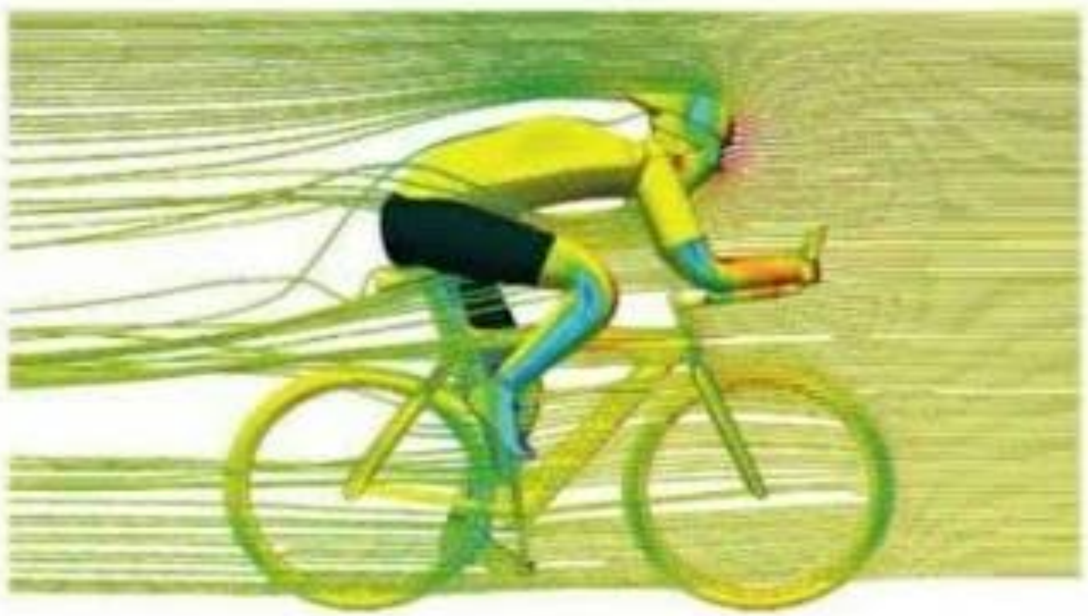
FLUID KINEMATICS



Hydraulics

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The Velocity Field

- The infinitesimal particles of a fluid are tightly packed together (as is implied by the continuum assumption).
- Thus, at a given instant in time, a description of any fluid property (such as density, pressure, velocity, and acceleration) may be given as a function of the fluid's location.

- This representation of fluid parameters as functions of the spatial coordinates is termed a ***field representation*** of the flow.

$$\boldsymbol{V} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})\hat{\boldsymbol{\imath}} + \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})\hat{\boldsymbol{\jmath}} + \boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})\widehat{\boldsymbol{k}}$$

where *u*, and *w* are the *x*, *y*, and *z* components of the velocity vector.

- By definition, the velocity of a particle is the time rate of change of the position vector for that particle.

- Shown in *Fig.1,* The position of particle *A* relative to the coordinate system is given by its ***position vector,*** which (if the particle is moving) is a function of time.
- The time derivative of this position gives the ***velocity*** of the particle,

$dr_A/dt = V_A$.

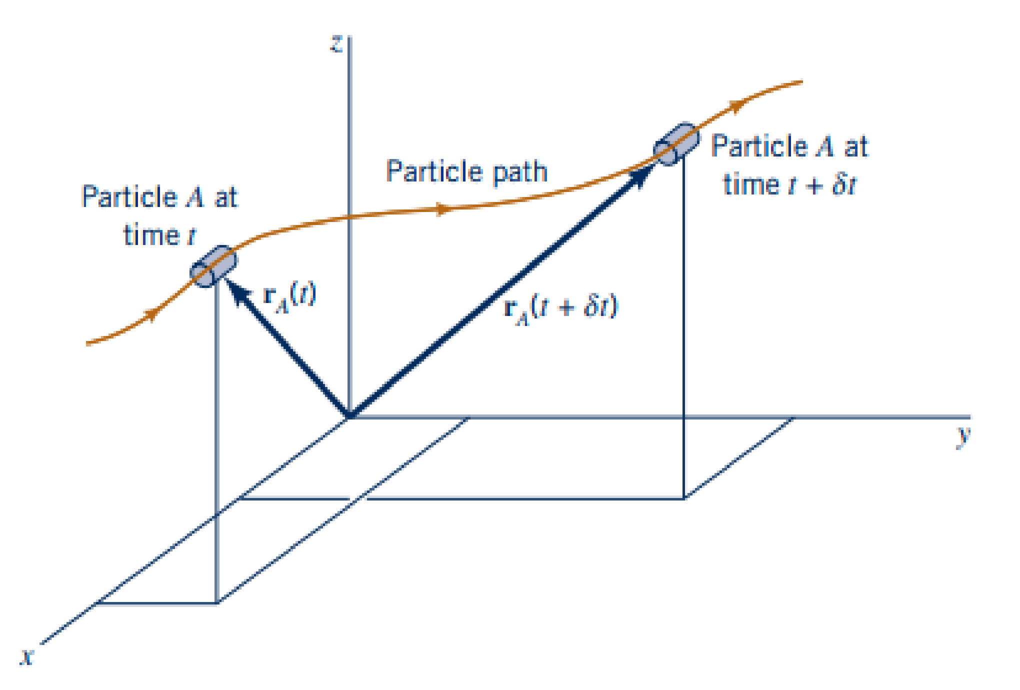


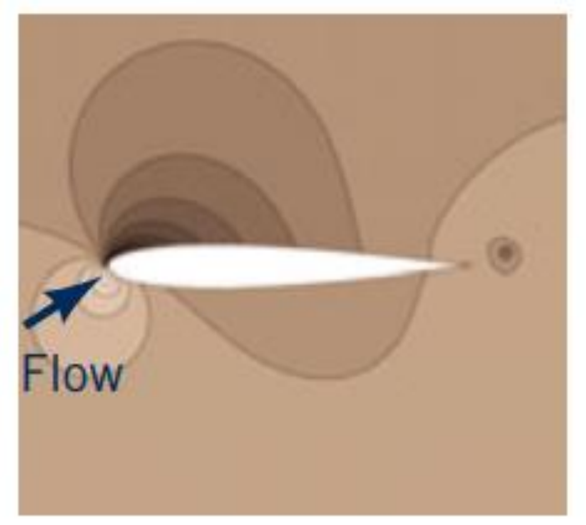
Fig.1

- By writing the velocity for all of the particles, we can obtain the field description of the velocity vector $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$.
- Since the velocity is a vector, it has both a direction and a magnitude. The magnitude of **V**, denoted $\boldsymbol{V} = |\boldsymbol{V}| = \sqrt{(\boldsymbol{u}^2 + \boldsymbol{v}^2 + \boldsymbol{w}^2)}$, is the speed of the fluid.

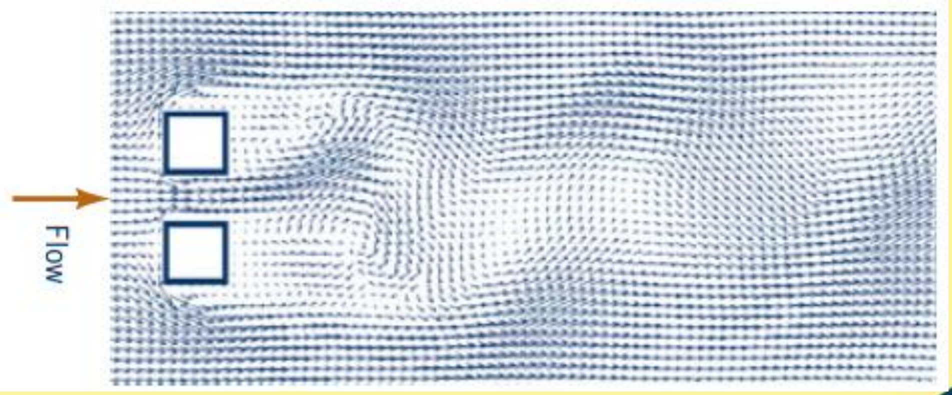
Eulerian Flow Descriptions

- There are two general approaches in analyzing fluid mechanics problems. The first method, called the ***Eulerian method.***
- In this case, the fluid motion is given by completely prescribing the necessary properties (pressure, density, velocity, etc.) as functions of space and time.
- From this method we obtain information about the flow in terms of what happens at fixed points in space as the fluid flows through those points.

➢ A typical Eulerian representation of the flow is shown by the figure in the margin which involves flow past an air foil at angle of attack.



➢ The pressure field is indicated by using a contour plot showing lines of constant pressure, with gray shading indicating the intensity of the pressure.



Lagrangian Flow Descriptions

- The second method, called the ***Lagrangian method,*** involves following individual fluid particles as they move about and determining how the fluid properties associated with these particles change as a function of time.
- That is, the fluid particles are “tagged” or identified, and their properties determined as they move.
- The difference between the two methods of analyzing fluid flow problems can be seen in the example of smoke discharging from a chimney.

One-, Two-, and Three-Dimensional Flows

- Generally, a fluid flow is a rather complex three-dimensional, time-dependent phenomenon. $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{u}\hat{\boldsymbol{\imath}} + \boldsymbol{v}\hat{\boldsymbol{\jmath}} + \boldsymbol{w}\hat{\boldsymbol{k}}$
- In almost any flow situation, the velocity field actually contains all three velocity components(*u*, *v,* and *w*, for example). In many situations the ***three-dimensional flow*** characteristics are important in terms of the physical effects they produce. For these situations it is necessary to analyze the flow in its complete three-dimensional character.

- The flow of air past an airplane wing provides an example of a complex three-dimensional flow.
- In many situations one of the velocity components may be small (in some sense) relative to the two other components. In situations of this kind it may be reasonable to neglect the smaller component and assume ***two-dimensional flow.*** That is, $\mathbf{V} = \boldsymbol{u}\hat{\boldsymbol{\imath}} + \boldsymbol{v}\hat{\boldsymbol{\jmath}}$, where ***u*** and $\boldsymbol{v}$ are functions of *x* and *y* (and possibly time, *t*).

- It is sometimes possible to further simplify a flow analysis by assuming that two of the velocity components are negligible, leaving the velocity field to be approximated as a ***one-dimensional flow*** field. That is, $\mathbf{V} = \boldsymbol{u}\hat{\boldsymbol{\imath}}$.

Steady and Unsteady Flows

CLASSIFICATION OF FLOW

- **Steady Flow:** Fluid flow conditions at any point do not change with time. For example

$$\frac{\partial V}{\partial t} = 0,\ \frac{\partial p}{\partial t} = 0,\ \frac{\partial \rho}{\partial t} = 0$$

In a steady flow steam line, path line and streak line are identical.

- **Unsteady Flow:** Flow parameters at any point change with time, e.g. $\frac{\partial V}{\partial t} \neq 0$

Uniform and Non-uniform Flows

- **Uniform Flow:** The flow is defined as uniform flow when in the flow field the **velocity and other hydrodynamic parameters do not change from point to point at any instant of time.**
- For a uniform flow, the velocity is a function of time only, which can be expressed in Eulerian description as

$$\vec{V} = V(t)$$

- Implication:
  - For a uniform flow, there will be no spatial distribution of hydrodynamic and other parameters.
  - **Any hydrodynamic parameter will have a unique value in the entire field**, irrespective of whether it changes with time – **unsteady uniform flow** OR does not change with time – **steady uniform flow.**

- **Non-Uniform Flow:** When the **velocity and other hydrodynamic parameters changes from one point to another** the flow is defined as **non-uniform**.
- Important points:
  - For a non-uniform flow, the changes with position may be found either in the direction of flow or in directions perpendicular to it.
  - Non-uniformity in a direction perpendicular to the flow is always encountered near solid boundaries past which the fluid flows.

Reason: All fluids possess **viscosity** which reduces the relative velocity (of the fluid w.r.t. to the wall) to zero at a solid boundary. This is known as **no-slip condition.**

Streamline

- In a fluid flow, a continuous line so drawn that it is tangential to the velocity vector at every point is known as a **streamline**.
- If the velocity vector $\boldsymbol{V} = \hat{\boldsymbol{i}}\boldsymbol{u} + \hat{\boldsymbol{j}}\boldsymbol{v} + \hat{\boldsymbol{k}}\boldsymbol{w}$
- Then the differential equation of a streamline is given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Practice Problem

In a flow the velocity vector is given by V= 3xi + 4yj – 7zk. Determine the equation of the streamline passing through a point M= (1, 4, 5).

Solution:

The equation of the streamline is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Here, u = $3x$, v = $4y$ and w=-$7z$

Hence $$\frac{dx}{3x} = \frac{dy}{4y} = -\frac{dz}{7z}$$

Considering the equations involving x and y, on integration

$\frac{1}{3}\ln x = \frac{1}{4}\ln y + \ln C_1'$ Where, $C_1' =$ a constant

Or, $y = C_1 x^{\frac{4}{3}}$ Where C_1 is another constant.

Similarly, by considering equations with x and z and on integration

$\frac{1}{3}\ln x = -\frac{1}{7}\ln z + \ln C_2'$ Where, $C_2' =$ a constant

$Z = \frac{C_2}{x^{\frac{7}{3}}}$ Where, C_2 is another constant.

Putting the coordinates of the point M (1, 4, 5). $C_1 = \frac{4}{(1)^{4/3}} = 4$ and $C_2 = 5 \text{ X } 1^{7/3} = 5$

The streamline passing through M is given by

$$\boldsymbol{y = 4x^{4/3}} \quad \textbf{and} \quad \boldsymbol{z = \frac{5}{x^{7/3}}}$$

Path lines

- A **path line** is the actual path traveled by an individual fluid particle over some time period.
- **Path lines** are the easiest of the flow patterns to understand. A path line is a Lagrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.

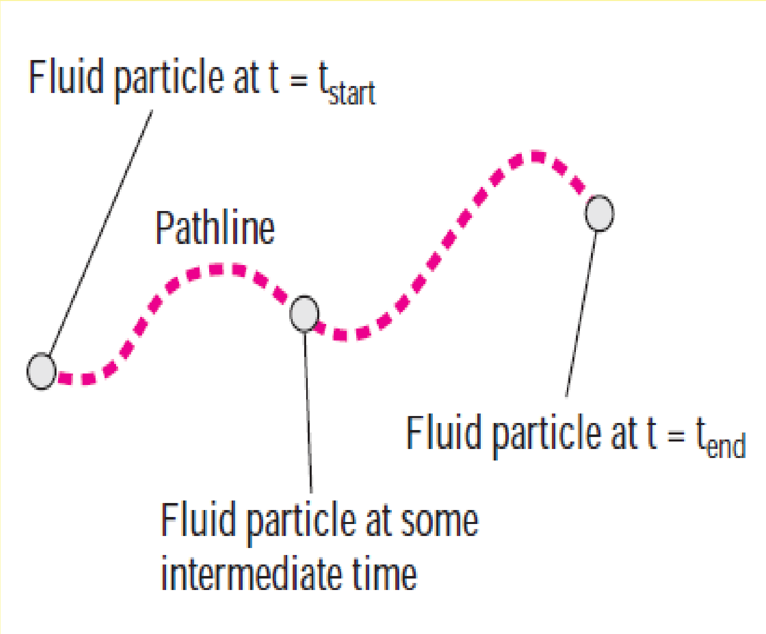


Fig.2

- Thus, a path line is the same as the fluid particle's material position vector (x particle(t), y particle(t), z particle(t)), traced out over some finite time interval.

Streak lines

- A **streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- **Streaklines** are the most common flow pattern generated in a physical experiment. If you insert a small tube into a flow and introduce a continuous stream of tracer fluid (dye in a water flow or smoke in an airflow), the observed pattern is a **streakline**.

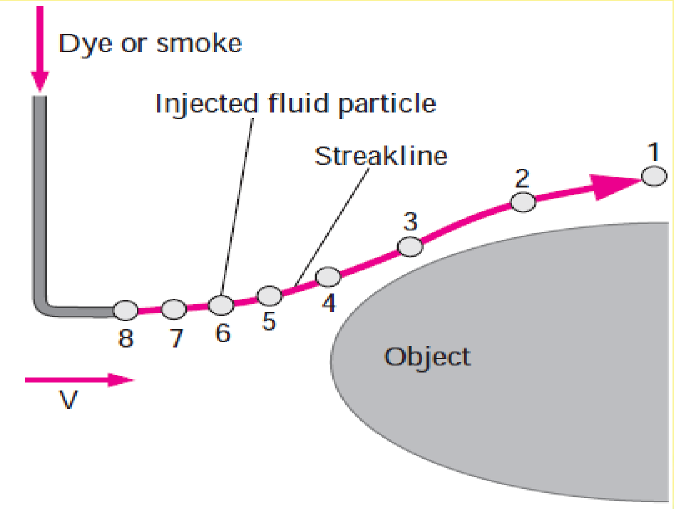


Fig.3

- Fig. 3 shows a **streakline** is formed by connecting all the circles into a smooth curve.

Stagnation Point

- A point of interest in the study of the kinematics of fluid is the occurrence of points where the fluid flow stops. When a stationary body is immersed in a fluid, the fluid is brought to a stop at the nose of the body. Such a point where the fluid flow is brought to rest is known as the **stagnation point**.

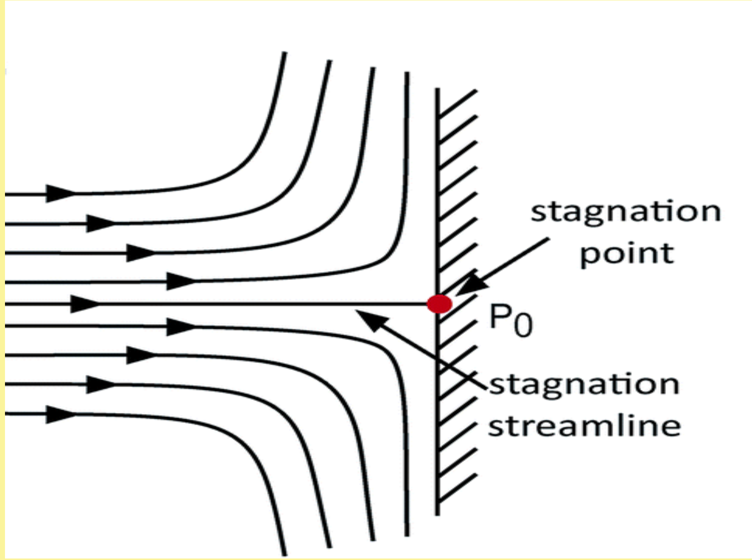


Fig.4

- Thus, a stagnation point is defined as a point in the flow field where the velocity is identically zero.
- This means that all the components of the velocity vector $\overline{\boldsymbol{V}}$ ***viz., u, v,*** and ***w*** are identically zero at the stagnation point.

Acceleration

- Acceleration is a vector.

  - In the natural co-ordinate system, viz., along and across a streamline (Fig. 5).

$$a = \frac{dV}{dt} \text{ and } \quad a = \sqrt{a_s^2 + a_n^2}$$

  - In the tangential direction: $a_s = \frac{\partial V_s}{\partial t} + V_s \frac{\partial V_s}{\partial s}$

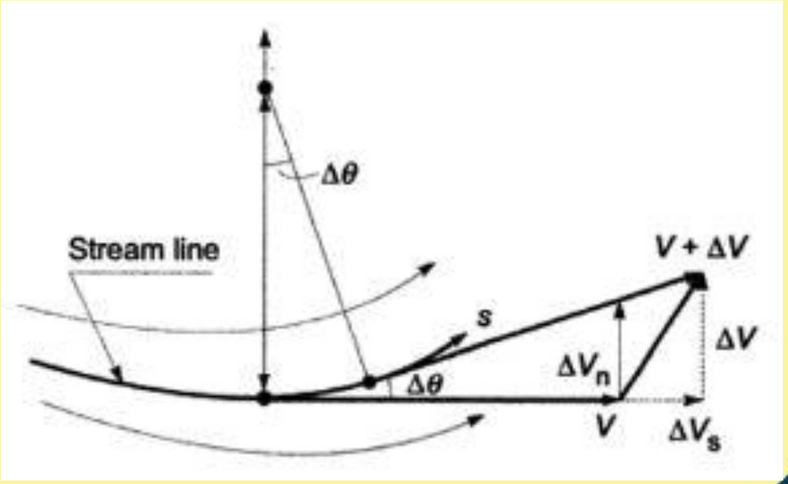


Fig.5

- In the normal direction : $a_n = \frac{\partial V_n}{\partial t} + \frac{V_s^2}{r}$

where r = radius of curvature of the streamline at the point

V_s = tangential component of the velocity V

And V_n = normal component of velocity generated due to change in direction.

- The terms $\frac{\partial \boldsymbol{V_s}}{\partial \boldsymbol{t}}$ and $\frac{\partial \boldsymbol{V_n}}{\partial \boldsymbol{t}}$ are called **local accelerations**.

  Also $\boldsymbol{V_s} \frac{\partial \boldsymbol{V_s}}{\partial \boldsymbol{s}}$ = tangential convective acceleration and $\frac{\boldsymbol{V_s^2}}{\boldsymbol{r}}$ = normal convective acceleration.

- In Cartesian co-ordinates: $\boldsymbol{V} = \hat{\boldsymbol{\imath}}\boldsymbol{u} + \hat{\boldsymbol{\jmath}}\boldsymbol{v} + \hat{\boldsymbol{k}}\boldsymbol{w}$
  - Acceleration a_x , a_y , and a_z in the x, y, z directions respectively are:
  - $a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$
  - $a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$
  - $a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$

Practice Problem

The velocity along the centreline of a nozzle of length L is given by

$$V = 2t\left(1 - \frac{x}{2L}\right)^2$$

where V= velocity in m/s, t = time in seconds from commencement of flow, x = distance from inlet to the nozzle. Find the convective acceleration, local acceleration and the total acceleration when t= 3s, x = 0.5 m and L = 0.8 m.

Solution:

(i) Local acceleration = $\frac{\partial V}{\partial t} = 2\left(1 - \frac{x}{2L}\right)^2$ at t = 3 s

and

x = 0.5 m, $\frac{\partial V}{\partial t} = 2\left(1 - \frac{0.5}{2\times 0.8}\right)^2$ = **0.945 m/s²**

(ii) Convective acceleration = $V\frac{\partial V}{\partial x} = 2t\left(1 - \frac{x}{2L}\right)^2 .2t.2\left(1 - \frac{x}{2L}\right)\left(-\frac{1}{2L}\right) = -\frac{4t^2}{L}\left(1 - \frac{x}{2L}\right)^3$

At 1 = 3 s and x= 0.5 m

Convective acceleration = $-\frac{4\times 3^2}{0.8}\left(1 - \frac{0.5}{2\times 0.8}\right)^3 = \mathbf{-14.623}$ **m/s²**

(iii) Total acceleration = (local + convective) acceleration = 0.945 — 14.623 = **— 13.68 m/s²**

Continuity equation

In One-dimensional Analysis

- In steady flow, mass rate of flow into stream tube is equal to mass rate of flow out of the tube

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

  - For incompressible fluid, under steady flow (Fig. 6).

$$A_1 V_1 = A_2 V_2$$

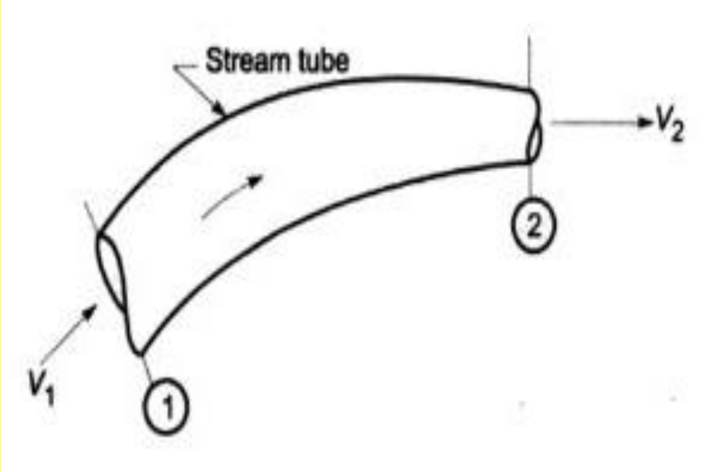


Fig:6

- When there is a variation of velocity across the cross section of a conduit, for an incompressible fluid discharge. (Fig. 7)

$$Q = \int_{A_1} v dA = \int_{A_2} v dA$$

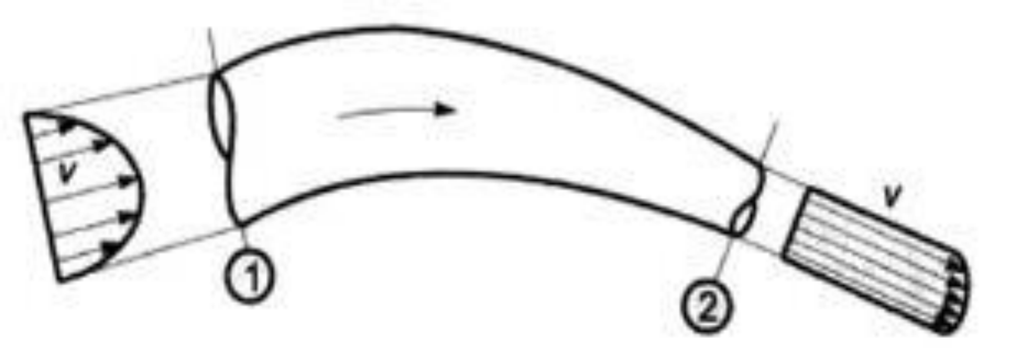
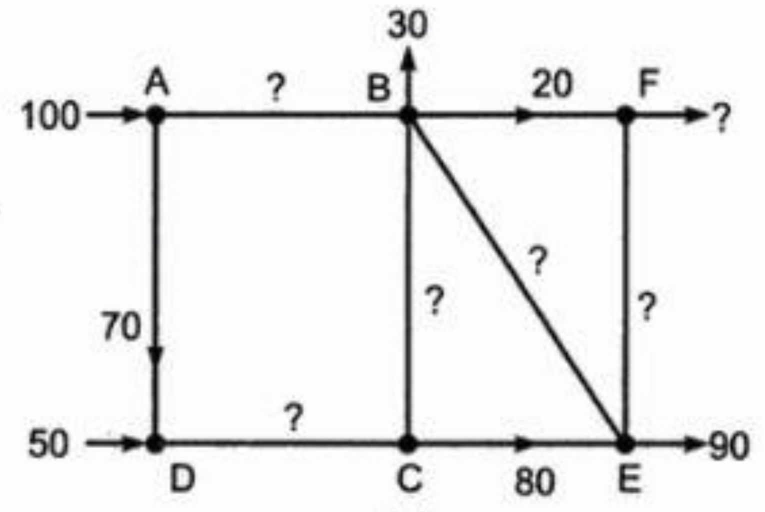


Fig: 7

Practice Problem

Fig shows a pipe network with junctions (nodes) at A, B, C, D, E, and F. The numerals in the fig indicates the discharges at the nodes or in the pipes as the case is and the arrows indicate the directions of flows. By continuity equation determine the missing discharge values and their direction in pipes AB, BC, CD, BE and EF at the node F.



(a)

By continuity criterion, the flow entering into a node must be equal to the flow going out of the node. Thus by considering flow into a node as positive, the algebraic sum of discharges at a node is zero.

Thus at node A:

$100 – 70 – Q_{AB} = 0$

Or Q_{AB} = **30 and Q_{AB} is from A to B**.

At node D: $70 + 50 – Q_{DC} = 0$

Q_{DC} = **120 and Q_{DC} is from D to C.**

At node C: $120 - 80 – Q_{CB} = 0$

Q_{CB} = **40 and Q_{CB} is from C to B.**

At node B: $30 + 40 - 30 – Q_{BE} -20 = 0$

Q_{BE} = **20 and Q_{BE} is from B to E.**

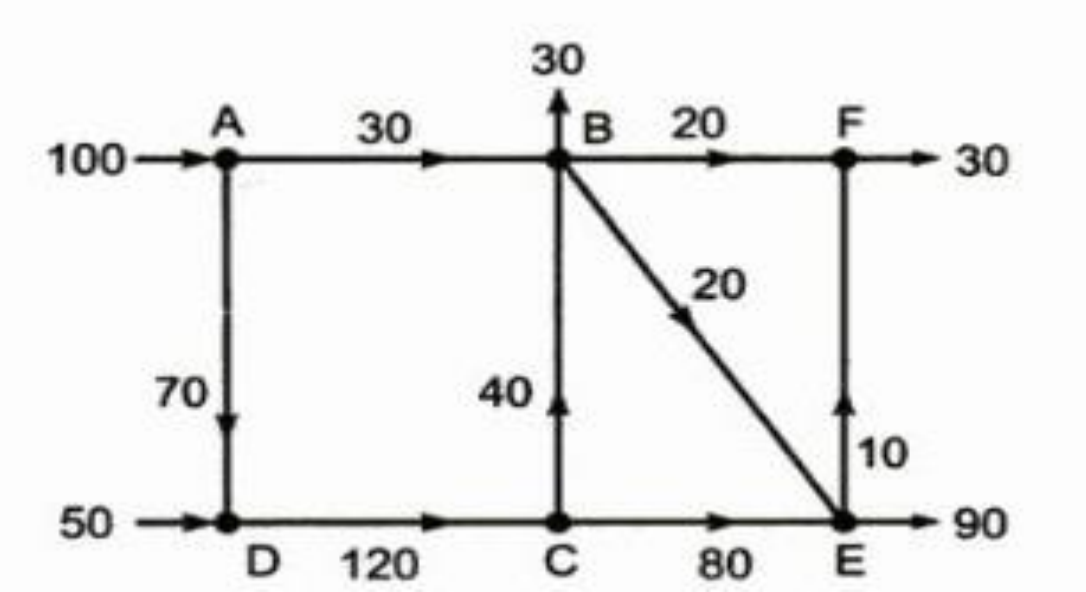
At node E: $80 + 20 – Q_{EF} - 90 = 0$

Q_{EF} = **10 and Q_{EF} is from E to F.**

At node F: $20 + 10 - Q_F = 0$

Q_F = **discharge out of node F = 30.**

The distribution of discharges are as in fig below.



It can be seen now that at each node the continuity equation is satisfied.

In Differential Form

- Cartesian co-ordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

- For incompressible fluid (dρ/dt = 0) and hence the above equation is simplified as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Rotational and irrotational action

- Consider a rectangular fluid element of sides *dx* and *dy* [(Fig. 8.(a)].
- Under the action of velocities acting on it let it undergo deformation as shown in Fig. 8.(b) in a time dt.

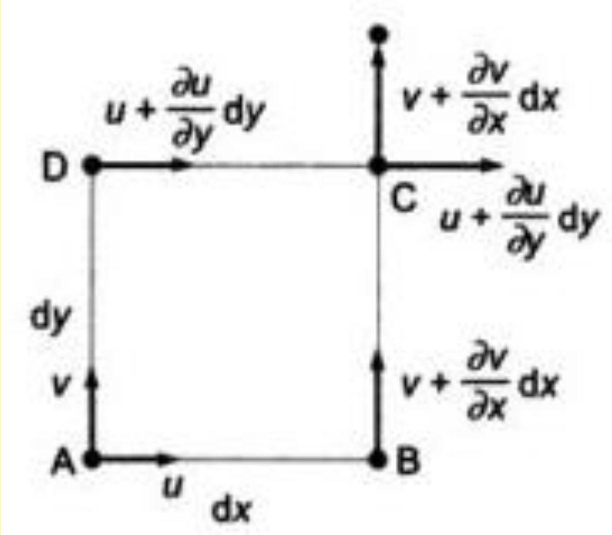


Fig.8.a

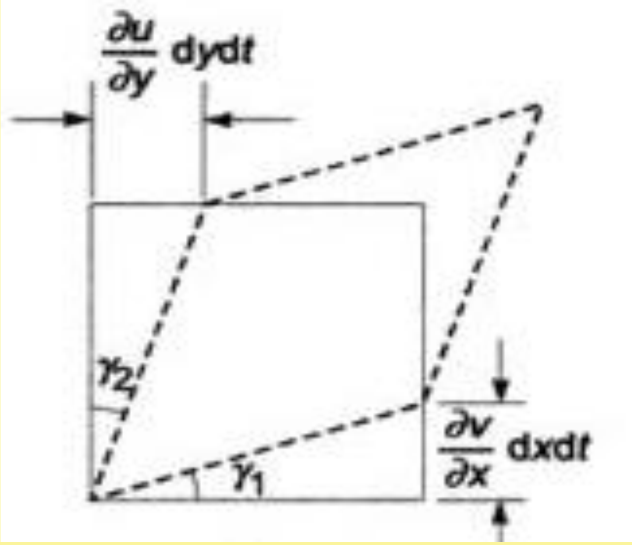


Fig.8.b

γ_1= angular velocity of element AB = $\frac{\partial v}{\partial x}$

γ_2 = angular velocity of element AD = $\frac{\partial u}{\partial y}$

- Considering the anticlockwise rotation as positive, the average of angular velocities of two mutually perpendicular elements is defined as the rate of rotation.
  - Thus rotation about z-axis

$$\boxed{\boldsymbol{\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}}$$

- Thus for a three-dimensional fluid element, three rotational components as given in the following are possible:

$$\text{About z axis, } \boldsymbol{\omega_z} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\text{About y axis, } \boldsymbol{\omega_y} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\text{About x axis, } \boldsymbol{\omega_x} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

- Fluid motion with one or more of the terms $\boldsymbol{\omega_z}$, $\boldsymbol{\omega_y}$ or $\boldsymbol{\omega_z}$ different from zero is termed ***rotational motion***.
- Twice the value of rotation about any axis is called as ***vorticity*** along that axis.
- Thus the equation (for vorticity along z-axis is $\zeta_z = 2\boldsymbol{\omega_z} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$
- A flow is said to be ***irrotational*** if all the components of rotation are zero,

viz. $\boldsymbol{\omega_z = \omega_y = \omega_z = 0}$

Practice Problem

For the following flows, determine the components of rotation about the various axes.

$u = xy^3z$, $v = - y^2 z^2$, $w = yz^2 - (y^3 Z^2 /2)$

Solution: The components of rotation about the various axes are:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(0 - 3xy^2z) = -\boldsymbol{\frac{3}{2}xy^2z}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \boldsymbol{\frac{1}{2}\left(z^2 - \frac{3y^2z^2}{2} + 2y^2z\right)}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(xy^3 - 0) = \frac{1}{2}\boldsymbol{xy}^3$$

Stream function

- In a two-dimensional flow consider two streamlines S_1 and S_2. The flow rate (per unit depth) of an incompressible fluid across the two streamlines is constant and is independent of the path, (path *a* or path *b* from A to B in Fig. 9).

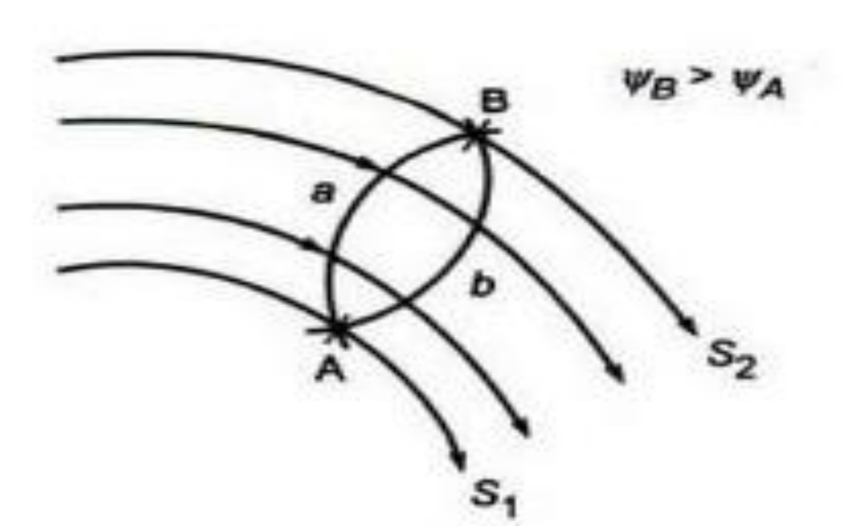


Fig.9

- A stream function Ψ is so defined that it is constant along a streamline and the difference of Ψ_s for the two streamlines is equal to the flow rate between them.
- Thus $\Psi_A - \Psi_B$= flow rate between S_1 and S_2. The flow from left to right is taken as positive, in the sign convention. The velocities *u* and *v* in *x* and *s* directions are given by

$$u = \frac{\partial \Psi}{\partial y} \text{ and } \mathrm{v} = -\frac{\partial \Psi}{\partial x}$$

- The stream function Ψ is defined as above for two dimensional flows only.

- For an irrotational flow, $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ and hence,

$$-\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} = 0$$

- That is, the Laplace equation $\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$

Potential function

- In irrotational flows, the velocity can be written as a gradient of a scalar function ϕ called velocity potential.

$$u = \frac{\partial \phi}{\partial x}, \text{v} = \frac{\partial \phi}{\partial y} \qquad \text{and w} = \frac{\partial \phi}{\partial z}$$

Considering the equation of continuity for an incompressible fluid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- And substituting the expressions for u, v and w in terms of φ

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- Thus the velocity potential satisfies the Laplace equation. Conversely, any function ϕ which satisfies he Laplace equation is a possible irrotational fluid flow case.
- Lines of constant ϕ arc called ***equipotential lines*** and it can be shown that these lines will form orthogonal grids with *Ψ = constant lines*. This fact is used in the construction of flow nets for fluid flow analysis.

[Note : Some authors define φ such that

$u = -\frac{\partial \phi}{\partial x}, \mathrm{v} = -\frac{\partial \phi}{\partial y}$ and $\mathrm{w} = -\frac{\partial \phi}{\partial z}$]

Relation between Ψ and φ for 2-dimensional flow

- φ exists for irrotational flow only.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial y}$$

$$\mathrm{v} = \frac{\partial \phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

- $\Psi =$ constant along a streamline.
- $\Phi =$ constant along an equipotential line which is normal to streamlines.

- By continuity equation

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

- By irrotational flow condition

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

Practice Problem

A velocity potential for a two-dimensional flow is given by $\phi = (x^2 — y^2) + 3xy$.

Calculate (i) the stream function and (ii) the flow rate between the streamlines passing through points (1, 1) and (1, 2)

Solution:

$\phi = (x^2 - y^2) + 3xy$ $\quad$ $u = \frac{\partial \Phi}{\partial x} = 2x + 3y = \frac{\partial \psi}{\partial y}$ $\quad$ $\Psi = 2xy + \frac{3}{2}y^2 + f(x)$ $\quad$ (i)

$v = \frac{\partial \Phi}{\partial y} = -2y + 3x = -\frac{\partial \psi}{\partial x}$ $\quad$ (ii) $\quad$ And from (i) $-\frac{\partial \psi}{\partial x} = -2y - f'(x)$

Thus $f'(x) = -3x$ and hence $\quad$ $f(x) = -\frac{3}{2}x^2$

The required stream function is $\quad$ $\Psi = 2xy — \frac{3}{2}(x^2 — y^2)$

At point (1, 1) Ψ_1 = $(2 — \frac{3}{2}(1 —1))$ = 2 units

At point (1, 2)

Ψ_2 = $[2 \text{ x } (1 \text{ x } 2) — \frac{3}{2}(1 — 4)]$ = 8.5 units

Flow rate between the stream lines passing through (1, 1) and (1, 2)

$\Delta\,\Psi = \Psi_2 - \Psi_1$ = (8.5 - 2.0) = **6.5 units**