ON LEECH TREES AND SOME RELATED CONCEPTS

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Abstract

- Leech labeling
- Known results about Leech trees
- Some classes of non-Leech trees
- Some related labelings

Graph Labeling

- Assign integer values
- To vertices – Vertex labeling
- To edges – Edge labeling
- Both – Total labeling

Types of labeling

- Graceful labeling: Vertices from 0 to m, so that edges receive 1 to m.
- Edge graceful labeling: Edges from 1 to q so that vertices receive 0 to p – 1.
- Harmonious labeling: Vertices from 0 to m – 1, so that edges receive 0 to m - 1.
- Graph colorings
- Leech labeling

Leech labeling

Tree → Unique path between every pair of vertices

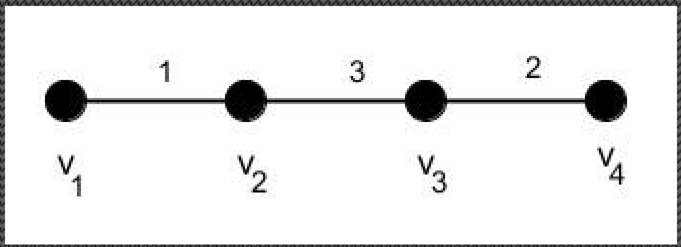
${}^{n}C_2 = n(n-1)/2$ paths

Assign distinct positive integer edge weights

Leech labeling → Path weights are exactly 1,2,…, ${}^{n}C_2$

Leech labeling exist → Leech Tree

Example



Origin of Leech labeling

John Leech

↓

Another tree labeling problem

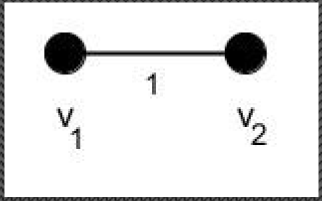
↓

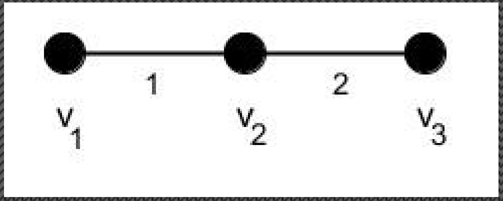
American Mathematics Monthly

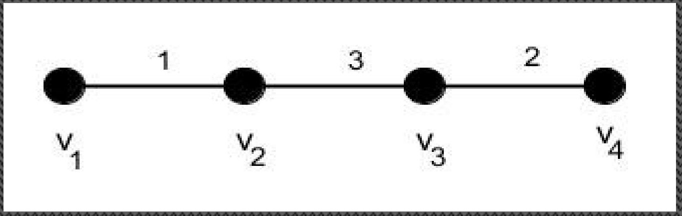
↓

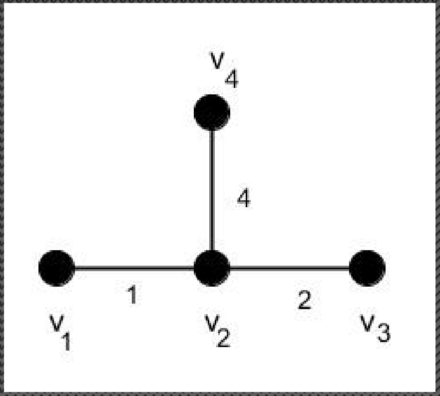
82 (1975), 923 – 925

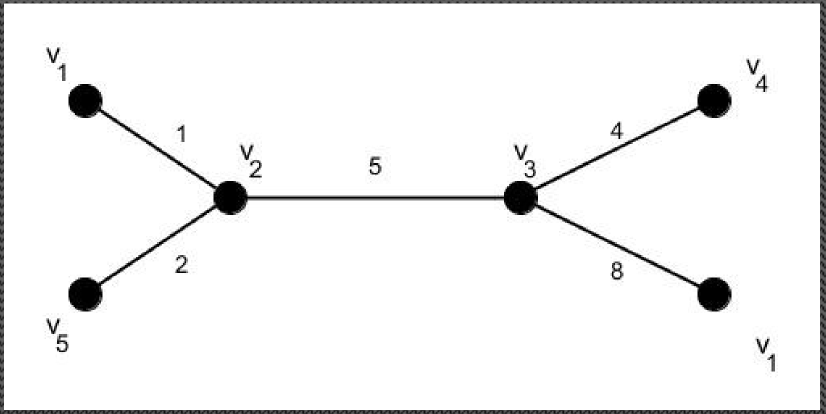
Known Leech trees





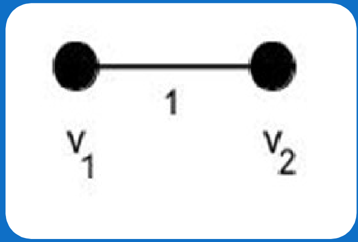




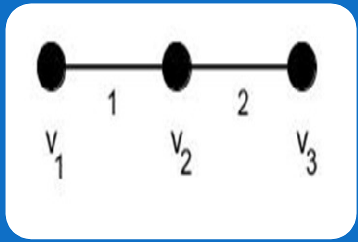


Properties of Leech Trees

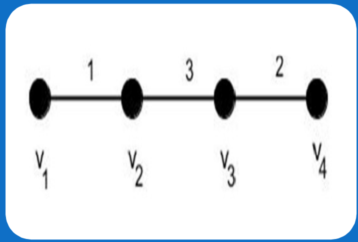
Edge of weight 1



Edge of weight 2



$^nC_2$ Path weight of a longest path



Leech Path

Path of weight ${}^{n}C_{2}$ is P_n

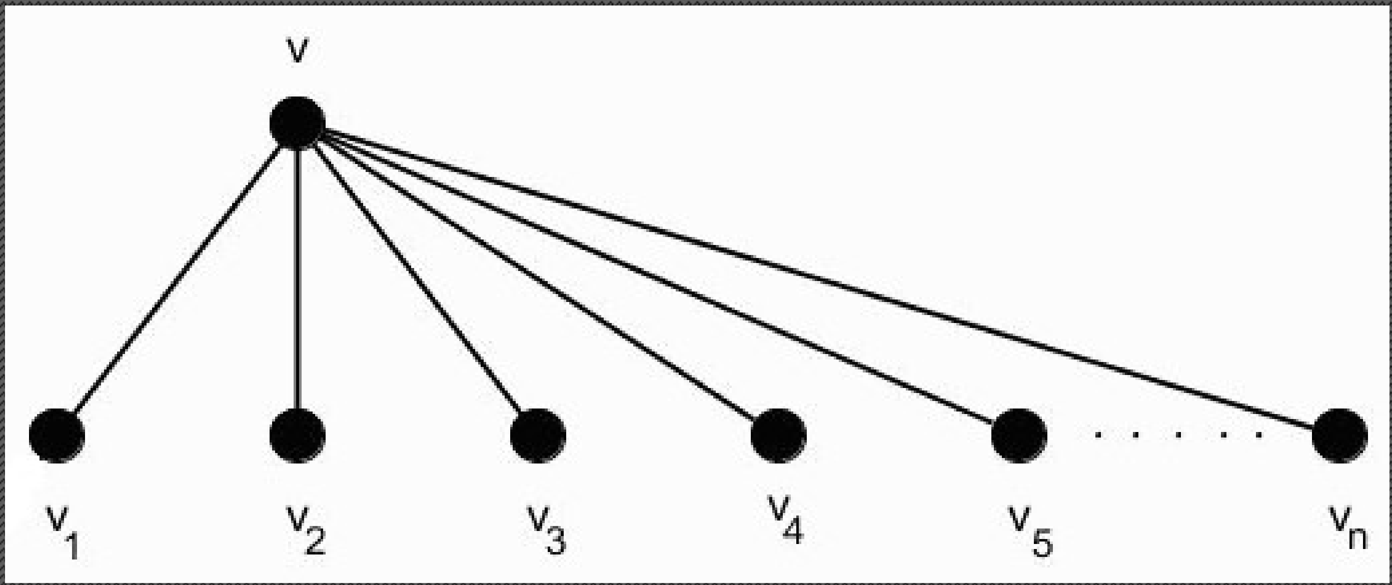
Weights of the n–1 edges → 1,2,…,n–1

Edge of weight 1 adjacent only to edge of weight n–1

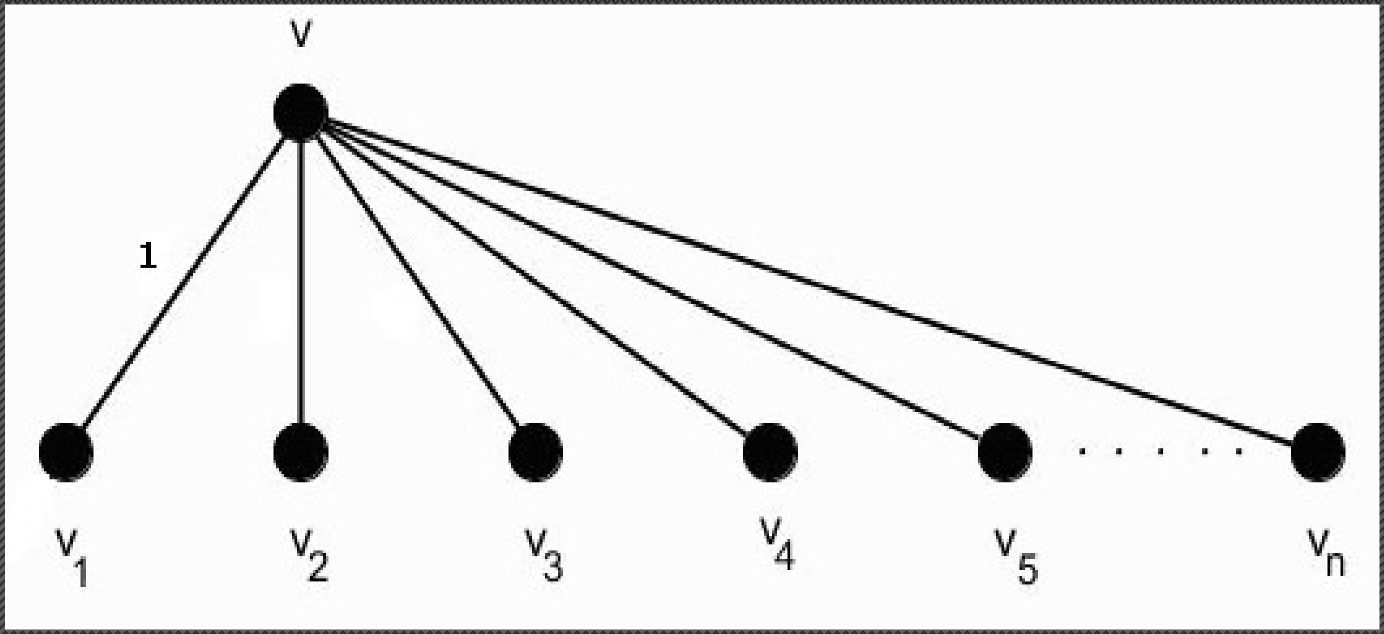
Edge of weight 2 adjacent only to edge of weight n–1

Leech path => n ≤ 4

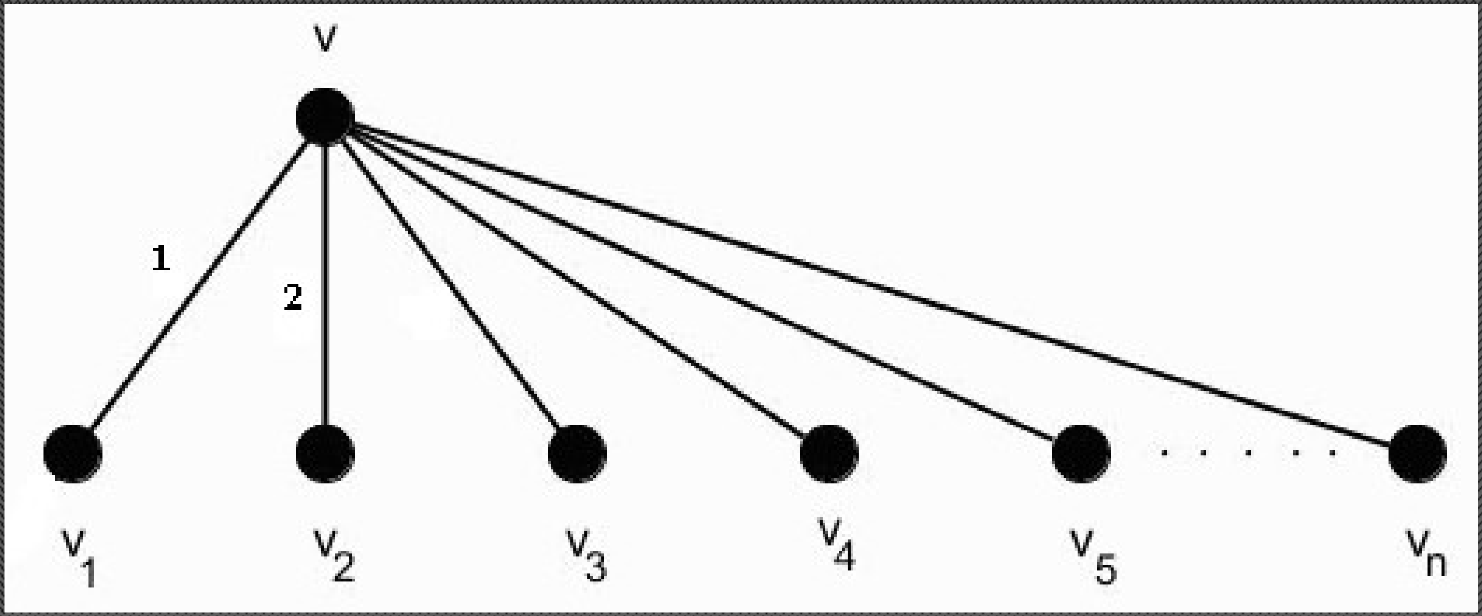
Leech Star



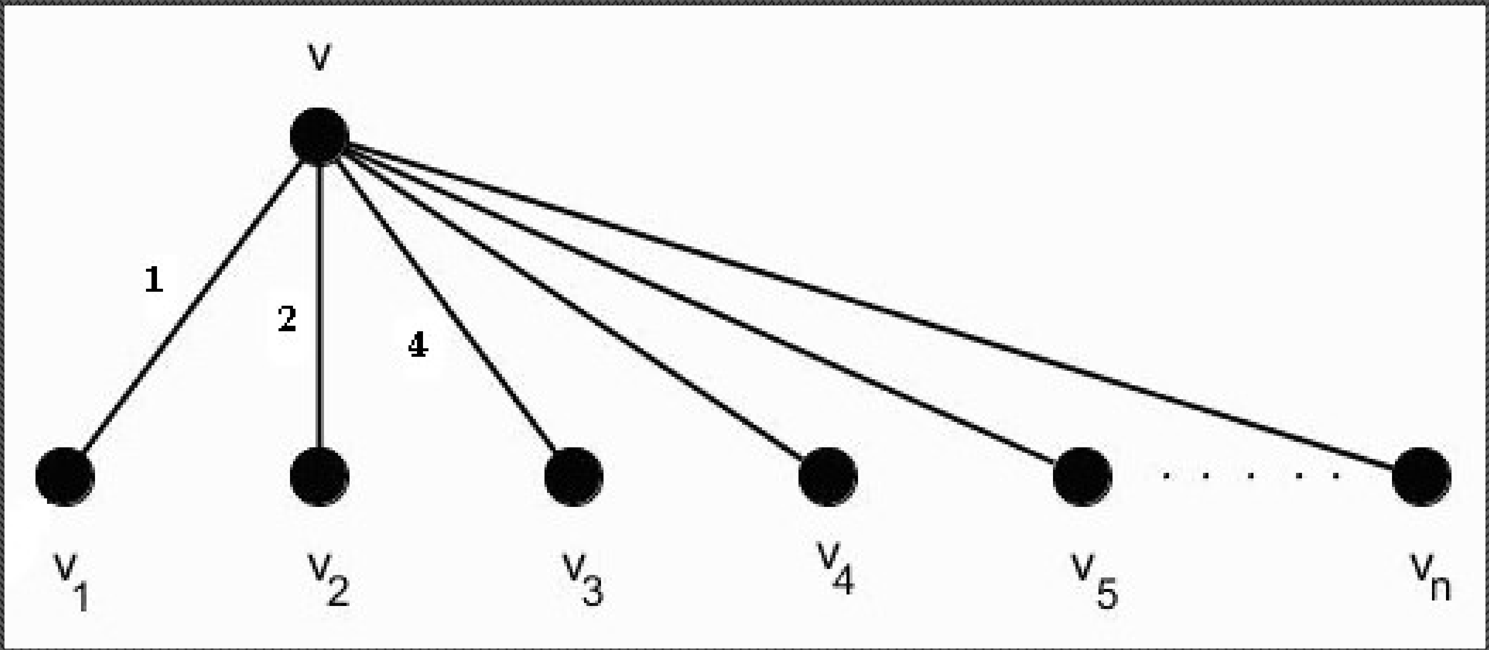
Leech Star



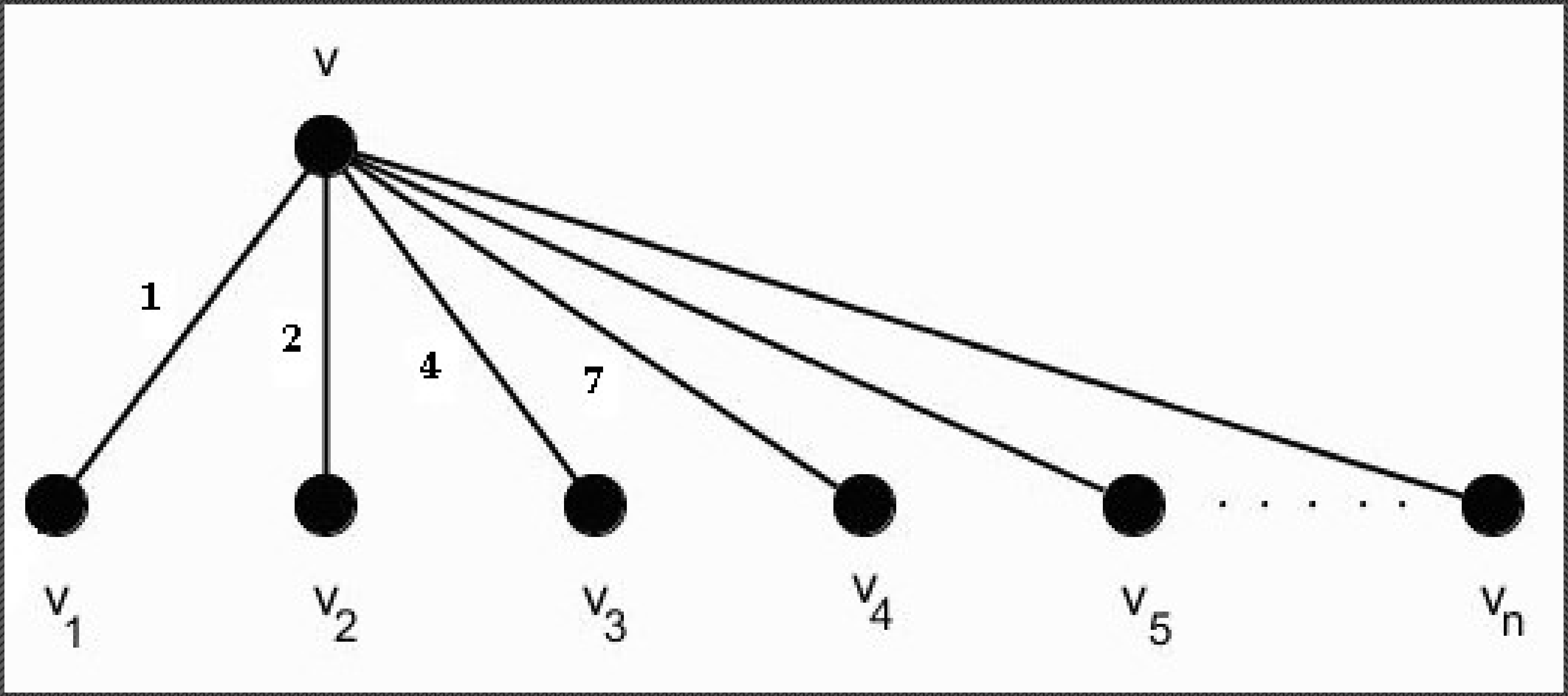
Leech Star



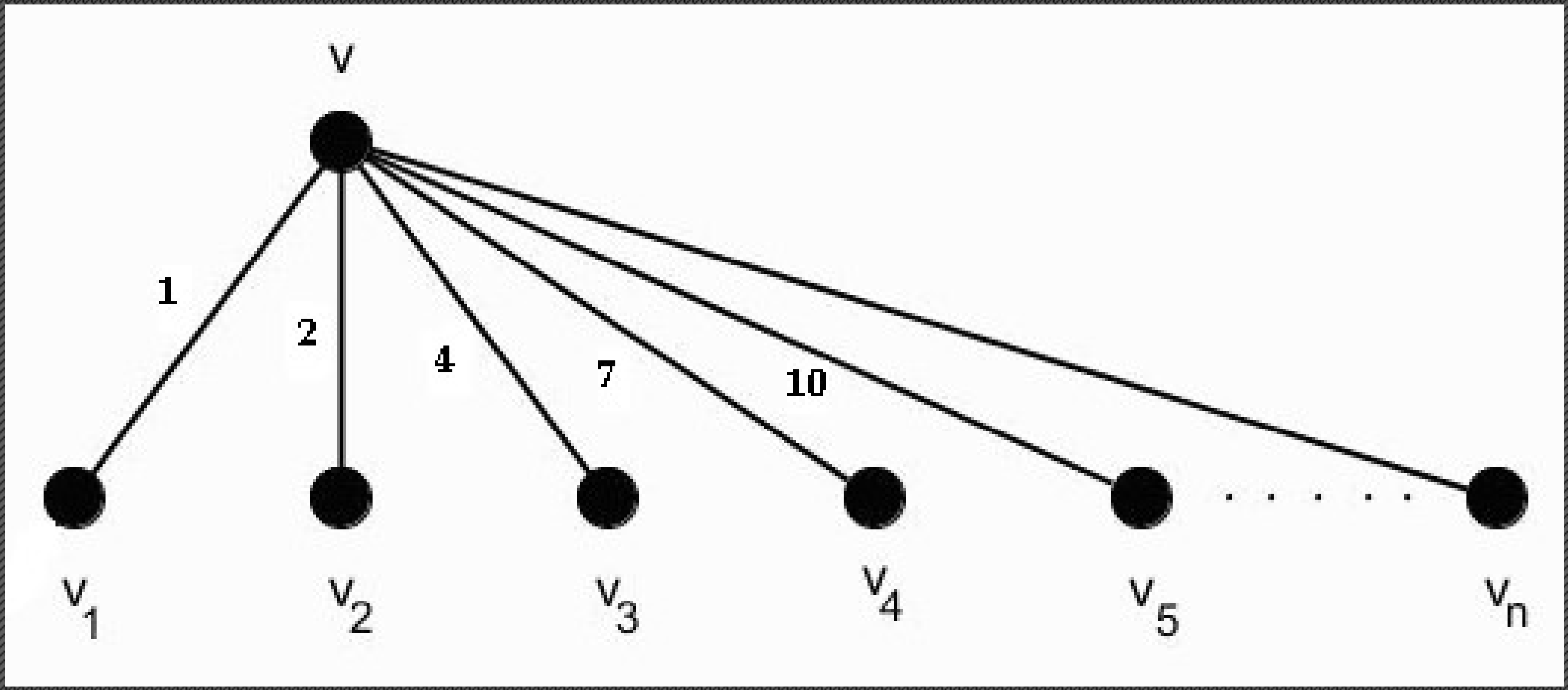
Leech Star



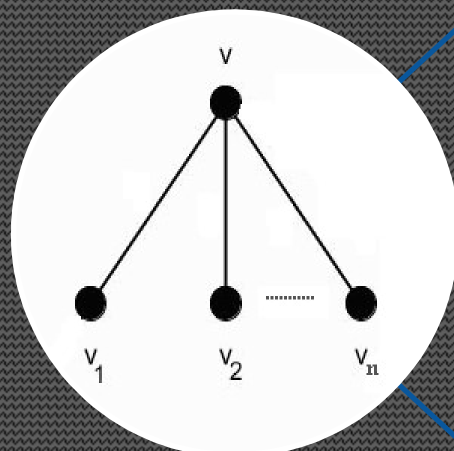
Leech Star



Leech Star



Leech Star



$K_{1,1} = P_2$

$K_{1,2} = P_3$

$K_{1,3}$

Leech Tree

H. Taylor

- Odd path sums in an edge-labeled tree
- Mathematics Magazine
- 50 (1977) (5) 258 – 259

H. Taylor

- Synchronization patterns and related problems in combinatorial analysis and graph theory
- USCEE Technical Report
- June, (1981), 6 – 8

Order of a Leech Tree

Distance between x and y, d(x,y) = Path weight of xy-path

Start with any vertex v

Color it Black

Vertices at even distance → Black

Vertices at odd distance → White

Order of a Leech Tree

B = Number of Black vertices, W = Number of White vertices

B + W = n

Path weight of xy-path is odd $\leftrightarrow$ x and y are of opposite color

Number of paths of odd weight = B.W

Leech labeling with ${}^{n}C_{2}$ even $\rightarrow$ Number of odd paths = n(n-1)/4

$(B-W)^2 = (B+W)^2 - 4BW \rightarrow (B-W)^2 = n$

Leech labeling with ${}^{n}C_{2}$ odd $\rightarrow$ Number of odd paths $= \frac{1}{2}\left(\frac{n(n-1)}{2}+1\right) = \frac{n^2-n+2}{4}$

$(B - W)^2 = n - 2$

Order of a Leech Tree

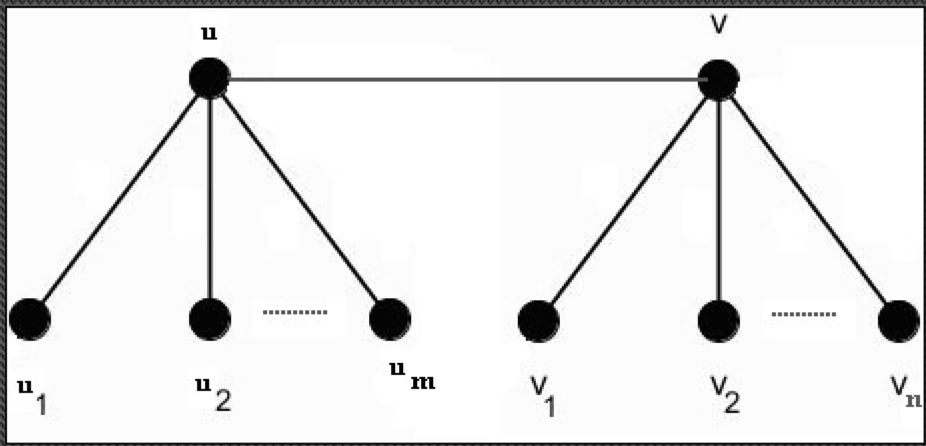
T is a Leech tree $\rightarrow$ $n = k^2$ or $k^2 + 2$

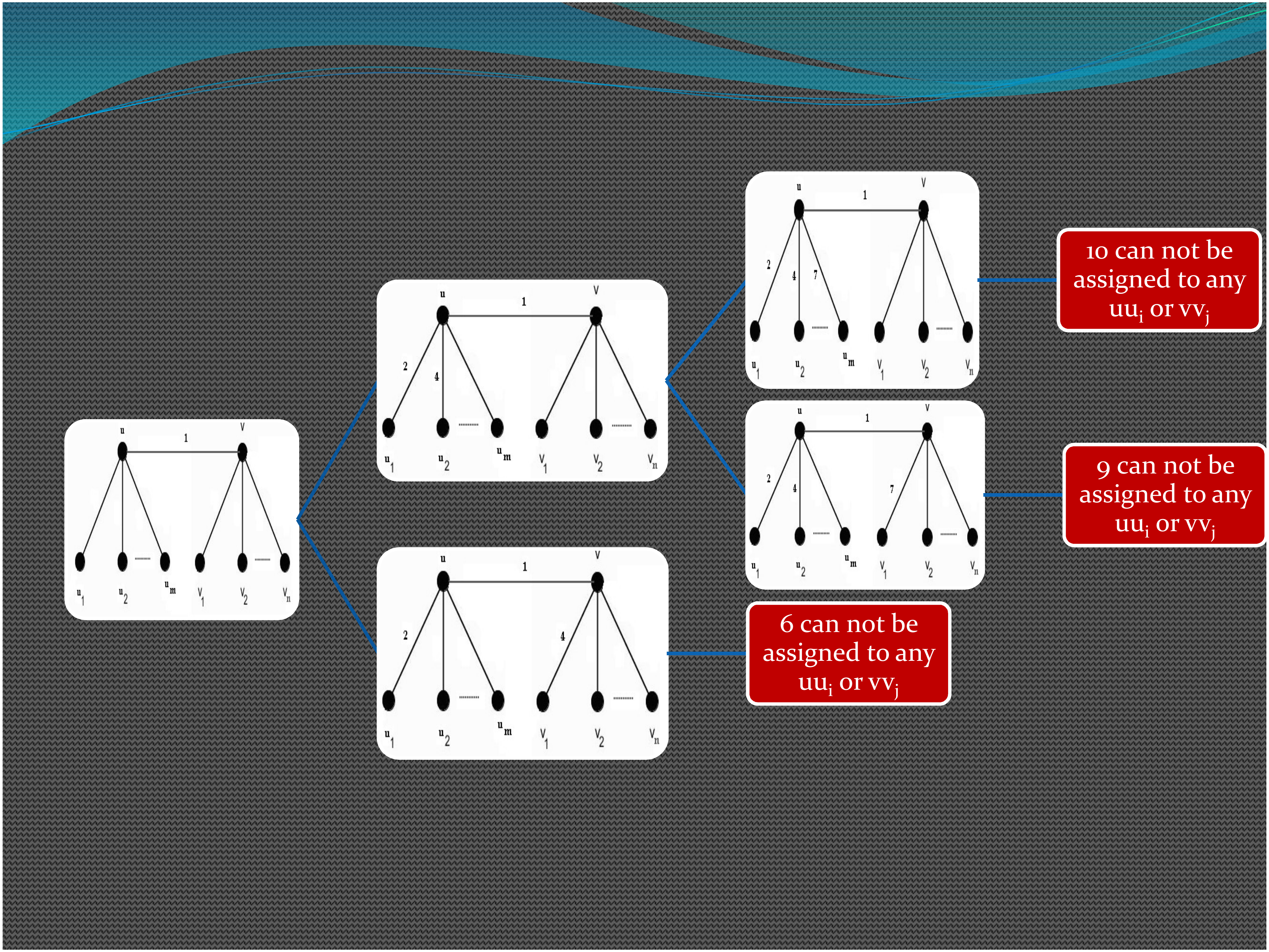
Bistar – $B_{m,n}$

Vertex set = $\{u,v,u_1,u_2,\ldots u_m,v_1,v_2,\ldots,v_n\}$

Edge set = $\{uv\}$ U $\{uu_i : i=1,2,\ldots,m\}$ U $\{vv_j : j=1,2,\ldots,n\}$

m & n are non-zero → Trees of diameter two




10 can not be assigned to any uu_i or vv_j
9 can not be assigned to any uu_i or vv_j
6 can not be assigned to any uu_i or vv_j

Leech Bistar

Similar argument for $uu_1 = 1$

Bistar Leech trees – $B_{0,0}$, $B_{1,1}$, $B_{2,2}$, $B_{1,0}$ & $B_{2,0}$

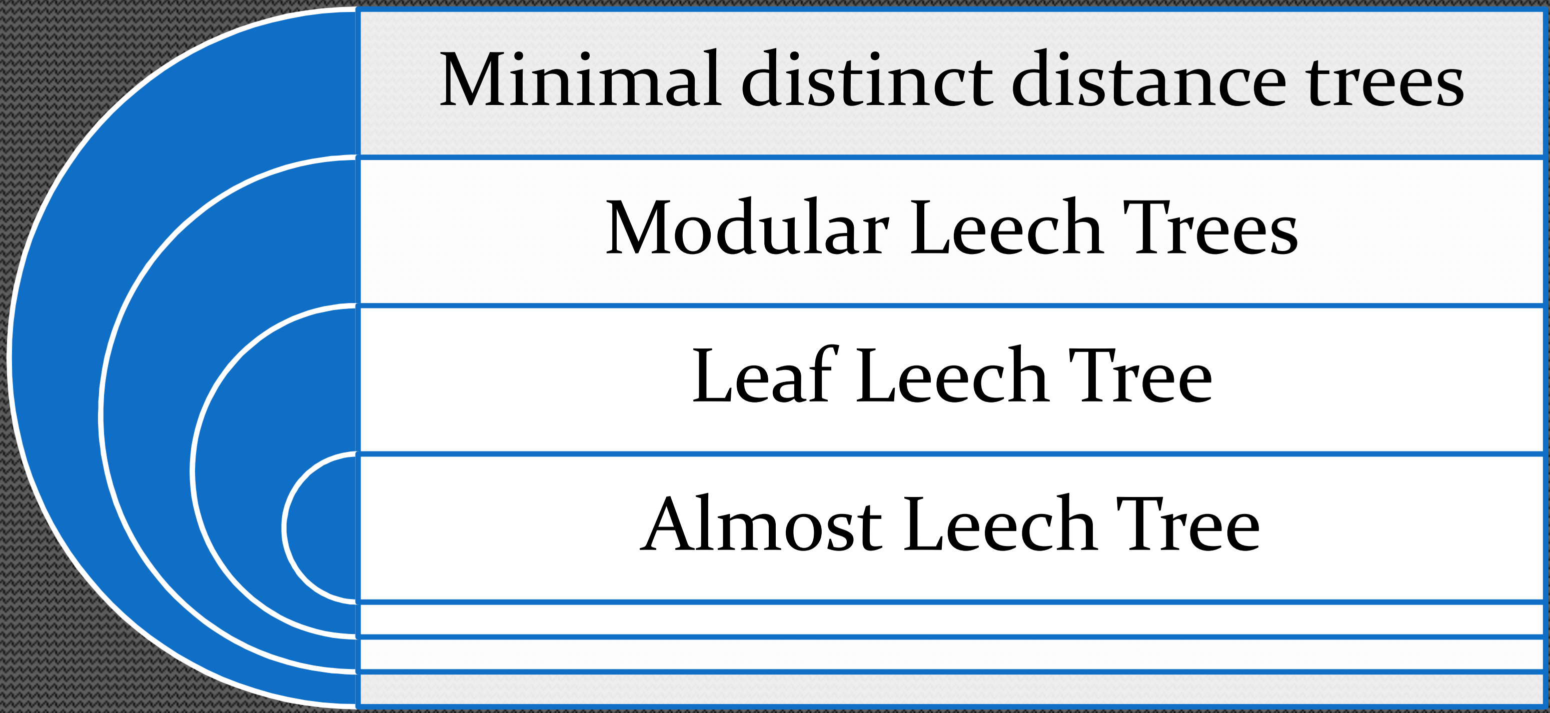
Leech Tree Conjecture

These bistars are the only Leech Trees

Non-Leech trees

Class	Proved by
Path P_n, where n > 5	J. Leech, 1975
Star $K_{1,n}$,where n > 4	
Trees with $n \neq k^2$ or k^2+2	H. Taylore, 1977
Trees with n = 9 or 11	L. Szekely, H. Wang, Y. Zhang, 2005
Bistar Bm,n, where at least one among m, n > 3	S. Arumugam, Seena Varghese, ALS (manuscript)
Path with a pendent vertex attached to a support vertex	
Tristar Bm,n,p, which are not bistars	Seena Varghese, ALS (manuscript)

Leech-like labeling



Minimal distinct distance trees

Distinct Distance Tree → Weighted distances are all distinct

Minimal Distance Distinct Tree → Minimize the maximum distance between vertices

M(n) = Maximum distance in a minimal distinct distance tree on n vertices

B. Calhoun, K. Ferland, L. Lister, and J. Polhill, Minimal distinct distance trees, JCMCC, 61 (2007), 33 - 57

Minimal distinct distance trees

- Leech trees are called as Perfect Distance Distinct Trees
- Computed $M(n)$ up to $n = 10$
- Determined a bound for $M(n)$ for $n > 10$
- Generalized to forests

Modular Leech Trees

Edge weight from Z_k

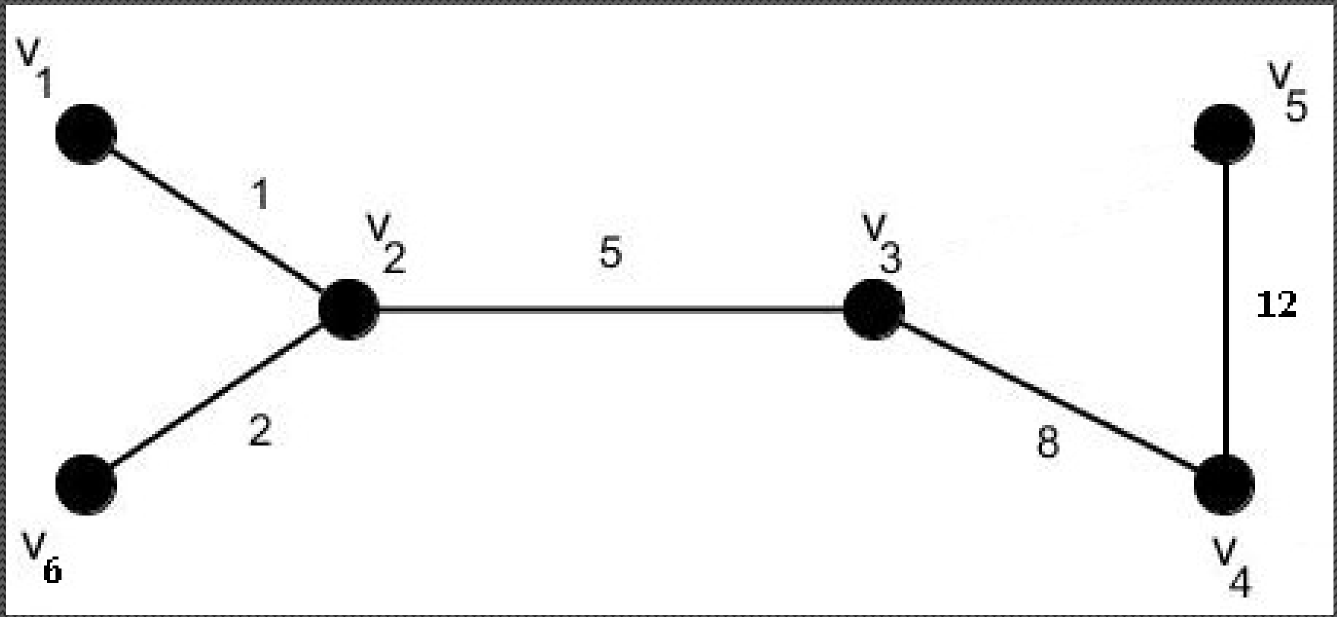
Path weight found as addition modulo k

Z_k-Leech labeling

Trees that admits Z_k-Leech labeling $\rightarrow$ Modular Leech Trees

D. Leach and M. Walsh, Generalized Leech trees, JCMCC, 78 (2011), 15-22

Example



Properties of Modular Leech Trees

Taylor's Condition : $n = k^2 + 2$ or $n \equiv 0$ or $1 \pmod 4$

Only one modular Leech Tree on $n = 8$ vertices

If T admits a modular Leech labeling, then for any edge e in G, there is a modular Leech labeling with $w(e) = 1$

Leaf Leech Tree

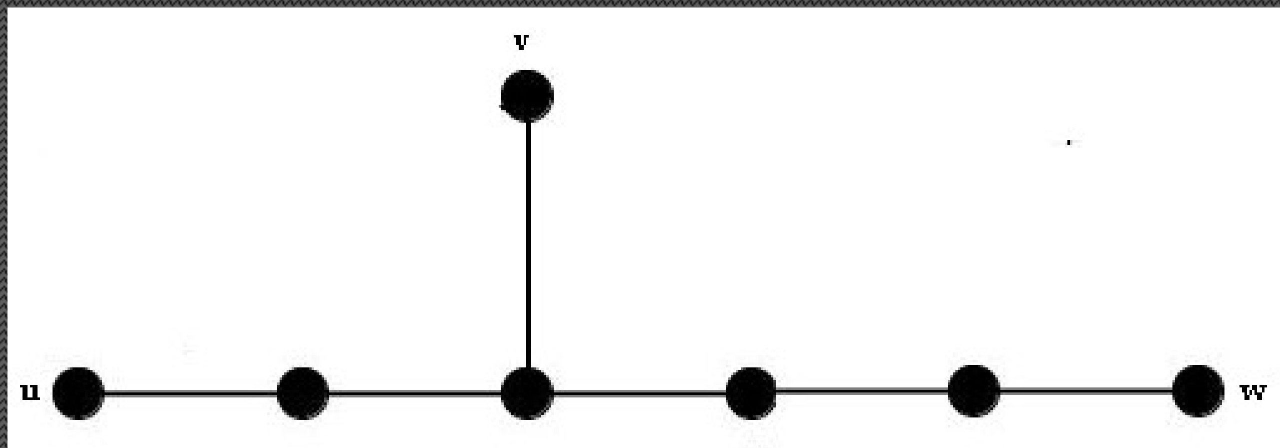
Leaf-Leech Tree: Distance between n leaves of a tree are exactly 3, 4, …, ${}^{n}C_2 + 2$

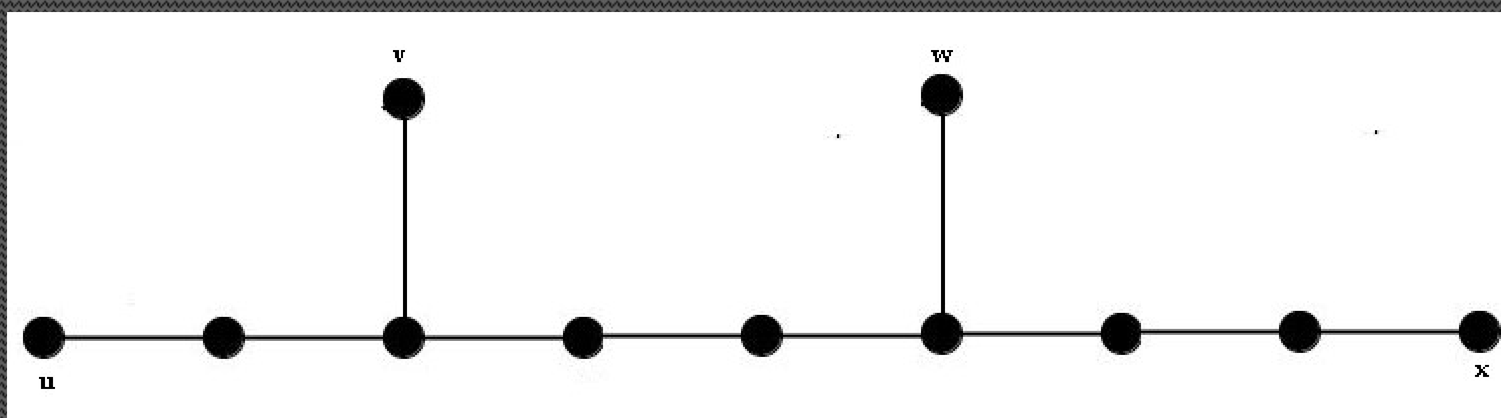
Distance between two leaves = 1 → $T = K_2$

Distance between two leaves u and v = 2 → d(u,w) = d(v,w) for every vertex w

So we consider distances from 3 onwards

Example





Properties of Leaf-Leech Trees

Taylor's condition : Number of leaves = k^2 or $k^2 + 2$

No star-like Leaf-Leach trees with more than 4 leaves

No leaf-Leach caterpillars with more than 4 leaves

There exist Leech tree on n vertices → There exist leaf-Leech tree with n leaves

Expansion of a tree

For every edge u_iu_j with weight w → subdivide u_iu_j into a path of length w

Attach a pendant vertex to every u_i

Expansion of Leech tree T, gives a leaf-Leach tree

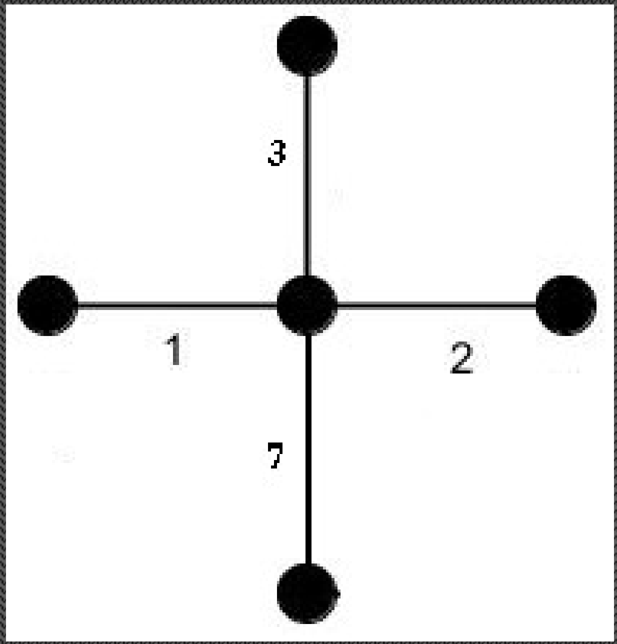
Almost Leech Tree

One path weight is missing

Obviously, one path weight will be repeated

Similarly, almost modular Leech Tree, almost leaf-Leech tree … can be defined

Example



Open problems

Does there exist any Leech tree other than the 5 known Leech trees?

Prove that there exist only finitely many Leech trees.

Determine $M(n)$ for $n > 10$

Find better bounds for $M(n)$ for $n > 10$

Identify modular Leech trees of order > 8

Does there exist any leaf-Leech tree which is not an extension of a Leech tree?


Thank You