Advanced Matrix Algebra and Applications - Python Module

18-22 September 2019

Problem Sheet - 1

1. Generate a random matrix A of order 4×4 and compute the following:
   (a) $det(A^2 + A)$
   (b) A^{-1}, if it exists
   (c) $\|A\|_1$

2. Define a function which returns matrices of one of the below structures by taking the order and the required structure as its inputs.
   (a) hermitian
   (b) stochastic

3. Define a function which computes $\|x\|_p$ for any $x \in \mathbb{R}^n$; $1 \leq p \leq \infty$. (do not use numpy.linalg.norm or anything to that effect)

4. Write a function which achieves the following:
   (a) Define a symmetric matrix A of order 3.
   (b) Find its spectral decomposition (XDX^{-1}).(you may use numpy.linalg.eig)
   (c) Verify that $A = XDX^{-1}$.

5. Consider a simple model as below to find the probability of it raining on a given day.
   Let $X_n = \begin{cases} 1, & \text{if it rains on the } n^{th} \text{ day} \\ 0, & \text{otherwise} \end{cases}$.
   Let $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$ where P_{ij} denotes the probability of $X_{n+1} = j$ given that $X_n = i$ where $i, j \in \{0, 1\}$ and $n \in \mathbb{N}$. Suppose that it was equally likely to rain or otherwise on the very first day of observation after building this model. We encode this information in the vector $\pi_0 = [0.5, 0.5]$, which is called as the probability distribution of X_0. Then, $\forall n \in \mathbb{N}, \pi_n = \pi_0 P^n$ is the probability distribution of X_n. Note that $P(X_n = i) = \pi_{n,i}$.
   (a) For each of the first 100 days, compute the probability of it raining i.e. compute $P(X_n = 1)$ for $n = 1, 2, \cdots, 100$.
   (b) Plot $P(X_n = 0)$ v\s $P(X_n = 1)$ for $n = 1, 2, ..., 100$. What do you observe?
   (c) Find the dominant eigenvalue and the corresponding eigenvector of P^T. Normalize this eigenvector in the sense that the sum of the components is 1. What do you observe?
   (d) What is/are your conclusion(s) based on this model?

6. Do the following for the iris data set. Let X be the data matrix.

   (a) Find the dimensions of X.

   (b) Find the rank of X and $X^T X$.

   (c) Is $X^T X$ invertible?

   (d) Find the spectral decomposition of $X^T X$.

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