_{n,k} \frac{x^n}{n!} = \frac{\mathrm{Li}_{-k}(1-e^{-x})}{1-e^{-x}},$$

- A surprising result is that $B_{k,n} = B_{n,k}$.
- There are many combinatorial interpretations for $B_{n,k}$.
- For example, the number of AOs of $K_{n,k}$ is $B_{n,k}$.
- A permutation $\pi \in S_{k+n}$ is said to be a (k,n)-Vesztergombi permutation if $-k \leq \pi_i - i \leq n$ for $1 \leq i \leq k+n$.
- The number of (k,n)-Vesztergombi permutations is $B_{n,k}$.

The first few poly-Bernoulli numbers

$n\backslash k$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

Forward difference operators

- Let Δ be the discrete (forward) difference operator, i.e. for any function $f(n)$, $\Delta(f(n)) = f(n+1) - f(n)$.
- The higher difference operators are obtained by composition.
- For example, $\Delta^2(f(n)) = f(n+2) - 2f(n+1) + f(n)$.
- Note that $\Delta^0(f(n)) = f(n)$.

Main result 3

Back to data

Theorem (A. and Bényi, 2021+)

The number of r-toppleable permutations in S_n is

$$t_r(n) = \Delta^{r-1}\left(B_{n-p+1-r,p}\right),$$

where $p = \lfloor (n+1)/2 \rfloor$ and Δ acts on the first index.

- We generalise this result to any position of adding the extra chip.
- We also characterise all possible final permutations and enumerate permutations toppling to these.

Focus on odd n

- For each statement, the results for odd and even n differ slightly.
- To make the presentation cleaner, we state the results only for odd n.
- This will avoid the presence of floors and ceilings all over the place.
- The corresponding results for even n are given in `arXiv:2010.11236`.

A useful lemma

Lemma

Suppose $\pi \in S_{2m+1}$ is r-toppleable. Then

1. *for each $1 \leq k \leq m+1$, the final move of chip k when toppling $\pi^{(r)}$ is to the left;*
2. *for each $m+2 \leq k \leq n+1$, the final move of chip k when toppling $\pi^{(r)}$ is to the right;*
3. *in the final move, chips $m+1$ and $m+2$ topple to their correct positions.*

(1) and (2) follow by induction on k.
(3) follows from the fact that the origin is vacant at the end.

Monotonicity

Theorem (A., Hathcock and Tetali, 2020+)

Let $\pi \in S_{2m+1}$.

1. *Suppose $2 \leq r \leq m+1$. Then π is $(r-1)$-toppleable if π is r-toppleable.*
2. *Suppose $m+2 \leq r \leq 2m$. Then π is $(r+1)$-toppleable if π is r-toppleable.*
3. *π is $(m+1)$-toppleable if and only if π is $(m+2)$-toppleable.*

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Ideas in the proof of the monotonicity theorem

- (1) and (2) are equivalent by symmetry. Focus on (1).
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- At each step of the toppling procedure, $\pi^{(r)}$ and $\pi^{(r-1)}$ continue to differ only in their positions of $r-1$ and r.
- This will be the case until we reach the point when $r-1$ and r are at the same position.
- At this point, the two topplings are coupled and the final result is identity.

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- The only problem with this argument is that we could have reached the final result without ever being coupled.
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- But we know that eventually $r - 1$ will end up to the left of r.

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$$\begin{array}{cccccc} & & & r & & \\ \dots & r-1 & \dots & a & b & \dots \\ \hline & j & \dots & 0 & 1 & \dots \end{array}$$

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- There are again two cases depending on whether $r >< a$.
- We use a double induction on j and n, and previous lemma to conclude the result.

The notion of a pass

- For $\pi \in S_{2m+1}$, let the number of chips at each site of L_n in $\pi^{(r)}$ be
$$p^{(r)} = (\_, 1, \dots, 1, \hat{2}, 1, \dots, 1, \_).$$
- Topple as follows:
$$\begin{aligned} p^{(r)} &\to (\_, 1, \dots, 1, 1, 2, \hat{\_}, 2, 1, 1, \dots, 1, \_) \\ &\to (\_, 1, \dots, 1, 2, \_, \hat{2}, \_, 2, 1, \dots, 1, \_) \\ &\to (\_, 1, \dots, 2, \_, 1, \hat{2}, 1, \_, 2, \dots, 1, \_). \end{aligned}$$
- At this point, we leave the origin unchanged and start to topple the vertices with 2 chips both on the left and right, until we reach the end.
- We then arrive at the configuration with chip counts given by
$$(1, \_, 1, \dots, 1, \hat{2}, 1, \dots, 1, \_, 1).$$

The notion of a pass

- Now, the extremal points cannot be modified by any further topplings and are fixed.
- We call this sequence of topplings the first pass.
- This consists of $2m+1$ individual topplings.
- Similarly, the second pass will be initiated by toppling the origin in a similar way, and we will end up with

$$(1,1,\_,1,\ldots,1,\hat{2},1,\ldots,1,\_,1,1).$$

- Continue this way until the configuration stabilizes.
- If n is odd, then we see that after $(n+1)/2$ passes, the configuration will freeze leaving the origin empty.

Observations about passes

- Every chip between vacancies topples at least once in every pass.
- If $\pi \in S_{2m+1}$ is toppleable, then for $1 \leq i \leq m+1$, i and $2m+2-i$ get fixed in their correct positions at the end of the i'th pass.
- For example:

$$\_\ \ 3\ \ 6\ \ \overset{1}{2}\ \ 4\ \ 5\ \ \_ \xrightarrow[\text{pass}]{\text{first}} 1\ \ \_\ \ 3\ \ \overset{2}{6}\ \ 4\ \ \_\ \ 5$$

$$\xrightarrow[\text{pass}]{\text{second}} 1\ \ 2\ \ \_\ \ \overset{3}{4}\ \ \_\ \ 6\ \ 5 \xrightarrow[\text{pass}]{\text{third}} 1\ \ 2\ \ 3\ \ \_\ \ 4\ \ 6\ \ 5$$

Positions of 1 and n

Lemma

If $\pi \in S_{2m+1}$ is toppleable, then 1 is in position at most $m+1$ in π.

Conversely, if 1 (resp n) is in position at most $m+1$ (at least $m+1$) in π, then 1 (resp $n+1$) is in the first (resp. last) position in $T(\pi, m+1)$.

Positions of 1 and n

Proof.

- Suppose 1 is to the right of the origin in $\pi^{(m+1)}$. Then, in the first pass, 1 will move exactly one position to the left (since it is smallest) at the end of the first pass. Therefore, 1 is not frozen in its correct position, which is the extreme left. So π cannot be toppleable.

Positions of 1 and n

Proof.

- Suppose 1 is to the right of the origin in $\pi^{(m+1)}$. Then, in the first pass, 1 will move exactly one position to the left (since it is smallest) at the end of the first pass. Therefore, 1 is not frozen in its correct position, which is the extreme left. So π cannot be toppleable.
- Conversely, suppose 1 is in a position on or to the left of center. Then it gets a partner at some point during the first pass. After that time, it keeps moving left for all future times until the first pass ends and gets placed at the extreme left, its correct position. A similar argument works for n.

□

A generalization of this idea proves the structure theorem.

Structure theorem

Theorem (A., Hathcock and Tetali, 2020+)

A permutation $\pi \in S_{2m+1}$ *is* $(m+1)-$*toppleable if and only if*

$$\begin{aligned} \pi_i &\leq m+i, \quad 1 \leq i \leq m, \\ \pi_i &\geq i-m, \quad m+1 \leq i \leq 2m+1. \end{aligned}$$

Equivalently,

$$\begin{aligned} \pi_i^{-1} &\in \{1, \ldots, m+i\}, \quad 1 \leq i \leq m+1 \\ \pi_i^{-1} &\in \{i-m, \ldots, 2m+1\}, \quad m+2 \leq i \leq 2m+1. \end{aligned}$$

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Main idea: The notion of a pass, previous lemma and induction.

Bijection

Lemma

Permutations $\pi \in S_{2m+1}$ such that $\pi_i \leq m + i$ for $1 \leq i \leq m$ and $\pi_i \geq i - m$ for $m + 1 \leq i \leq 2m + 1$ are in bijection with permutations in S_{2m+1} whose excedance set is $\{1, \ldots, m\}$.

Proof idea.

$$(\pi_1, \ldots, \pi_m | \pi_{m+1}, \ldots, \pi_{2m+1}) \\ \to \sigma = 2m + 2 - (\pi_m, \ldots, \pi_1 | \pi_{2m+1}, \ldots, \pi_{m+1}).$$

□

Illustration for $m = 2$

Consider $\pi \in S_5$ such that $\pi_1 = 3$ and

1. $\pi_i \leq 2 + i$ for $1 \leq i \leq 2$, and
2. $\pi_i \geq i - 2$ for $3 \leq i \leq 5$.

Toppleable permutations	Permutations σ with $\pi_2 = 3$ and excedance set $\{1, 2\}$
31245	53124
31254	53214
31425	53142
31524	53241
32145	43125
32154	43215
34125	23145

Proof of main result

Proof.

- By the monotonicity result, we see that $\pi \in S_{2m+1}$ is toppleable if it is $(m+1)$-toppleable.
- According to the structure theorem, $\pi_i \leq m + i$ for $1 \leq i \leq m$ and $\pi_i \geq i - m$ for $m + 1 \leq i \leq 2m + 1$.
- Now, the previous lemma proves that the number of such permutations is $a_{2m+1,m}$ bijectively, completing the proof.

$\square$

Back to HMP toppling

Theorem (Lemma 12, Hopkins, McConville and Propp)

Starting with n chips at the origin, the position of chip k lies between $-\lfloor (n+1-k)/2 \rfloor$ and $\lfloor k/2 \rfloor$ for $1 \leq k \leq n$ at all times.

- When n is odd, $n = 2m+1$, the final configuration will contain single chips in all positions $-m$ through m.
- We now apply this condition to count permutations arising from this condition switching positions from $[-m, m]$ to $[n]$.
- For n even, the only permutation that appears as a result of toppling is id.
- We also consider this case, although it is not directly relevant to the toppling problem.

Collapsed permutations

Definition

We say that a permutation $\pi \in S_n$ is collapsed if

$$\pi_k^{-1} \geq \begin{cases} \lceil k/2 \rceil & n \text{ odd}, \\ 1 + \lfloor k/2 \rfloor & n \text{ even} \end{cases} \quad \text{and} \quad \pi_k^{-1} \leq \lceil n/2 \rceil + \lfloor k/2 \rfloor.$$

Let G_n be the subset of collapsed permutations in S_n

- For $n = 2m + 1$,

i	1	2	3	$\dots$	$2m$	$2m+1$
Position of $i \geq$	1	1	2	$\dots$	m	$m+1$
Position of $i \leq$	$m+1$	$m+2$	$m+2$	$\dots$	$2m+1$	$2m+1$

- For example, $G_3 = \{123, 132, 213\}$ and $G_4 = \{1234, 1324\}$.

Seidel triangle for the Genocchi numbers

- To state our results, we recall a well-known combinatorial triangle.
- The Seidel triangle is the triangular sequence $S_{n,k}$ for $n \geq 1$ given by

$$\begin{aligned} S_{1,1} &= 1, \\ S_{n,k} &= 0, \quad k < 2 \text{ or } (n+3)/2 < k, \\ S_{2n,k} &= \sum_{i \geq k} S_{2n-1,i}, \\ S_{2n+1,k} &= \sum_{i \leq k} S_{2n,i}. \end{aligned}$$

First few rows

$n \backslash k$	2	3	4	5	6
1	1				
2	1				
3	1	1			
4	2	1			
5	2	3	3		
6	8	6	3		
7	8	14	17	17	
8	56	48	34	17	
9	56	104	138	155	155
10	608	552	448	310	155

Genocchi numbers of the first kind

- The numbers on the rightmost diagonal are the Genocchi numbers of the first kind, g_{2n}.
- They counts permutations in S_{2n-3} whose excedence set is $\{1, 3, \ldots, 2n-5\}$.
- For example, $g_8 = 17$:

$$21435, 21534, 21543, 31425, 315, 24, 31542, 32415, 32514,$$
$$32541, 41523, 41532, 42513, 42531, 51423, 51432, 52413, 52431.$$

- The exponential generating function of g_{2n} is given by

$$\sum_{n\geq 0} g_{2n} \frac{x^{2n}}{(2n)!} = x \tan\left(\frac{x}{2}\right).$$

Odd collapsed permutations

Theorem

The number of collapsed permutations in S_{2n+1} is g_{2n+4}.

- Define a bijection $f : G_{2n+1} \to S_{2n+1}$ which send

$$\pi \mapsto \sigma = (\sigma_1, \ldots, \sigma_{2n+1})$$

such that

1. $\sigma_{2i} = \pi_i, \sigma_{2i-1} = \pi_{n+1+i}$ for $1 \leq i \leq n$, and
2. $\sigma_{2n+1} = \pi_{n+1}$.

- The bijection for $n = 1$ is illustrated below:

G_3	S_3 with excedence set $\{1\}$
132	213
123	312
213	321

Genocchi numbers of the second kind

- The numbers on the leftmost diagonal are the median Genocchi numbers or Genocchi numbers of the second kind, H_{2n+1}.
- They count among other things, ordered pairs $((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$ such that $0 \leq a_k \leq k$ and $1 \leq b_k \leq k$ for all k and $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\} = [n-1]$.
- For example, $H_7 = 8$:

$$((0,0),(1,2)),\ ((0,1),(1,2)),\ ((0,2),(1,1)),\ ((0,2),(1,2)),$$
$$((1,0),(1,2)),\ ((1,1),(1,2)),\ ((1,2),(1,1)),\ ((1,2),(1,2)).$$

- In terms of the Genocchi numbers of the first kind, we have

$$H_{2n+1} = \sum_{i=0}^{n} g_{2n-2i} \binom{n}{2i+1}.$$

Normalized median Genocchi numbers

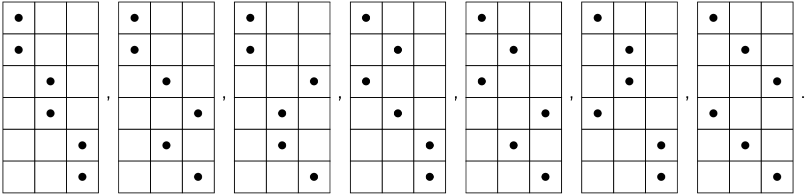
- Although it is not clear either from the above definition or the formula, H_{2n+1} is always divisible by 2^n.
- The numbers $h_n = H_{2n+1}/2^n$ are called the normalized median Genocchi numbers.
  The first few numbers of this sequence are
  $$\{h_n\}_{n=0}^{7} = \{1, 1, 2, 7, 38, 295, 3098, 42271\}.$$
- A classical combinatorial interpretation for these are certain configurations first defined by Hippolyte Dellac in 1900.

Dellac configuration

Definition

A Dellac configuration of order n is a $2n \times n$ array containing $2n$ points, such that every row has a point, every column has two points, and the points in column j lie between rows j and $n+j$, both inclusive, $1 \leq j \leq n$.

For example, when $n = 3$, the 7 Dellac configurations are



Even collapsed permutations

Theorem

The number of collapsed permutations in S_{2n} is given by H_{2n-1}.

- Both $2i$ and $2i+1$ have to lie in positions between $i+1$ and $i+n$, both inclusive, for $1 \leq i \leq n-1$.
- Thus, $\#G_{2n}$ is divisible by 2^{n-1}.
- Focus on $\pi \in G_{2n}$ such that $2i$ precedes $2i+1$ in one-line notation for all i.
- Since $\pi_1 = 1$ and $\pi_{2n} = 2n$ are forced, we focus on $(\pi_2, \ldots, \pi_{2n-1})$.

Bijection

- Construct a configuration C of points on an $(2n-2) \times (n-1)$ array as follows:
- For $2 \leq i \leq 2n-1$, place a point in position $(i-1, \lfloor \pi_i/2 \rfloor)$.
- C is a Dellac configuration and this can be inverted.
- For example, the permutation $1\underbrace{243657}8 \in G_8$ is in bijection with

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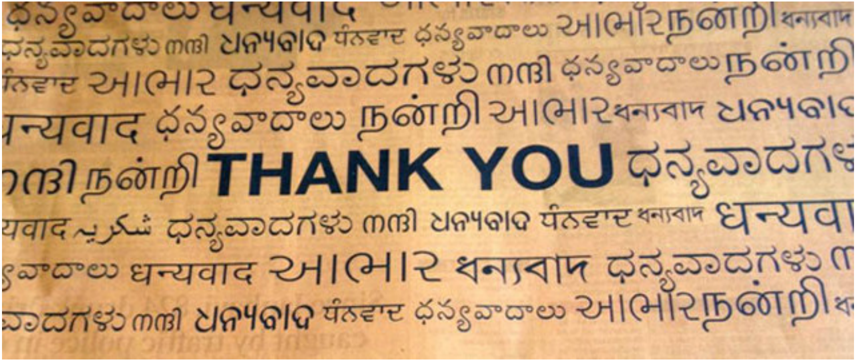
THANK YOU
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