Join of hypergraps and their spectra

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- m-Uniform Hypergraph.
- Degree of a vertex.
- Regular Hypergraph.
- Paths in hypergraphs is a sequence $v_1 e_1 v_2 e_2 \dots v_l e_l$ of distinct vertices and edges satisfying $v_i,\ v_{i+1} \in e_i$.
- Connected Hypergraph.

Adjacency Matrix

- Let $\mathcal{H}(V, E, W)$ be a hypergraph with the vertex set $V = \{1, 2, \dots, n\}$, the edge set E and with an weight function $W : E \to \mathbb{R}_{\geq 0}$ defined by $W(e) = w_e$ for all $e \in E$. The adjacency matrix $A_{\mathcal{H}}$ of $\mathcal{H}$ is defined as

$$(A_{\mathcal{H}})_{ij} := \sum_{e \in E; i,j \in e} \frac{w_e}{|e| - 1}.$$

  This definition is adopted from the definition of the adjacency matrix of an unweighted non-uniform hypergraph defined in In Press, Linear Algebra and its Applications, 2020. DOI: 10.1016/j.laa.2020.01.012.

- Thus for an m-uniform hypergraph $\mathcal{H}$ we have $(A_{\mathcal{H}})_{ij} = \dfrac{1}{m-1} \sum_{e \in E, i,j \in e} w_e$. If we take $w_e = 1$, then $(A_{\mathcal{H}})_{ij} = \dfrac{d_{ij}}{m-1}$ where d_{ij} is the codegree of the vertices i, j, i.e., the number of edges containing both the vertices i and j.

Let $\mathcal{H}(V, E, W)$ be an m-uniform weighted hypergraph with n vertices.

- We say a partition $\pi = \{C_1, C_2, \ldots, C_k\}$ of V is an equitable partition of V if for any $p, q \in \{1, 2, \ldots, k\}$ and for any $i \in C_p$,

$$\sum_{j, j \in C_q} (A_{\mathcal{H}})_{ij} = b_{pq},$$

  where b_{pq} is a constant depends only on p and q.
  For an equitable partition with k-number of cells we define the quotient matrix B as $(B)_{pq} = b_{pq}$, for $1 \leq p, q \leq k$.
- The characteristic matrix P of order $n \times k$ as follows

$$(P)_{ij} = \begin{cases} 1 & \text{if vertex i} \in C_j, \\ 0 & \text{otherwise.} \end{cases}$$

- We have $A_{\mathcal{H}}P = PB$ and so $A_{\mathcal{H}}^k P = PB^k$ for any $k \in \mathbb{N}$. Therefore $f(A_{\mathcal{H}})P = Pf(B)$, for any polynomial $f(x)$. If $f(A_{\mathcal{H}})$=0, then $Pf(B) = 0$ and which gives $f(B) = 0$. Again $A_{\mathcal{H}}$ being a real symmetric matrix, is diagonalizable. So minimal polynomial of $A_{\mathcal{H}}$ is product of linear polynomials. Hence minimal polynomial of B is also product of linear polynomials. Therefore B is also diagonalizable.
- For each $\lambda \in spec(B)$ with the multiplicity r, $\lambda \in spec(A_{\mathcal{H}})$ with the multiplicity atlest r.

Let $\mathcal{H}_1(V_1, E_1, W_1)$ and $\mathcal{H}_2(V_2, E_2, W_2)$ be two m-uniform hypergraphs. The join $\mathcal{H}(V, E, W) := \mathcal{H}_1 \oplus^{w_{12}} \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the hypergraph with the vertex set $V = V_1 \cup V_2$, edge set $E = \cup_{i=0}^{2} E_i$, where $E_0 = \{e \subseteq V : |e| = m, e \cap V_i \neq \phi, \forall i = 1, 2\}$ and the weight function $W : E \to \mathbb{R}_{\geq 0}$ is defined by

$$W(e) = \begin{cases} W_1(e) & \text{if } e \in E_1, \\ W_2(e) & \text{if } e \in E_2, \\ w_{12} & \text{otherwise,} \end{cases}$$

where w_{12} is a real non-negative constant.

Weighted joining of a set of uniform hypergraphs

Let $S = \{\mathcal{H}_i(V_i, E_i, W_i) : 1 \leq i \leq k\}$, $|V_i| = n_i$, be a set of m-uniform hypergraphs, $k \leq m$. Using the set S, of hypergraphs we construct a new m-uniform hypergraph $\mathcal{H}(V, E, W)$ where $V = \cup_{i=1}^{k} V_i$, $E = \cup_{i=0}^{k} E_i$, $E_0 = \{e \subseteq V : e \cap V_i \neq \phi, \forall i = 1, 2, \ldots, k, |e| = m\}$ and $W : E \to \mathbb{R}_{\geq 0}$ defined by

$$W(e) = \begin{cases} W_i(e) & \text{if } e \in E_i \text{ for } i = 1, 2 \ldots, k, \\ w_s & \text{otherwise,} \end{cases}$$

where w_s is a real non-negative constant. The resultant hypergraph $\mathcal{H}$ is called the join of a set S of m-uniform hypergraphs $\mathcal{H}_i$'s and it is denoted as $\mathcal{H} := \oplus_{\mathcal{H}_i \in S}^{w_s} \mathcal{H}_i$.

Adjacency matrix of the join $\mathcal{H}$

The adjacency matirx $A_{\mathcal{H}}$ of $\mathcal{H}$ can be expressed as

$$(A_{\mathcal{H}})_{ij} = \begin{cases} \left(A_{\mathcal{H}_p}\right)_{ij} + w_s d_{pp}^{S(m)} & \text{if } i,j \in V_p,\ i \neq j, \\ 0 & i = j, \\ w_s d_{pq}^{S(m)} & \text{if } i \in V_p, j \in V_q, p \neq q. \end{cases}$$

where

$$\begin{aligned} d_{pp}^{S(m)} :&= \frac{1}{m-1}(\text{number of new edges containing two fixed vertices } l_1, l_2 \text{ from } V_p) \\ &= \frac{1}{m-1} \sum_{\substack{i_p \geq 0, i_j \geq 1, (j \neq p) \\ i_1 + i_2 + \cdots + i_k = m-2}} \binom{n_1}{i_1}\binom{n_2}{i_2}\cdots\binom{n_{p-1}}{i_{p-1}}\binom{n_p-2}{i_p}\binom{n_{p+1}}{i_{p+1}}\cdots\binom{n_k}{i_k}. \end{aligned}$$

$$\begin{aligned} d_{pq}^{S(m)} &:= \frac{1}{m-1}(\text{number of new edges containing two fixed vertices one from } V_p \text{ and another from} \\ &= \frac{1}{m-1} \sum_{\substack{i_p, i_q \geq 0, i_j \geq 1, (j \neq p,q) \\ i_1 + i_2 + \cdots + i_k = m-2}} \binom{n_1}{i_1}\binom{n_2}{i_2}\cdots\binom{n_{p-1}}{i_{p-1}}\binom{n_p-1}{i_p}\binom{n_{p+1}}{i_{p+1}}\cdots \\ &\quad \binom{n_{q-1}}{i_{q-1}}\binom{n_q-1}{i_q}\binom{n_{q+1}}{i_{q+1}}\cdots\binom{n_k}{i_k}. \end{aligned}$$

Let $\mathcal{H}(V, E, W)$ be an m-uniform hyperghraph. We call the hypergraph $\mathcal{H}_b(V_b, E_b, W_b)$, $V_b = \{1, 2, \dots, n\}$ as a backbone of $\mathcal{H}$ if $\mathcal{H}$ can be constructed by a set $S = \{\mathcal{H}_i(V_i, E_i, W_i)\colon\ i = 1, 2, ..., n\}$, of m-uniform hypergraphs, $(m \geq max\{|e| : e \in E_b\})$ with the following operations :

1. Replace vertex i of $\mathcal{H}_b$ by $\mathcal{H}_i$, for $i = 1, 2, \dots, n$.
2. For each edge $e = \{j_1, j_2, \dots, j_k\} \in E_b$, take $S_e = \{\mathcal{H}_{j_i} : i = 1, 2, \dots, k\}$ and apply the operation $\oplus_{S_e}^{W_b(e)}$ defined above.

We call the hypergraphs $\mathcal{H}_i$'s as participants on the backbone $\mathcal{H}_b$ to form the hypergraph $\mathcal{H}$. Thus the adjacency matrix for $\mathcal{H}$ can be written as

$$(A_{\mathcal{H}})_{ij} = \begin{cases} (A_{\mathcal{H}_p})_{ij} + \sum\limits_{e \in E_b, p \in e} W_b(e) d_{pp}^{S_e(m)} & \text{if } i, j \in V_p,\ i \neq j, \\ 0 & \text{if } i = j, \\ \sum\limits_{e \in E_b, p, q \in e} W_b(e) d_{pq}^{S_e(m)} & \text{if } i \in V_p, j \in V_q, p \neq q. \end{cases}$$

- A matrix A is called reducible if there exists a permutation matrix P such that

$$PAP^t = \begin{bmatrix} (B)_{k\times k} & C \\ 0 & (D)_{(n-k)\times(n-k)} \end{bmatrix}$$

  otherwise A is said to be irreducible.
- $\mathcal{H}$ is connected iff $A_{\mathcal{H}}$ is irreducible.

Perron-Frobenius Theorem Let A be a non-negative irreducible matrix. Then

- A has a positive eigenvalue λ with positive eigenvector.
- λ is simple and for any other eigenvalue μ of A, $|\mu| \leq \lambda$.

Schur complement

Let A be an $n \times n$ matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices. If A_{11} and A_{22} are invertible, then

$$det(A) = det(A_{11})det(A_{22} - A_{12}A_{11}^{-1}A_{21}) \quad (1)$$

$$= det(A_{22})det(A_{11} - A_{21}A_{11}^{-1}A_{12}). \quad (2)$$

Theorem

Let $\mathcal{H}_b(V_b, E_b, W_b)$ be a hypergraph with the vertex set $V_b = \{1, 2, \ldots, n\}$ and let $\{\mathcal{H}_i(V_i, E_i, W_i) : i = 1, 2, \ldots, n\}$ be a collection of regular m-uniform hypergraphs $(m \geq \{|e| : e \in E_b\})$. Let $\mathcal{H}(V, E, W)$ be the m-uniform hypergraph constructed by taking $\mathcal{H}_b$ as backbone hypergraph and $\mathcal{H}_i$'s as participants. Then for any non-Perron eigenvalue λ of $A_{\mathcal{H}_p}$ with multiplicity l, $\lambda - \sum\limits_{e \in E_b, p \in e} W_b(e) d_{pp}^{S_e(m)}$ is an eigenvalue of $A_{\mathcal{H}}$ with the multiplicity atleast l.

Proof.

Let (λ, f) be an eigenpair of $A_{\mathcal{H}_p}$, such that f is orthogonal to the constant vector $[1, 1, 1, \ldots, 1]^t$. We define $f^* : V \to \mathbb{R}$ by

$$f^*(v) = \begin{cases} f(v) & \text{if } v \in V_p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus f^* is an eigenvector of $A_{\mathcal{H}}$ corresponding to the eigenvalue $\lambda - \sum\limits_{e \in E_b, p \in e} w_e d_{pp}^{S_e(m)}$. Since $\sum_{i \in V_p} f(i) = 0$, thus the proof folows. □

When $\mathcal{H}_i(V_i, E_i, W_i)$'s are regular, the partition $\pi = \{V_1, V_2, \ldots, V_n\}$ forms an equitable partition for $\mathcal{H}$. In particular if $\mathcal{H}_i$'s are r_i regular then the quotient matrix B is as follows

$$(B)_{pq} = \begin{cases} r_p + (n_p - 1) \sum\limits_{e \in E_b, p \in e} W_b(e) d_{pp}^{S_e(m)} & \text{if } p = q, \\ n_q \sum\limits_{e \in E_b, p, q \in e} w_b(e) d_{pq}^{S_e(m)} & \text{otherwise.} \end{cases} \tag{3}$$

Let $\{g_i | i = 1, 2, \ldots, n\}$ be a set of linearly independent eigenvectors of B. Then $\{Pg_i | i = 1, 2, \ldots, n\}$ is also a set of linearly independent eigenvectors of $A_{\mathcal{H}}$. Now from the proof of Theorem 1 we have $N := \sum\limits_{i=1}^{n} n_i - n$ linearly independend eigenvectors $\{f_i^* | i = 1, 2, \ldots, N\}$ of $A_{\mathcal{H}}$. So we have a set $\{Pg_i | i = 1, 2, \ldots, n\} \cup \{f_i^* | i = 1, 2, \ldots, N\}$ of $\sum\limits_{i=1}^{n} n_i$ eigenvectors of $A_{\mathcal{H}}$. Now we show that this set is linearly independent. Here for all i, j

$$\begin{aligned} < f_i^*, Pg_j > &= \sum_{p=1}^{n} (\sum_{k \in V_p} f_i^*(k)) C_{j_p} \quad [\because Pg_j(k) = C_{j_p}, \text{ constant for all } k \in V_p.] \\ &= 0 \quad [\because \sum_{k \in V_p} f_i^*(k) = 0.] \end{aligned}$$

Therefore in Theorem 1 the remaining n eigenvalues of $A_{\mathcal{H}}$ can be obtained from B.

Corollary

Let $S = \{\mathcal{H}_i(V_i, E_i, W_i) : 1 \leq i \leq k \leq m\}$ be a set of m-uniform hypergraphs where $\mathcal{H}_i$ are r_i-regular. Let $\mathcal{H} = \oplus_{\mathcal{H}_i \in S}^{w_s} \mathcal{H}_i$. Then for any non-perron eigenvalue λ of $A_{\mathcal{H}_i}$ with the multiplicity l, $\lambda - w_s d_{ii}^{S(m)}$ is an eigenvalue of $A_{\mathcal{H}}$ with multiplicity atleast l.

Note that the remaining k eigenvalues can be obtained from the quotient martrix B defined as

$$(B)_{pq} = \begin{cases} r_p + (n_p - 1) w_s d_{pp}^{S(m)} & \text{if } p = q, \\ n_q w_s d_{pq}^{S(m)} & \text{otherwise.} \end{cases}$$

Example

Take $\mathcal{H}_i = \overline{K}_{n_i}^m$, $1 \leq i \leq k \leq m$, $S = \{\mathcal{H}_i : i = 1, 2, \ldots, k\}$ and $w_s = 1$. Then $\oplus_{\mathcal{H}_i \in S}^{\mathbf{1}} \mathcal{H}_i = K_{n_1, n_2, \ldots, n_k}^m$, which is the weak m-uniform k-partite complete hypergraph. Using the above corollary we get that for any $i = 1, 2, \ldots, k$; $-d_{pp}^{S(m)}$ is an eigenvalue of $A_{K_{n_1, n_2 \ldots, n_k}^m}$ with the multiplicity atleast $(n_i - 1)$ for $i = 1, 2, \ldots, k$. The remaining eigenvalues of $A_{K_{n_1, n_2, \ldots, n_k}^m}$ are the eigenvalues of the quotient matrix B, defined as

$$(B)_{pq} = \begin{cases} (n_p - 1) d_{pp}^{S(m)} & \text{if } p = q, \\ n_q d_{pq}^{S(m)} & \text{otherwise.} \end{cases}$$

- In the above example, if we take $k = m$, we have 0 as an eigenvalue of $A_{K^m_{n_1,n_2,\ldots,n_m}}$ with the multiplicity atleast $\sum\limits_{i=1}^{m} n_i - m$. Here, the quotient matrix formed by the equitable partition $\pi = \{V_1, V_2, \ldots, V_m\}$ and which is given by

$$B = \frac{1}{m-1}\begin{bmatrix} 0 & s_1 & \cdots & s_1 & s_1 \\ s_2 & 0 & \cdots & s_2 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_m & s_m & \cdots & s_m & 0 \end{bmatrix},$$

where $s_i = \prod\limits_{j=1, j\neq i}^{m} n_j$.

- Note that $\alpha(\neq 0) \in spec(K^m_{n_1,n_2,\ldots,n_m})$ if and only if $r^{m-1}\alpha \in spec(K^m_{rn_1,rn_2,\ldots,rn_m})$ for $r \in \mathbb{N}$.

- Let $\mathcal{H} = K^m_{\underbrace{n_1, n_1, \ldots, n_1}_{l_1}, \underbrace{n_2, n_2, \ldots, n_2}_{l_2}}$, where $l_1 + l_2 = m$. Then the quotient matrix B for the equitable partition formed by the m-parts of $\mathcal{H}$ can be written as $B = \frac{1}{m-1}B'$, where B' is given by

$$\begin{bmatrix} s_1(J_{l_1} - I_{l_1}) & s_1 J_{l_1 \times l_2} \\ s_2 J_{l_2 \times l_1} & s_2(J_{l_2} - I_{l_2}) \end{bmatrix},$$

where $s_1 = n_1^{l_1 - 1} n_2^{l_2}$, $s_2 = n_1^{l_1} n_2^{l_2 - 1}$.

- We have the characteristic polynomial of B' as follows

$$\begin{aligned} f_{B'}(x) &= det(B' - xI) \\ &= det\Big(s_2 J_{l_2} - (s_2 - x)I\Big) det\Big(s_1 J_{l_1} - (s_1 + x)I - s_1 s_2 J_{l_1 \times l_2}(s_2 J_{l_2} - (s_2 + x)I)^{-1} J_{l_2 \times l_1}\Big) \\ &= (-1)^{m-2}(x + s_1)^{l_1 - 1}(x + s_2)^{l_2 - 1}(x - a^+)(x - a^-), \end{aligned}$$

where $a^{\pm} = \frac{1}{2}\Big[s_1(l_1 - 1) + s_2(l_2 - 1) \pm \sqrt{\{s_1(l_1 - 1) + s_2(l_2 - 1)\}^2 + 4 s_1 s_2 (l_1 + l_2 - 1)}\Big]$.

Thus the eigenvalues of $\mathcal{H}$ are $\frac{-s_i}{m-1}$ with the multiplicity $l_i - 1$ for $i = 1, 2$ and $\frac{a^{\pm}}{m-1}$ with the multiplicity 1

Using a result (1) from R. B. Bapat, M. Karimi. Integral complete multipartite graphs. Linear Algebra and its Applications, 549: 1-11, 2018 we have the following result

Proposition

Characteristic polynomial of $K^m_{n_1,n_2,\ldots,n_m}$ *is* $x^{n-m}\left(x^m - \sum_{i=2}^{m} \frac{i-1}{(m-1)^i}\sigma_i(s_1,s_2,\ldots,s_m)x^{m-i}\right)$

where $n = \sum_{i=1}^{m} n_i$, $s_i = \prod_{j=1, j\neq i}^{m} n_j$ *and* $\sigma_i(s_1,s_2,\ldots,s_m) = \sum_{1\leq j_1<j_2<\cdots<j_i\leq m} s_{j_1}s_{j_2}\ldots s_{j_i}$.

- From the above result it is clear that the quotient matrix B is non-singular. Hence the multiplicity of eigenvalue 0 of $K^m_{n_1,n_2,\ldots,n_m}$ is $n-m$.

Let $\mathcal{H}_0(V_0, E_0)$ be an m-uniform hypergraph with the edge set $E_0 = \{e_1, e_2, \dots, e_k\}$ and $|V_0| = n_0$. Also let $\mathcal{H}_i(V_i, E_i)$ $(i = 1, 2, \dots, k)$ be m-uniform hypergraphs. For each $i = 1, 2, \dots, k$ we consider $e_i \oplus \mathcal{H}_i$. The new hypergraph is known as the edge corona of hypergraphs and we write it by $\mathcal{H} = \mathcal{H}_0 \square^k \mathcal{H}_i$. When $|V_i| = n_1$ for all $i = 1, \dots, k$, we write $D_i = A_{\mathcal{H}_i} + c(J_{n_1} - I_{n_1})$, take $D = diag(D_1, D_2, \dots, D_k)$, $R =$ vertex-edge incidence matrix, for $\mathcal{H}_0$, $1_{n_1} = [1, 1, \dots, 1]$ row vector of length n_1, $a = \binom{m+n_1-2}{m-2} - 1$,
$b = \dfrac{1}{m-1}\binom{n+n_1-2}{m-2}$, $c = \binom{m+n_1-2}{m-2} - \binom{n_1-2}{m-2}$.

Theorem

Let $\mathcal{H}_0(V_0, E_0)$ be an m-uniform hypergraph with $|V_0| = n$, $|E_0| = k$. Let $\{\mathcal{H}_i(V_i, E_i) : 1 \le i \le k, |V_i| = n_1\}$ be a set of m-uniform hypergraphs. Then the the characteristic polynomial of $A_{\mathcal{H}}$ for the edge corona $\mathcal{H} = \mathcal{H}_0 \square^k \mathcal{H}_i$ is as follows

$$f_{\mathcal{H}}(x) = det(D - xI_{kn_1})det(\{(a+1)A_{\mathcal{H}_0} - xI_n - b^2(R \otimes 1_{n_1})(D - xI_{kn_1})^{-1}(R^T \otimes 1_{n_1}^T)\}). \quad (4)$$

Corollary

Let $\mathcal{H}_i$'s be r_1-regular hypergraphs and $spec(A_{\mathcal{H}_i}) = \{\lambda_i^{(1)}, \lambda_i^{(2)}, \ldots, \lambda_i^{(n_1)}(= r_1)\}$. Then the characteristic polynomial of $\mathcal{H}$ can be given by

$$f_{\mathcal{H}}(x) = \{r_1 + (n_1 - 1)c - x\}^{k-n} det\Big(\beta_1(x)A_{\mathcal{H}_0} - b^2 n_1 D_d + \beta_2(x)I_n\Big) \prod_{i=1}^{k} \prod_{j=1}^{n_1-1} (\lambda_i^{(j)} - c - x), \tag{5}$$

where $D_d = diag(d_1, d_2, \ldots, d_n)$ where d_i denote the degree of vertices of $A_{\mathcal{H}_0}$, $\beta_1(x) = (a+1)\{r_1 + (n_1 - 1)c - x\} - (m-1)b^2 n_1$, $\beta_2(x) = x\{x - r_1 - (n_1 - 1)c\}$.

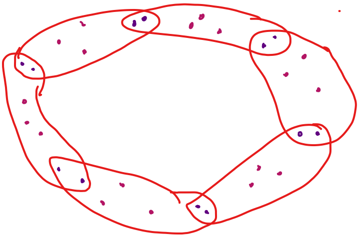
Corollary

Let $\mathcal{H}_i$'s be the hypergraphs mentioned in the above corollary and $\mathcal{H}_0$ be r-regular with $spec(A_{\mathcal{H}_0}) = \{\mu_1, \mu_2, \ldots, \mu_n(= r)\}$. Then the adjacency eigenvalues of $\mathcal{H}$ are $r_1 + (n_1 - 1)c$ with the multiplicity $k - n$, $\lambda_i^{(j)}$ with the multiplicity one, for all $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_1 - 1$ and $\beta_j^{\pm}$ with the multiplicity one for $j = 1, 2, \ldots, n$, where

$$\beta_j^{\pm} = \frac{1}{2}\left[r_1 + (n_1 - 1)c + (a+1)\mu_j \pm \sqrt{\{r_1 + (n_1 - 1)c - (a+1)\mu_j\}^2 + 4b^2 n_1\{(m-1)\mu_j + r\}}\right].$$

Generalized s-loose path $P^{(m)}_{L(s;n)}$ is an m-uniform hypergraph with the vertex set $V = \{1, 2, \ldots, m + (n-1)(m-1)\}$ and edge set the $E = \{\{i(m-s)+1, i(m-s)+2, \ldots, i(m-s)+m\} : i = 0, 1, \ldots, n-1\}$ **[Peng2016]**. For $s = 1$, $P^{(m)}_{L(1;n)}$ is known as loose path. Similarly, generalized s-loose cycle $C^{m}_{L(s;n)}$ is an m-uniform hypergraph with the vertex set $V = \{1, 2, \ldots, n(m-s)\}$ and the edge set $\{\{i(m-s)+1, \ldots, i(m-s)+m\} : i = 0, 1, \ldots, n-1\} \cup \{n(m-s)-s+1, n(m-s)-s+2, \ldots, n(m-s), 1, 2, \ldots, s\}$.

Loose cycle



$s=2$, $n=6$,

$m=7$

$C^{(7)}_{L(2;6)}$

Theorem

The adjacency eigenvalues of an s-loose cycle $C^{(m)}_{L(s;n)}$, *are*

1. $\dfrac{-2}{2s-1}$ *with the multiplicity* $n(s-1)$ *and* $\dfrac{2}{2s-1}(s-1+s\cos\dfrac{2\pi i}{n})$ *with the multiplicity one, for* $i = 1, 2, \ldots, n$, *when* $m = 2s$ *and*
2. $\dfrac{-1}{m-1}$ *with the multiplicity atleast* $n(m-2s-1)$, $\dfrac{-2}{m-1}$ *with the multiplicity atleast* $n(s-1)$ *and* γ_i^+, γ_i^- *with the multiplicity atleast one, where,*

$$\gamma_i^{\pm} = \frac{1}{2}\left[m-3+2s\cos\frac{2\pi i}{n} \pm \sqrt{(m-3+2s\cos\frac{2\pi i}{n})^2 + 8(m-s-1+s\cos\frac{2\pi i}{n})}\right],$$

for $i = 1, 2, \ldots, n$, *when* $m \geq 2s+1$.

Proof.

- **Case** $m = 2s$: Let $\mathcal{H} = C^{(2s)}_{L(s;n)}$. In the Theorem 1 take $\mathcal{H}_b = C_n$, the cycle graph over n vertices, and $\mathcal{H}_i = K_s$, the complete graph with s-vertices with each edge weight two. Then the resultant hypergraph is a graph, G(say). Hence $A_{\mathcal{H}} = \frac{1}{2s-1}A_G$. Thus $\frac{-2}{2s-1}$ is an eigenvalue of $A_{\mathcal{H}}$ with the multiplicity atleast $n(s-1)$. The quotient matrix is $B = \frac{1}{2s-1}\{sA_{C_n} + 2(s-1)I_n\}$. The remaining eigenvalues of $A_{\mathcal{H}}$ are $\frac{2}{2s-1}(s-1+s\cos\frac{2\pi i}{n})$ for $i = 1, 2, \ldots, n$.
- **Case** $m \geq 2s+1$: Let $\mathcal{H}_b = C_n \square^n K_1$ and $\mathcal{H} = C^{(m)}_{L(s;n)}$. We take the vertices of $\mathcal{H}_b$ as $V(C_n) = \{1, 2, \ldots, n\}$ and $V(G_b) \backslash V(C_n) = \{n+1, n+2, \ldots, 2n\}$. For $i = 1, 2, \ldots, n$, we take $\mathcal{H}_i = K_s$, the complete graph with s vertices with edge weight 1, and for $i = n+1, \ldots, 2n$, take $G_i = K_{m-2s}$ with edge weight 2. Considering $\mathcal{H}_b$ as backbone graph with each edge weight one and G_i's as participants, we get a graph G (say). Then $A_{\mathcal{H}} = \frac{1}{m-1}A_G$. Now using the Theorem 1 we get the eigenvalues of A_G which are -1 with the multiplicity atleast $n(m-2s-1)$ and -2 with multiplicity $n(s-1)$.

$\square$

Proof.

Next using Equation (3) we have the remaining $2n$ eigenvalues are the eigenvalues of the quotient matrix B given by

$$(B)_{pq} = \frac{1}{m-1}\begin{cases} r_q & \text{if } p = q, \\ n_q & \text{if } p \sim q \text{ in } G_B, \\ 0 & \text{otherwise,} \end{cases}$$

where $r_q = 2s-2, n_q = s$ for $q = 1, 2, \ldots, n$ and $r_q = m-2s-1, n_q = m-2s$ for $q = n+1, \ldots, 2n$. We write

$$B = \frac{1}{m-1}\begin{bmatrix} (2s-2)I_n + sA_{C_n} & t(I_n+Y) \\ s(I_n+Y^t) & (t-1)I_n \end{bmatrix},$$

where $t = m-2s$, A_{C_n} is the adjacency matrix of an n-cycle C_n and Y is the $n \times n$ circulant matrix with the first row $[0, 0, \ldots, 1]$. We suppose $B = \dfrac{1}{m-1}B'$. Then using Lemma 9 and the fact $(I_n+Y)(I_n+Y^t) = 2I_n + A_{C_n}$ we have the cahracteristic polynomial of B', as follows

$$\begin{aligned} f_{B'}(x) &= det(B' - xI_n) \\ &= det(\{(t-1-x)I_n\})det(\{sA_{C_n} + (2s-2-x)I_n - \frac{st}{t-1-x}(I_n+Y)(I_n+Y^t)\}) \\ &= det(\{x^2 - (2s+t-3)x - (2s+2t-2)\}I_n - (s+sx)A_{C_n}). \end{aligned} \tag{6}$$

$\square$

The eigenvalues of A_{C_n} are $\mu_i = 2\cos\frac{2\pi i}{n}$, $i = 1, 2 \ldots, n$. Thus from the Equation (6) we have

$$f_{B'}(x) = \prod_{i=1}^{n}\{x^2 - (2s + t - 3 + s\mu_i)x - (2s + 2t - 2 + s\mu_i)\}$$
$$= \prod_{i=1}^{n}(x - \gamma_i^+)(x - \gamma_i^-). \quad (7)$$

Question: What are the adjacency eigenvalues of $C^m_{L(s;n)}$ for $m \leq 2s - 1$?

Lemma

For a square matrix A we have

$$det(A + \sum_{i=1,n} u_{ii}E_{ii}) = det(A) + \sum_{i=1,n} u_{ii}det(A(i|i)) + u_{11}u_{nn}det(A(1, n|1, n)), \quad (8)$$

where $A(i|j)$ is the matrix obtained from A by deleting the i-th row and j-th column, respectively, and $E_{i,j}$ is the matrix with 1 in (i, j)-th position and zero elsewhere.

Theorem

The adjacency eigenvalues of an s-loose path $P^m_{L(s,n)}$ *are*

1. $\dfrac{-1}{m-1}$ *with the multiplicity atleast* $n(m-1)-2s(n-1)$, $\dfrac{-2}{m-1}$ *with the multiplicity atleast* $(n-1)(s-1)$ *and* $\dfrac{\alpha_i}{m-1}$ *with the multiplicity one, for* $i=1,2,\dots,2n-1$, *where* α_i*'s are the zeros of the polynomial*
$$\frac{(m-s-1-x)^2 f_1(x) + 2s^2(1+x)(m-s-1-x)f_2(x) + s^4(1+x)^2 f_3(x)}{m-2s-1-x},$$
where
$$f_j(x) = \prod_{i=1}^{n-j}\{x^2-(m-3+2s\cos\frac{\pi i}{n-j+1})x-2(m-s-1-x+s\cos\frac{\pi i}{n-j+1})\}.$$
for $j=1,2,3$, *when* $m\geq 2s+1$ *and*
2. $\dfrac{-1}{m-1}$ *with the multiplicity* $2(s-1)$, $\dfrac{-2}{m-1}$ *with multiplicity* $(n-1)(s-1)$ *and* $\dfrac{\beta_i}{2s-1}$ *with the multiplicity one, where* β_i *are the zeros of the polynomial* $(x-s+1)^2t_1(x)+2s^2(x-s+1)t_2(x)+s^4t_3(x)$, *where* $t_j(x)=\prod_{i=1}^{n-j}(2s-2-x+2s\cos\dfrac{\pi i}{n-j+1})$, *for* $j=1,2,3$ *when* $m=2s$.

- Let $\mathcal{H}(V,E)$ be an m-uniform hypergraph. A subhypergraph induced by $V^{'} \subset V$ is the hypergraph $\mathcal{H}[V^{'}]$ with the vertex set $V^{'}$ and edge set $E^{'} = \{e : e \in E, e \subset V^{'}\}$. Now for $V^{'} \subset V$ and a hypergraph $\mathcal{H}''(V'', E'')$, we denote $V^{'} \oplus \mathcal{H}^{''}$ as the hypergraph with the vertex set $V^{'} \cup V^{''}$ and edge set $E(\mathcal{H}[V^{'}] \oplus \mathcal{H}^{''}) \cup E(\mathcal{H})$.
- Let $\mathcal{H}_0(V_0, E_0)$ be an m-uniform hypergraph and $\pi = \{V_0^{(1)}, V_0^{(2)}, \ldots, V_0^{(k)}\}$ be a partition of $V_0 = \{1, 2, \ldots, n(= pk)\}$ with $V_0^{(i)} = \{(i-1)p+1, (i-1)p+2, \ldots, ip\}$ for $i = 1, 2, \ldots, k$. Also let $\{\mathcal{H}_i(V_i, E_i) : 1 \leq i \leq k\}$ be a set of m-uniform hypergraphs with $|V_i| = n_i$. For each $i = 1, 2, \ldots, k$, we take p copies, $\{\mathcal{H}_i^{(j)}(V_i^{(j)}, E_i^{(j)}) : j = 1, 2 \ldots, p\}$, of $\mathcal{H}_i(V_i, E_i)$. Then we consider $V_0^{(i)} \oplus \mathcal{H}_i^j(V_i^{(j)}, E_i^{(j)})$ for all $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, p$. This gives us an m-uniform hypergraph $\mathcal{H}_\pi(V, E)$ which is called generalized corona of hypergraphs and we write $\mathcal{H}_\pi = \mathcal{H}_0 \circ_p^k \mathcal{H}_i$.
- Here, we consider the case when $n_i = n_1$ and $\mathcal{H}_i$'s are r_1-regular. Now to find the characteristic polynomial we have the following theorem. We denote $a = \dfrac{p}{m-1}\left[\binom{n+n_1-2}{m-2} - \binom{n}{m-2}\right]$, $b = \binom{n+n_1-2}{m-1}$, $c = \dfrac{1}{m-1}\left[\binom{n+n_1-2}{m-2} - \binom{n_1-2}{m-2}\right]$, $D_i = A_{\mathcal{H}_i} + c(J_{n_1} - I_{n_1})$, $D = diag(I_p \otimes D_1, I_p \otimes D_2, \ldots, I_p \otimes D_k)$ and $S = I_k \otimes J_{p \times pn_1}$. The kronecker product $A \otimes B$ between two matrices $A = (a_{ij})$ and $B = (b_{pq})$ is defined as the partition matrix $(a_{ij}B)$. For matrices A, B, C and D we have $AB \otimes CD = (A \otimes C)(B \otimes D)$, when multiplication makes sense.

Theorem

Characteristic polynomial of $A_{\mathcal{H}_\pi}$ can be expressed as

$$f_{\mathcal{H}_\pi}(x) = \left(\prod_{i=1}^{k} det(D_i - xI_{n_1})\right)^p det\left(A_{\mathcal{H}_0} + I_k \otimes ((a - \frac{b^2 pn_1}{r_1 + (n_1-1)c - x})J_p - (a+x)I_p)\right). \quad (9)$$

Corollary

Let p=1 and $spec(A_{\mathcal{H}_0}) = \{\mu_i : i = 1, 2 \ldots, n\}$, $spec(A_{\mathcal{H}_i}) = \{\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_{n_1}^{(i)}(= r_1)\}$. *Then the adjacency eigenvalues of* $\mathcal{H}_\pi = \mathcal{H}_0 \circ_1^n \mathcal{H}_i$ *are* $\lambda_j^{(i)} - c$ *with the multiplicity one for* $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, n_1 - 1$ *and* $\alpha_i^{\pm}$ *with the multiplicity one for* $i = 1, 2, \ldots, n$ *where* $\alpha_i^{\pm} = \frac{1}{2}\left[r_1 + (n_1-1)c + \mu_i \pm \sqrt{\{r_1 + (n_1-1)c - \mu_i\}^2 + 4b^2 n_1}\right]$.

Corollary

Let $k = 1$ and $\mathcal{H}_0$ be r_0-regular. Let $spec(A_{\mathcal{H}_0}) = \{\mu_1, \mu_2, \dots, \mu_n(= r_0)\}$, $spec(A_{\mathcal{H}_1}) = \{\lambda_1, \lambda_2, \dots, \lambda_{n_1}(= r_1)\}$ and $\mathcal{H}_\pi = \mathcal{H}_0 \circ^1 \mathcal{H}_1$. Then the adjacency eigenvalues of $\mathcal{H}_\pi$ are given by $r_1 + (n_1 - 1)c$ with the multiplicity $n - 1$, λ_i with the multiplicity n for $i = 1, 2, \dots, n_1 - 1$, μ_j with the multiplicity one for $j = 1, 2, \dots, n - 1$ and $\alpha^\pm$ with the multiplicity one where

$$\alpha^\pm = \frac{1}{2}\left[r_1 + (n_1 - 1)c + r_0 + (n - 1)a \pm \sqrt{\{r_1 - r_0 + (n_1 - 1)c - (n - 1)a\}^2 + 4b^2 nn_1}\right].$$

Example

Let $V_0 = \{1, 2, \ldots, 8\}$, $E_0^{(1)} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}, \{2, 4, 6\}, \{7, 8, 3\}, \{7, 8, 4\}, \{7, 8, 5\}\}$ and $E_0^{(2)} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 1\}, \{2, 4, 6\}, \{7, 8, 1\}, \{7, 8, 2\}, \{7, 8, 6\}\}$. Then $\mathcal{H}_0(V_0, E_0^{(1)})$ and $\mathcal{G}_0(V_0, E_0^{(1)})$ are non-isomorphic cospectral 3-uniform hypergraphs.

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THANK YOU.