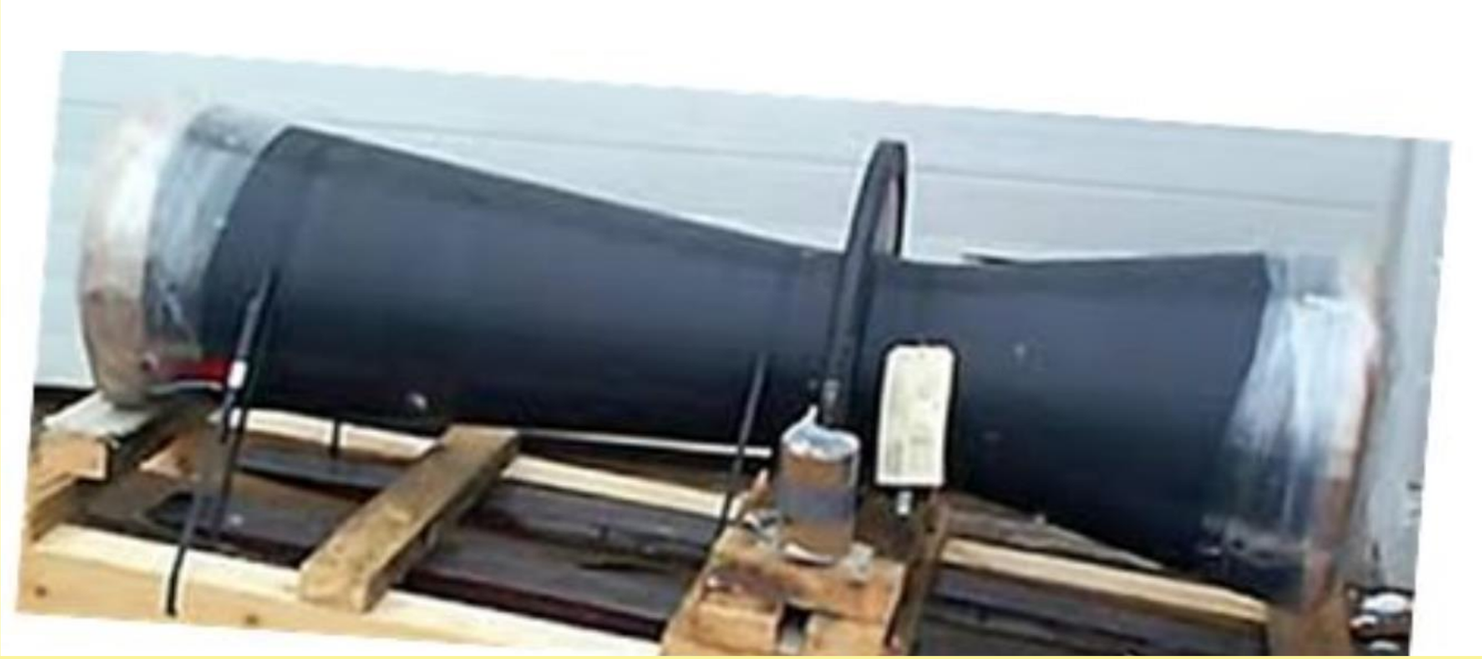
Bernoulli's Equation



Hydraulics

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Bernoulli Along a Streamline

$$-\nabla p = \rho \mathbf{a} + \rho g \hat{\mathbf{k}}$$

$$-\frac{\partial p}{\partial s} = \rho a_s + \rho g \frac{dz}{ds}$$

Separate acceleration due to gravity. Coordinate system may be in any orientation!

k is vertical, s is in direction of flow, n is normal.

Component of g in s direction

Note: No shear forces! Therefore flow must be frictionless.

Steady state (no change in p wrt time)

Bernoulli Along a Streamline

$$-\frac{\partial p}{\partial s} = \rho a_s + \gamma \frac{dz}{ds}$$

Can we eliminate the partial derivative?

Chain rule

$$a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = \frac{\partial V}{\partial s}V$$

Write acceleration as derivative wrt s

$$dp = \frac{\partial p}{\partial s}ds + \frac{\partial p}{\partial n}dn$$

$dn \to 0$ (n is constant along streamline, dn=0)

$\therefore dp/ds = \partial p/\partial s$ and $dV/ds = \partial V/\partial s$

$$-\frac{dp}{ds} = \rho V\frac{dV}{ds} + \gamma \frac{dz}{ds}$$

Integrate F=ma Along a Streamline

$$-\frac{dp}{ds} = \rho V \frac{dV}{ds} + \gamma \frac{dz}{ds}$$

Eliminate *ds*

$$dp + \rho V dV + \gamma dz = 0$$

Now let's integrate…

$$\int \frac{dp}{\rho} + \int V dV + g \int dz = 0$$

But density is a function of _pressure_.

$$\int \frac{dp}{\rho} + \frac{1}{2} V^2 + gz = C_p$$

If density is constant…

$$p + \frac{1}{2} \rho V^2 + \gamma z = C_{p'}$$

Bernoulli Equation

- Assumptions needed for Bernoulli Equation
  - Frictionless
  - Steady
  - Constant density (incompressible)
  - Along a streamline
- Eliminate the constant in the Bernoulli equation?

  Apply at two points along a streamline.
- Bernoulli equation does not include
  - Mechanical energy to thermal energy
  - Heat transfer, Shaft Work

Bernoulli Equation

The Bernoulli Equation is a statement of the conservation of Mechanical Energy

$$\underbrace{\frac{p}{\rho} + gz}_{\text{p.e.}} + \underbrace{\frac{1}{2}V^2}_{\text{k.e.}} = C_p$$

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C_{p''}$$

$\frac{p}{\gamma} =$ Pressure head

$z =$ Elevation head

$\frac{V^2}{2g} =$ Velocity head

$\frac{p}{\gamma} + z =$ Hydraulic Grade Line / Piezometric head

$\frac{p}{\gamma} + z + \frac{V^2}{2g} =$ Energy Grade Line / Total head

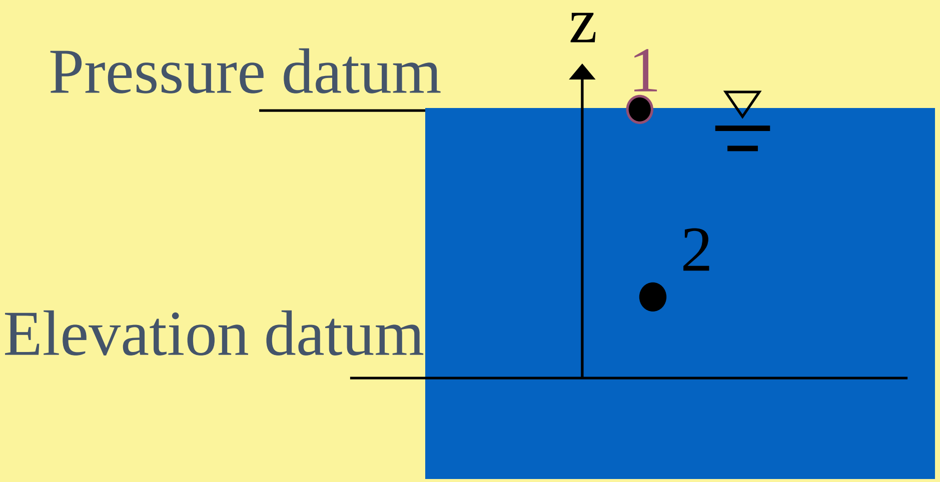
Bernoulli Equation: Simple Case (*V* = 0)

- Reservoir (*V* = 0)
  - Put one point on the surface, one point anywhere else

$$\frac{p}{\gamma} + z + \cancel{\frac{V^2}{2g}} = C_{p"}$$

$$\cancel{\frac{p_1}{\gamma}} + z_1 = \frac{p_2}{\gamma} + z_2$$

$$z_1 - z_2 = \frac{p_2}{g}$$



We didn’t **<u>cross</u>** any streamlines so this analysis is okay!

<u>Same as we found using statics</u>

Bernoulli Equation: Simple Case (p = 0 or constant)

- What is an example of a fluid experiencing a change in elevation, but remaining at a constant pressure? ___Free jet___

$$\cancel{\frac{p_1}{\gamma}} + z_1 + \frac{V_1^2}{2g} = \cancel{\frac{p_2}{\gamma}} + z_2 + \frac{V_2^2}{2g}$$

$$z_1 + \frac{V_1^2}{2g} = z_2 + \frac{V_2^2}{2g}$$

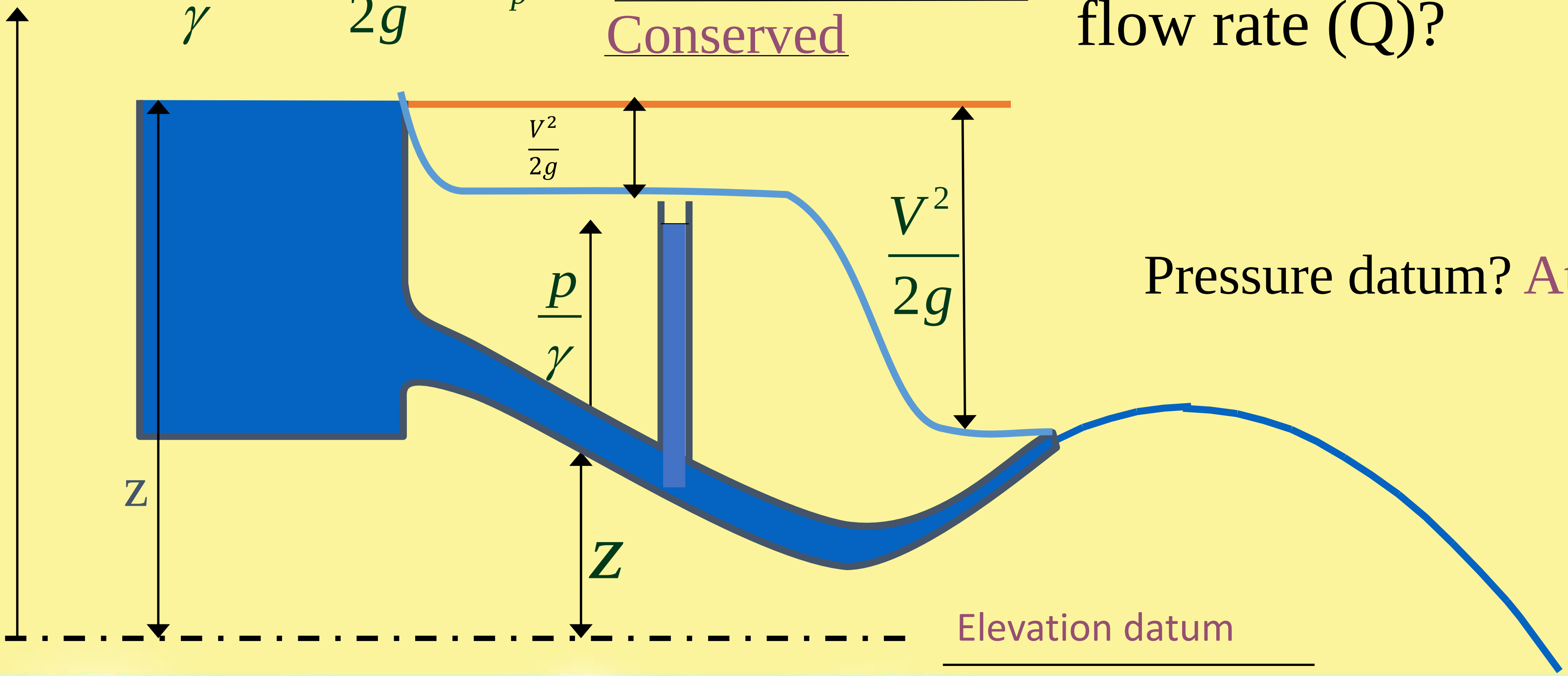
$$V_2 = \sqrt{2g\left(z_1 - z_2\right) + V_1^2}$$

Hydraulic and Energy Grade Lines (neglecting Mechanical losses for now)

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C_{p''}$$

Mechanical Energy Conserved

The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?

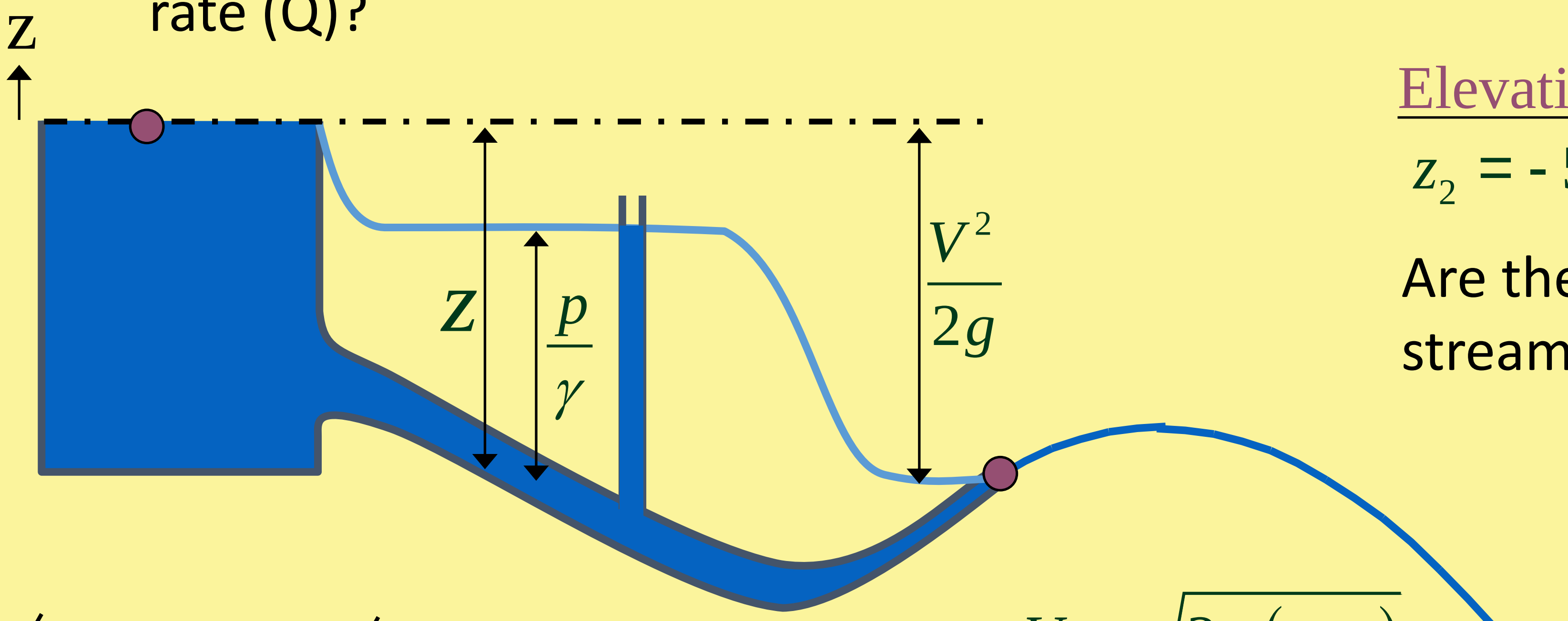


Pressure datum? Atmospheric pressure

Elevation datum

Jet Solution

The 2 cm diameter jet is 5 m lower than the surface of the reservoir. What is the flow rate (Q)?



Elevation datum

z_2 = - 5 m

Are the 2 points on the same streamline?

$$\cancel{\frac{p_1}{\gamma}} + \cancel{\frac{V_1^2}{2g}} + \cancel{z_1} = \cancel{\frac{p_2}{\gamma}} + \frac{V_2^2}{2g} + z_2$$

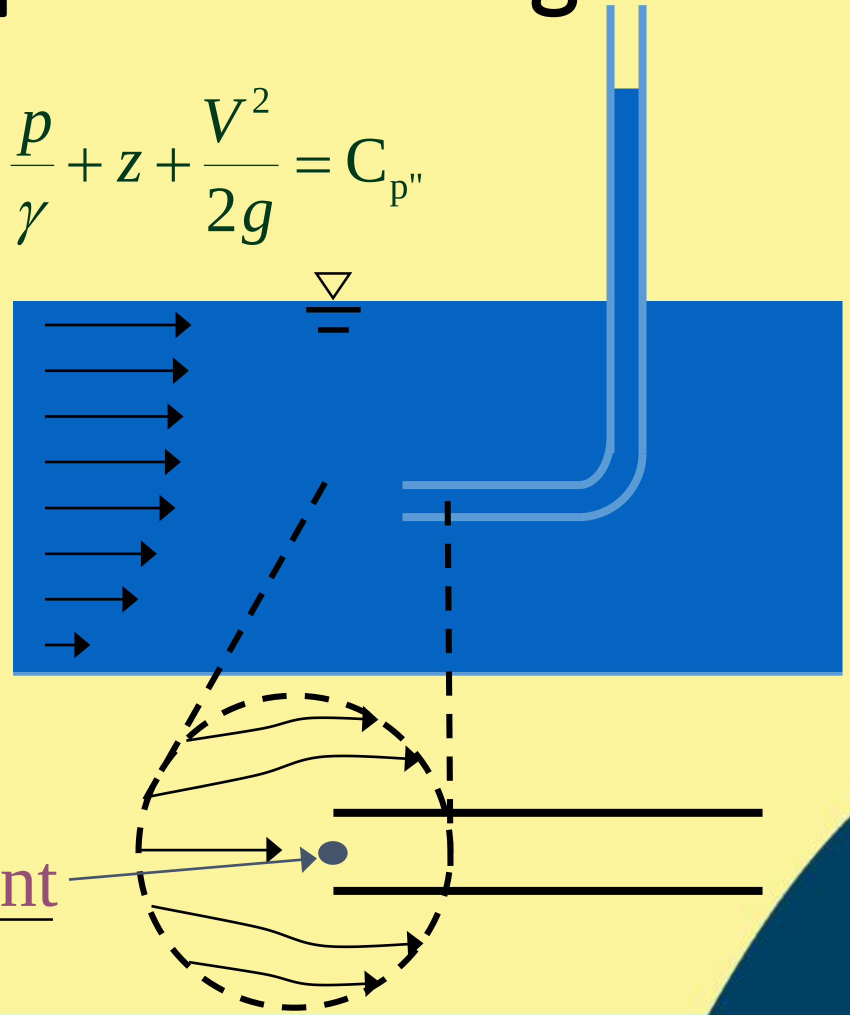
$$V_2 = \sqrt{2g\left(-z_2\right)}$$

$$Q = V_2 \frac{\pi d_2^2}{4} = \frac{\pi d_2^2}{4}\sqrt{2g\left(-z_2\right)}$$

Bernoulli Equation Application: Stagnation Tube

- What happens when the water starts flowing in the channel?
- Does the orientation of the tube matter? Yes!
- How high does the water rise in the stagnation tube?
- How do we choose the points on the streamline?

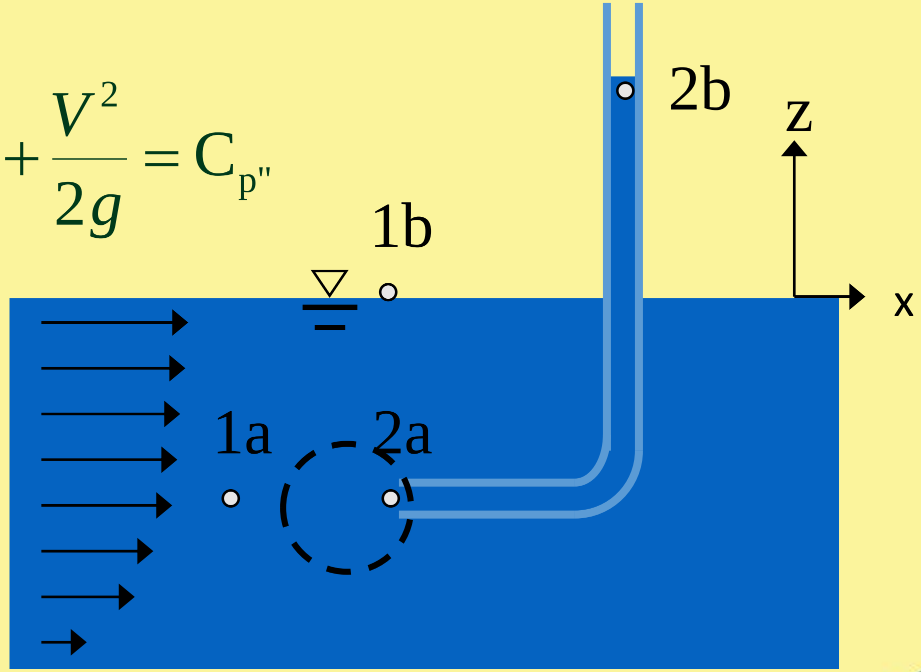
$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C_{p''}$$



Bernoulli Equation Application: Stagnation Tube

- 1a-2a V = f(Δp)
  - Same streamline
- 1b-2a V = f(Δp)
  - Crosses || streamlines
- 1a-2b V = f(z_2)
  - Doesn't cross streamlines

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C_{p''}$$



In all cases we don't know p_1

$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$\frac{V_1^2}{2g} = z_2$$

$$V_1 = \sqrt{2gz_2}$$

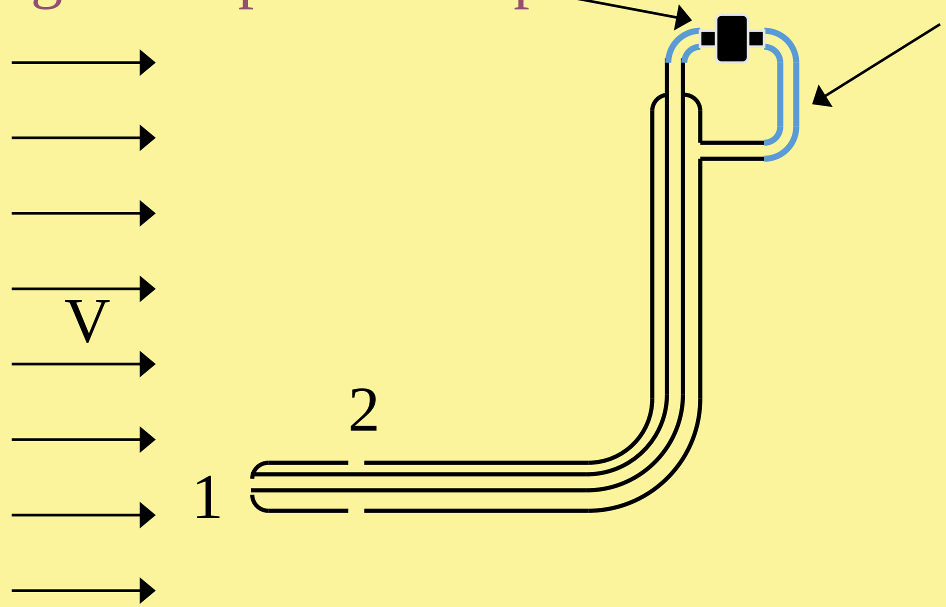
Pitot Tubes

- Used to measure air speed on airplanes
- Can connect a differential pressure transducer to directly measure $V^2/2g$
- Can be used to measure the flow of water in pipelines

Point measurement!

→

Pitot Tube



$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$$

$$V_1 = 0$$

$$z_1 = z_2$$

$$V = \sqrt{\frac{2}{\rho}\left(p_1 - p_2\right)}$$

Connect two ports to differential pressure transducer. Make sure Pitot tube is completely filled with the fluid that is being measured.

Solve for velocity as function of pressure difference

Relaxed Assumptions for Bernoulli Equation

- Frictionless (velocity not influenced by viscosity)

  Small energy loss (accelerating flow, short distances)

- Steady

  Or gradually varying

- Constant density (incompressible)

  Small changes in density

- Along a streamline

  Don't cross streamlines

Bernoulli Normal to the Streamlines

$$-\frac{\partial p}{\partial n} = \rho a_n + \rho g \frac{dz}{dn}$$

$$a_n = \frac{V^2}{R}$$

R is local radius of curvature

n is toward the center of the radius of curvature

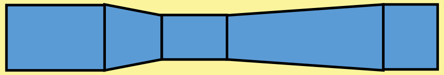
0 (s is constant normal to streamline)

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn$$

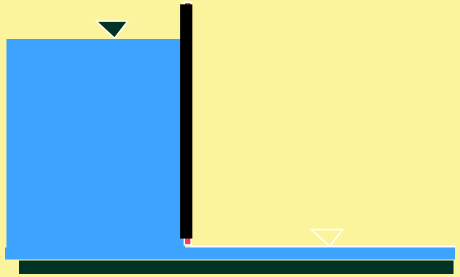
$$\therefore dp/dn = \partial p/\partial n$$

$$-\frac{dp}{dn} = \rho \frac{V^2}{R} + \rho g \frac{dz}{dn}$$

Bernoulli Equation Applications

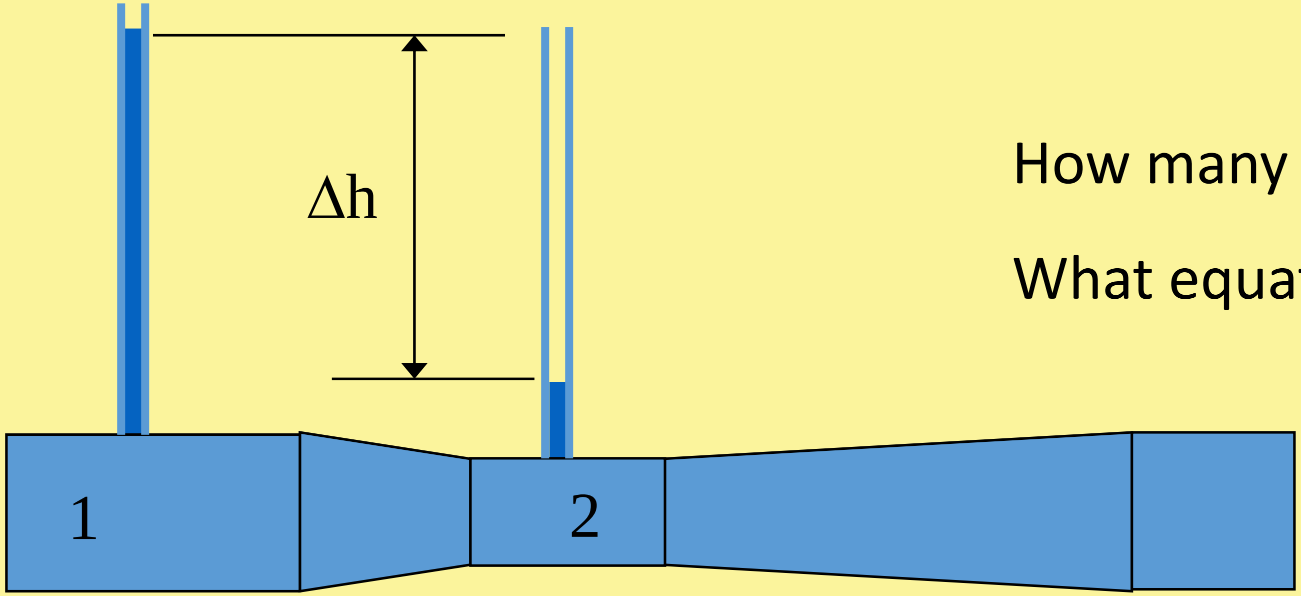
- Stagnation tube
- Pitot tube
- Free Jets
- Orifice
- Venturi 
- Sluice gate
- Sharp-crested weir

Applicable to contracting streamlines (accelerating flow).



Example: Venturi

How would you find the flow (Q) given the pressure drop between point 1 and 2 and the diameters of the two sections? You may assume the head loss is negligible. Draw the EGL and the HGL over the contracting section of the Venturi.



How many unknowns?

What equations will you use?

Example Venturi

$$\frac{p_1}{\gamma_1} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma_2} + z_2 + \frac{V_2^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} = \frac{V_2^2}{2g}\left[1 - \left(\frac{d_2}{d_1}\right)^4\right]$$

$$V_2 = \sqrt{\frac{2g(p_1 - p_2)}{\gamma\left[1 - (d_2/d_1)^4\right]}}$$

$$\boldsymbol{Q = C_d A_2 \sqrt{\frac{2g(p_1 - p_2)}{\gamma[1 - (d_2/d_1)^4]}}}$$

$$Q = VA$$

$$V_1 A_1 = V_2 A_2$$

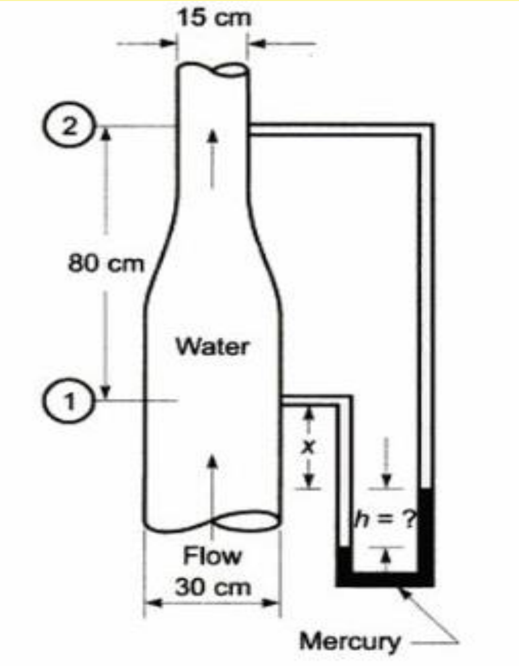
$$V_1 \frac{\pi d_1^2}{4} = V_2 \frac{\pi d_2^2}{4}$$

$$V_1 d_1^2 = V_2 d_2^2$$

$$V_1 = V_2 \frac{d_2^2}{d_1^2}$$

Practice Problem

Water flows up a tapered pipe as shown in Fig. below. Find the magnitude and direction of the deflection h of the differential mercury manometer corresponding to a discharge of 120 L/s. The friction in the pipe can be completely neglected.



Solution:

Let S = Relative density of mercury. For the manometer:

Considering the elevation of section 1 as datum

$$\frac{p_1}{\gamma} + x + h = \frac{p_2}{\gamma} + 0.8 + x + Sh$$

$$\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) - 0.8 = (S - 1)h$$

= (13.6 — 1) h = 12.6h

By continuity criterion, Q = $\frac{\pi}{4}$ x $(0.30)^2$ x V_1

= $\frac{\pi}{4}$ x $(0.15)^2$ x V_2 = 0.120 m^3/s

V_1 = 1.6977 m/s, V_2 = 6.79 m/s

By Bernoulli equation for points 1 and 2,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right) + 0 - 0.8 = \frac{V_2^2 - V_1^2}{2g} = \frac{(6.79)^2 - (1.6977)^2}{2 \times 9.81} = 2.2034$$

$$\left(\frac{p_1 - p_2}{\gamma}\right) - 0.8 = 12.6h = 2.2034$$

Therefore h =(2.2034/12.6) = 0.175 m = 17.5 cm