MA60053 - Computational Linear Algebra
Matrix and vector norms

M. Rajesh Kannan

Department of Mathematics,
Indian Institute of Technology Kharagpur,
email: rajeshkannan1.m@gmail.com, rajeshkannan@maths.iitkgp.ac.in

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Definition

Let V be vector space over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|.\| : V \longrightarrow [0,\infty)$ is called a norm on V if it satisfies the following conditions:

(i) $\|\lambda x\| = |\lambda|\|x\|$ *forall* $\lambda \in \mathbb{F}$ *and* $x \in V$,

(ii) $\|x\| = 0$ *if and only if* $x = 0$,

(iii) $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$.

Example

- $V = \mathbb{R}^n$, for $1 \leq p < \infty$, $\|x\|_p = \{\sum_{i=1}^{n} |x_i|^p\}^{\frac{1}{p}}$.
- $V = \mathbb{R}^n$, $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$.
- $V = \mathbb{R}^n$ and A be an $n \times n$ positive definite matrix, $\|x\|_A = \sqrt{\langle Ax, x\rangle_2}$ (Exercise)

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Let V be a vector space with a norm $\|.\|$. A sequence of vectors $\{x_n\} \in V$ converges to a vector in $x \in V$ with respect to the norm $\|.\|$, if $\|x_n - x\| \to 0$.

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Theorem (Equivalence of norms)

If $\|.\|_1$ and $\|.\|_2$ are two norms on $\mathbb{R}^n$, then there exits positive constants c and d such that $c\|x\|_1 \leq \|x\|_2 \leq d\|x\|_1$ for all $x \in \mathbb{R}^n$.

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Convergence in $\mathbb{R}^n$ with respect to a norm implies convergence in any other norm on $\mathbb{R}^n$.

Theorem

Let $x \in \mathbb{R}^n$. *If* $1 \leq p \leq q \leq \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

- $\|x\|_p \geq \|x\|_q$,
- $\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$.

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Corollary

$$\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty.$$

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Theorem

- $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$.
- $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

Theorem

$$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty.$$

Matrix norms

Definition (Matrix norms)

A matrix norm is a mapping $\|.\| : \mathbb{R}^{n\times n} \longrightarrow [0,\infty)$ *which satisfies the following:*

(i) $\|.\|$ *is a norm, and*

(ii) $\|AB\| \leq \|A\|\|B\|$ *for all* $A, B \in \mathbb{R}^{n\times n}$.

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Note: $\|I\| \geq 1$**. If** A **is invertible, then** $1 \leq \|AA^{-1}\| \leq \|A\|\|A^{-1}\|$ **.**

Example

- $\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$, $A \in \mathbb{R}^{n\times n}$ is a norm on $\mathbb{R}^{n\times n}$. [Frobenius norm]
- NOT all norms on $\mathbb{R}^{n\times n}$ are matrix norms. For,

$$\|A\|_\infty = \max_{1\leq i,j\leq n} |a_{ij}|$$

is a norm, but not a matrix norm.

Induced norm or operator norm

If $\|.\|$ is a norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

defines a norm on $\mathbb{R}^{n \times n}$. Equivalently,

$$\|A\| = \max_{\|x\| = 1} \|Ax\|.$$

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On $\mathbb{R}^{n \times n}$, for each $1 \leq p \leq \infty$,

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Example

On $\mathbb{R}^{n \times n}$, for each $1 \leq p \leq \infty$,

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is an induced norm. What about Frobenius norm?

Induced norms vs Matrix norms

Theorem

If $\|.\|$ is an induced norm on $\mathbb{R}^{n\times n}$, then $\|Ax\| \leq \|A\|\|x\|$, for all $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. The inequality is sharp.

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NOT all matrix norms are induced.

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Definition (Matrix p-norms)

For $1 \leq p \leq \infty$, the norm on $\mathbb{R}^{n\times n}$ by the p-norm on $\mathbb{R}^n$ is called the matrix p-norm.

$$\|A\|_p = \max_{x\neq 0} \frac{\|AX\|_p}{\|x\|_p} =$$

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$$\|A\|_p = \max_{x\neq 0} \frac{\|AX\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p.$$

Computing p-norms are hard.

Theorem

- $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}.|$ *[Column sum norm]*
- $\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$. *[Row sum norm]*
- $\|A\|_2 = [\lambda_{max}(A^T A)]^{\frac{1}{2}}$, *where* $\lambda_{max}(A^T A)$ *is the largest eigenvalue of* $A^T A$. *[Spectral norm]*
- $\|A\|_F = [\mathrm{Trace}(A^T A)]^{\frac{1}{2}}$, *where* $\mathrm{Trace}(A^T A)$ *is the trace of the matrix* $A^T A$.
- $\|A\|_2 \le \|A\|_F \le \sqrt{n}\|A\|_2$.
- *If A is symmetric positive semidefinite such that $A = C^T C$, then* $\|A\|_2 = \|C\|_2^2$.

We will prove some more interesting properties of norms after doing SVD!

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then

$$\|A\|_2 = \max_{\|x\| = 1} |\langle Ax, x\rangle|.$$

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive semidefinite matrix. Then,

$$\lambda_{max}(A) = \max_{\|x\| = 1} \langle Ax, x\rangle,$$

and

$$\lambda_{min}(A) = \min_{\|x\| = 1} \langle Ax, x\rangle.$$