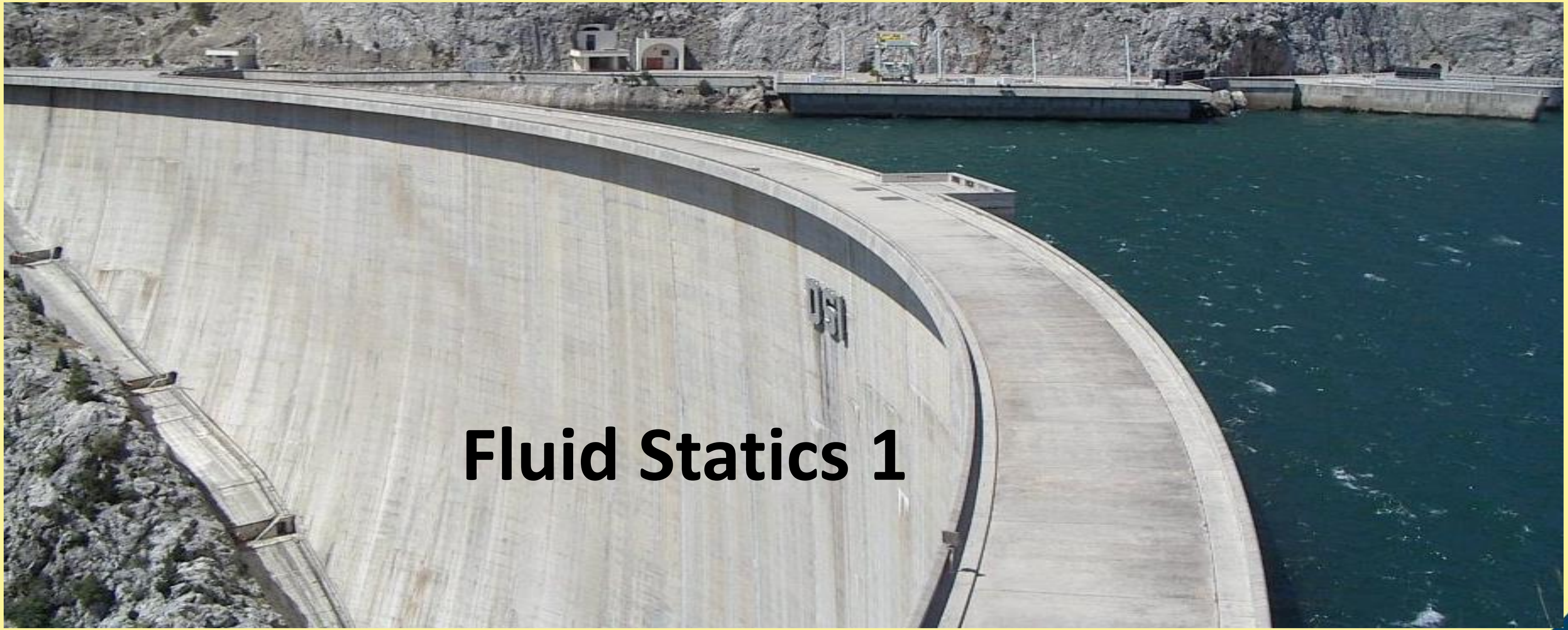

Fluid Statics 1

Hydraulic Engineering

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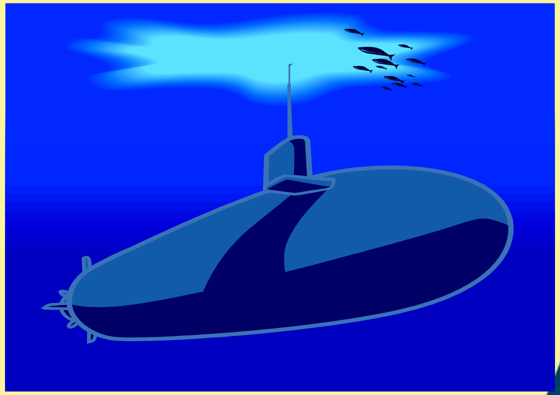
Variation of pressure with depth in a liquid

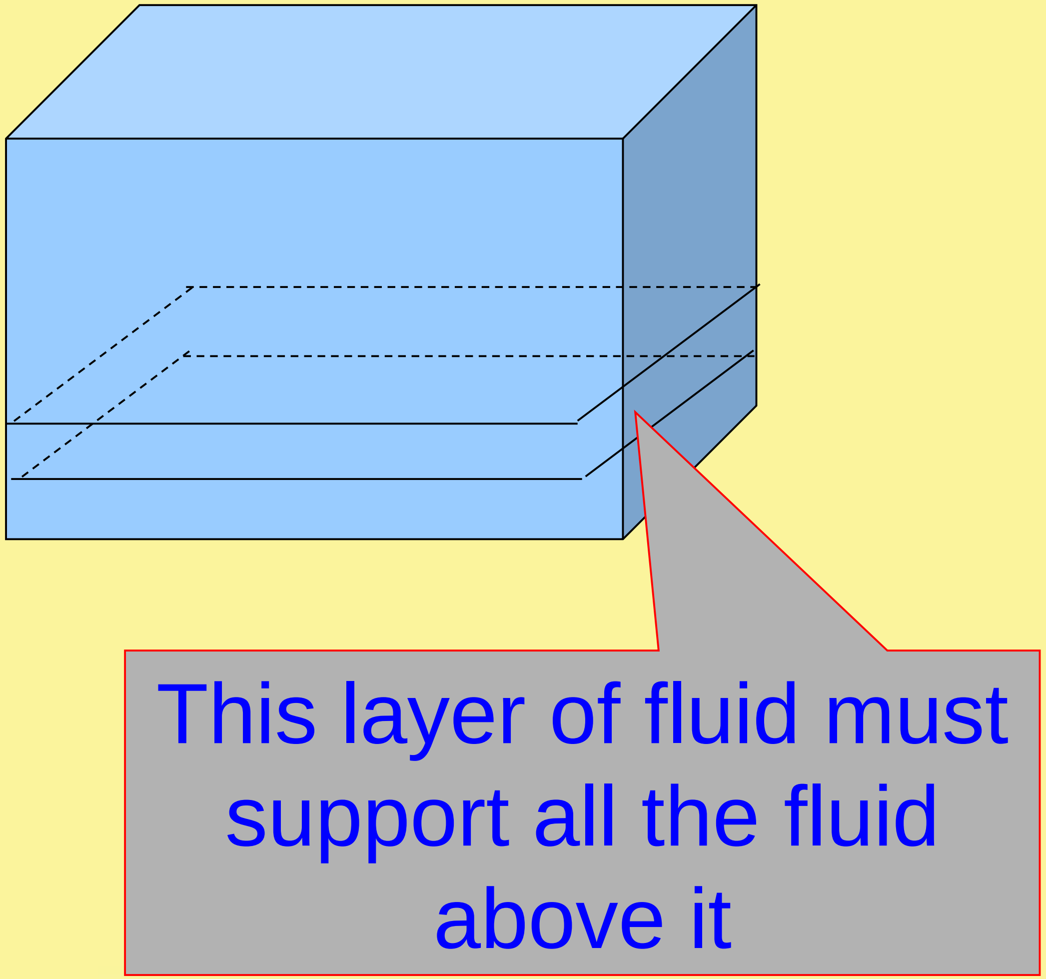
- Anybody that does scuba diving knows that the pressure increases as then dive to greater depths

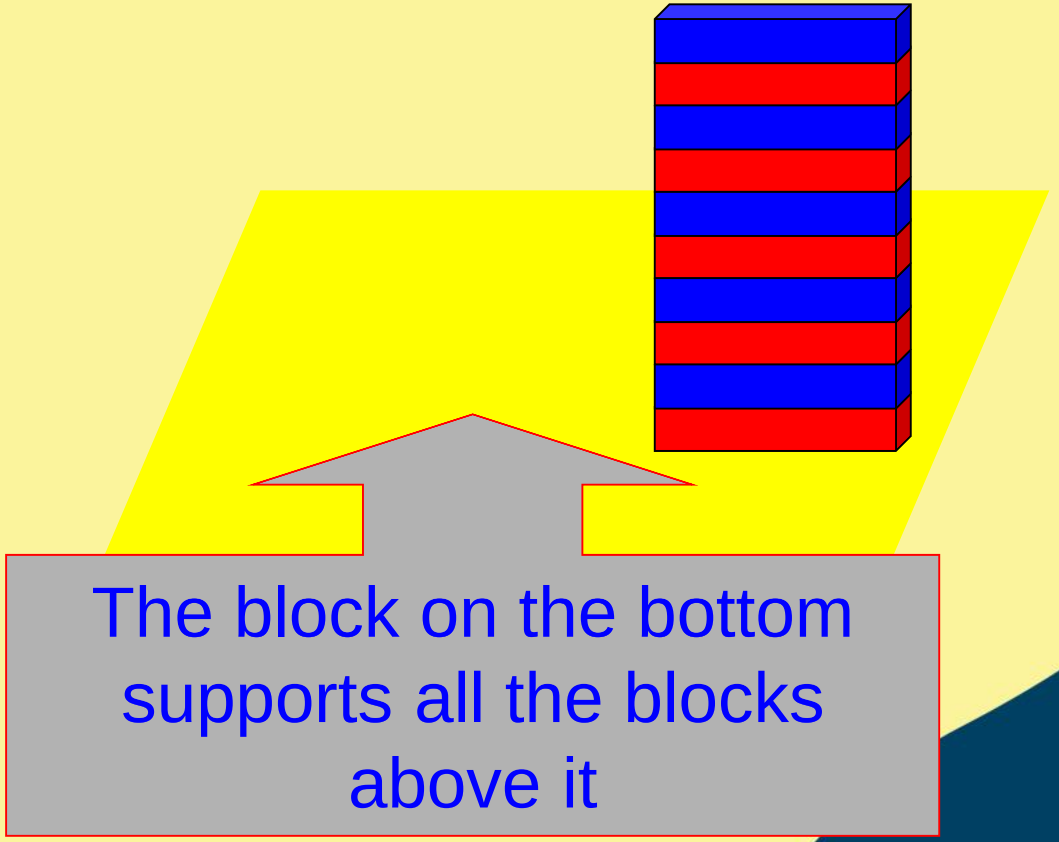
- The increasing water pressure with depth limits how deep a submarine can go



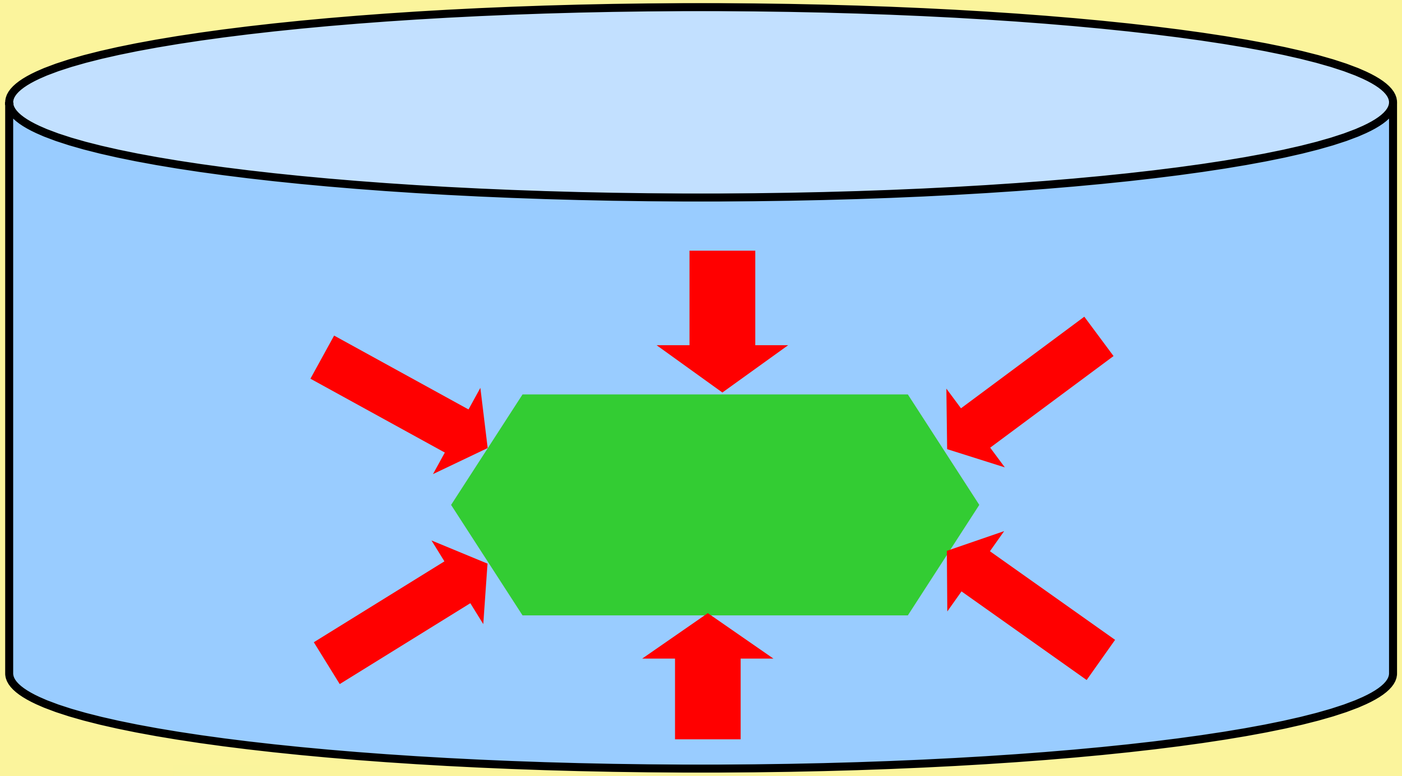
crush depth 2200 ft

Why does P increase with depth?

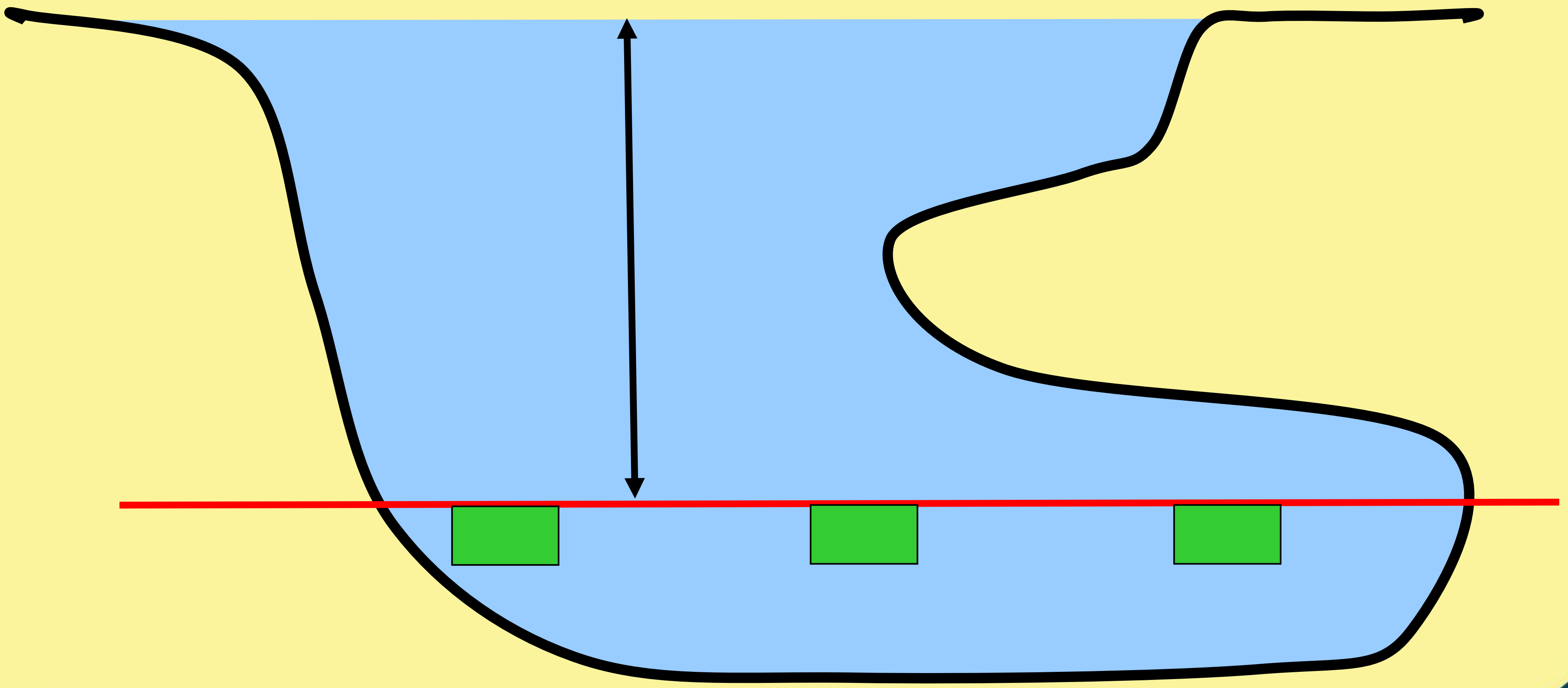


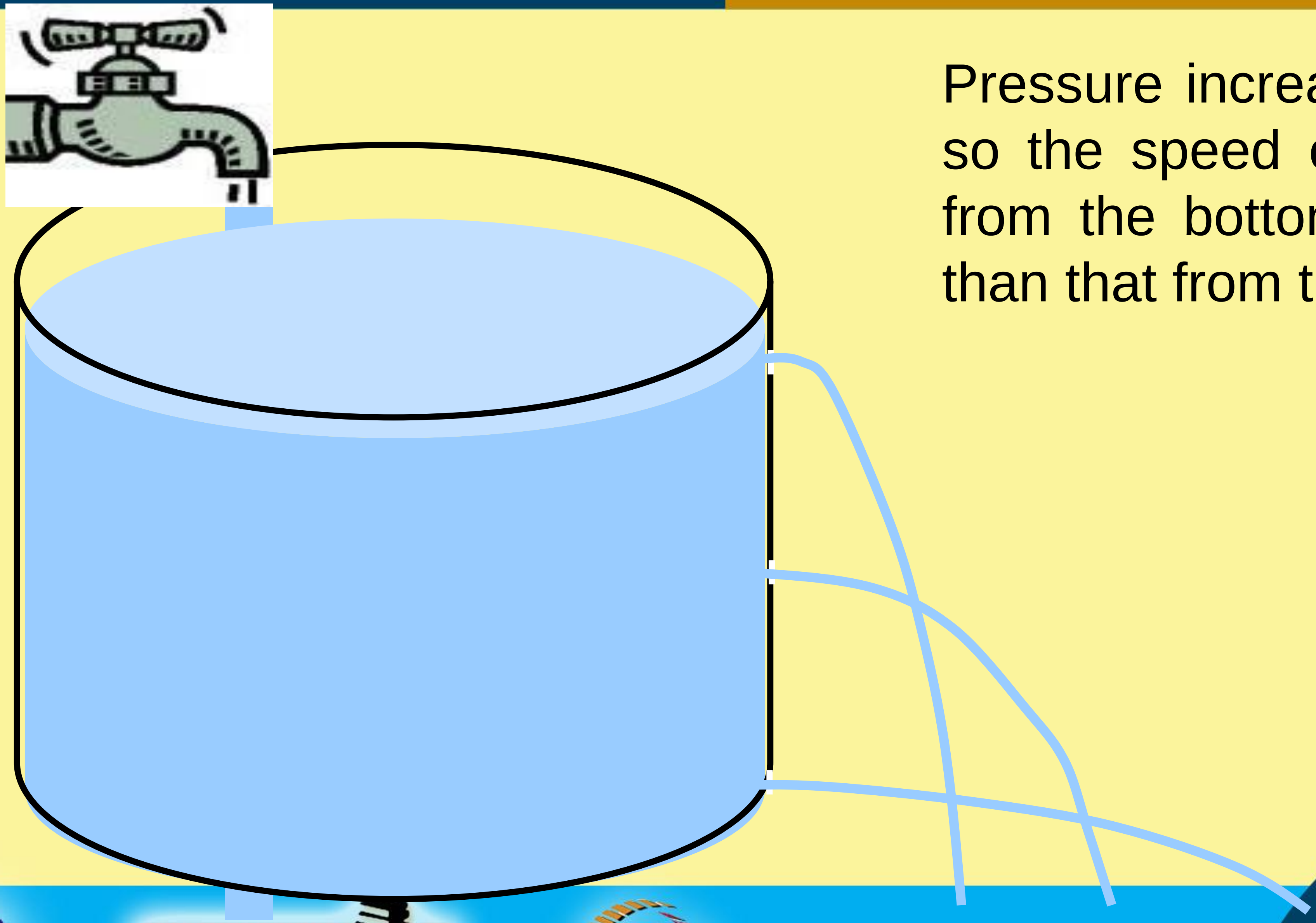


Pressure is always perpendicular to the surface of an object



Pressure depends only on depth





Pressure increases with depth, so the speed of water leaking from the bottom hole is larger than that from the higher ones.

Definitions and Applications

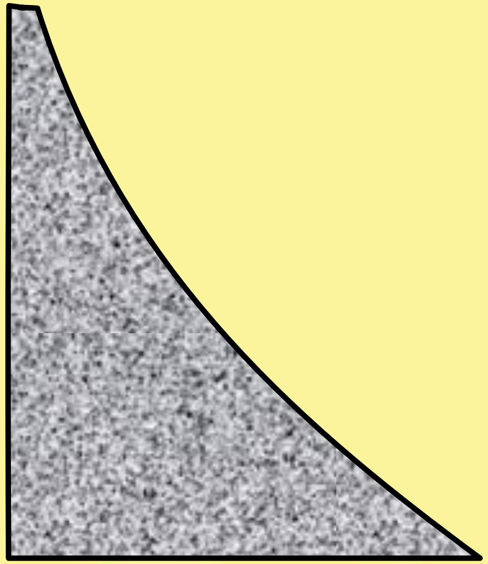
- Statics: no relative motion between adjacent fluid layers.
  - Shear stress is zero
  - Only pressure can be acting on fluid surfaces
- Gravity force acts on the fluid (body force)
- Applications:
  - Pressure variation within a reservoir
  - Forces on submerged surfaces
  - Tensile stress on pipe walls
  - Buoyant forces

Motivation?

- What are the pressure forces behind the Bhakra Nangal Dam?

Upstream face of the Dam

Crest thickness: 9.1 m
Base thickness: 191 m
WHY???



Upstream face of Bhakra Nangal Dam

What do we need to know?

- Pressure variation with direction
- Pressure variation with location
- How can we calculate the total force on a submerged surface?

Pressure Variation with Direction(Pascal's law)

Equation of Motion

F = ma

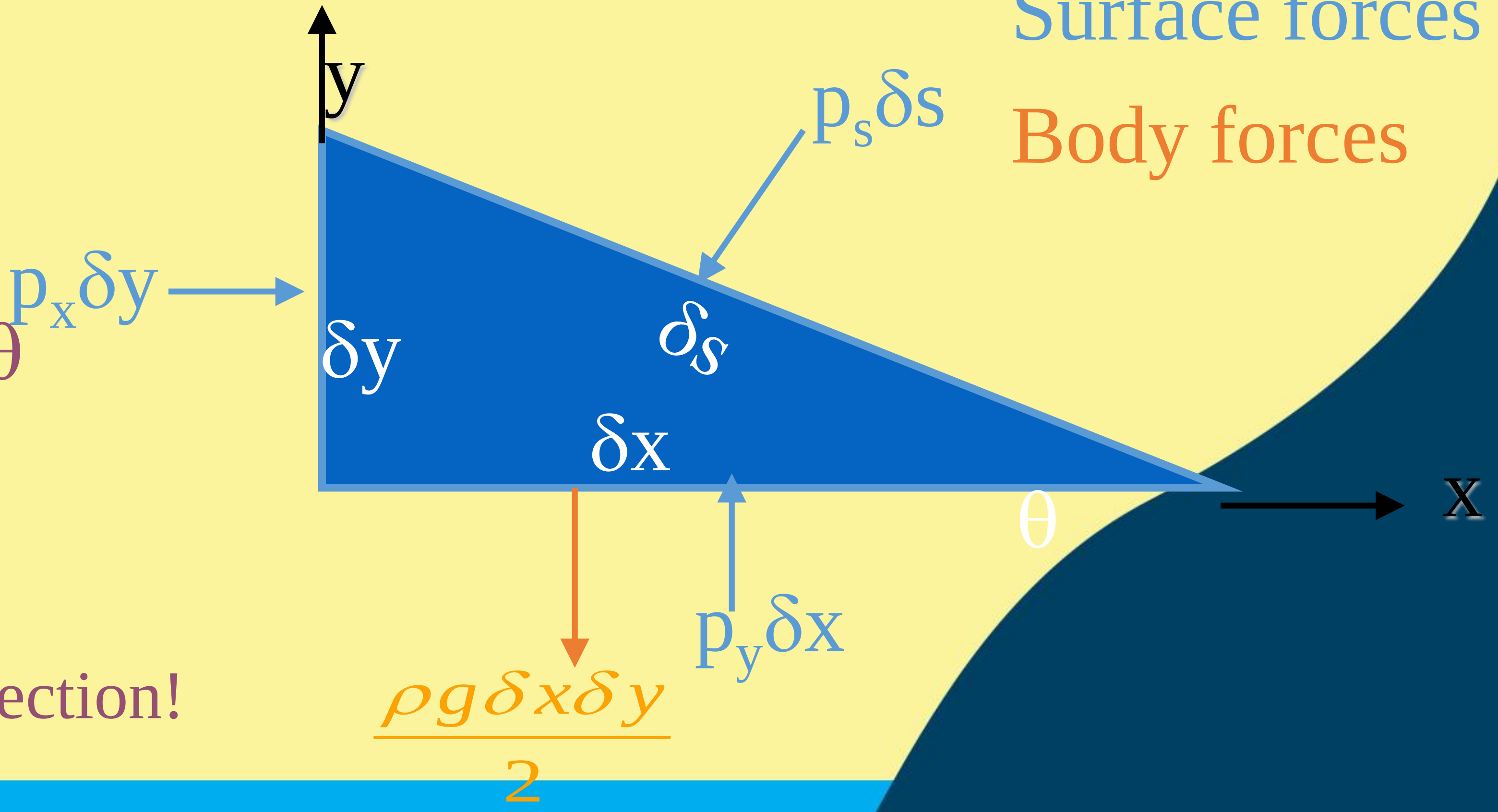
$$ma_x = \rho \frac{\delta x \delta y}{2} a_x = 0$$

$$\sum F_x = p_x \delta y - p_s \delta s \sin\theta$$

$$\delta s \sin\theta = \delta y$$

$$p_x \cancel{\delta y} - p_s \cancel{\delta y} = 0$$

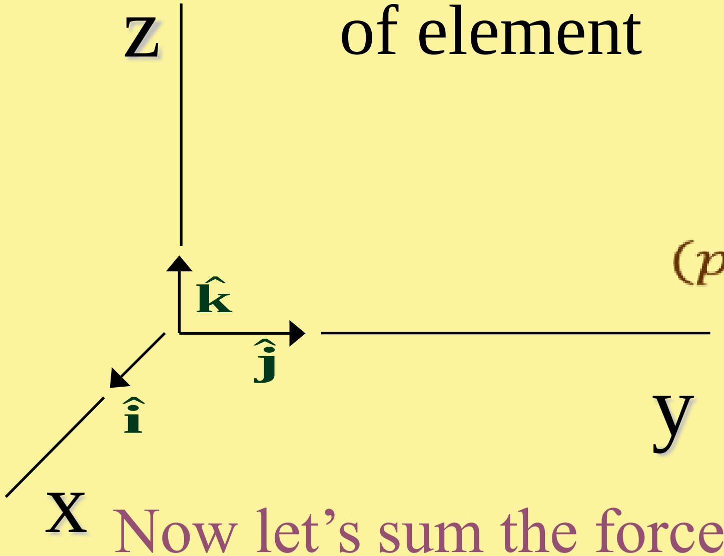
Pressure is independent of direction!



Pressure Field (pressure variation with location)

Small element of fluid in pressure gradient with arbitrary acceleration.

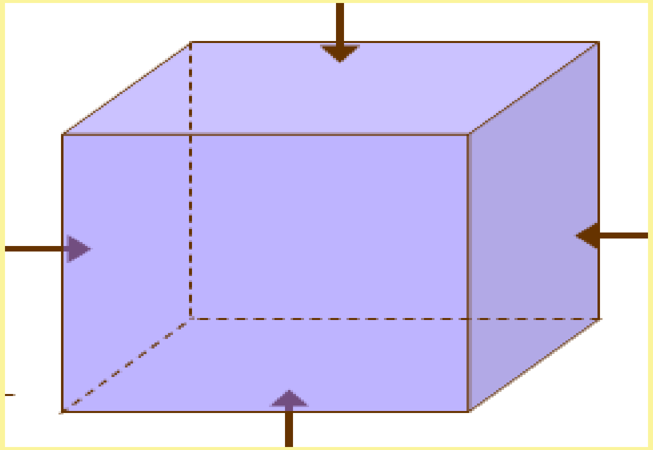
Pressure is *p* at center of element



Forces acting on surfaces of element

$$\left(p + \frac{\delta p}{\delta z}\frac{\delta z}{2}\right)\delta x \delta y$$

$$\left(p - \frac{\delta p}{\delta y}\frac{\delta y}{2}\right)\delta x \delta z$$



$$\left(p + \frac{\delta p}{\delta y}\frac{\delta y}{2}\right)\delta x \delta z$$

$$\left(p - \frac{\delta p}{\delta z}\frac{\delta z}{2}\right)\delta x \delta y$$

Mass…

$$\delta m = \rho \delta x \delta y \delta z$$

Same in x!

Now let's sum the forces in the y direction

Simplify the expression for the force acting on the element

$$\delta F_y = \left(\cancel{p} - \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z - \left(\cancel{p} + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z$$

Same in xyz!

$$\delta F_y = -\frac{\partial p}{\partial y}\delta x \delta y \delta z \qquad \delta F_x = -\frac{\partial p}{\partial x}\delta x \delta y \delta z \qquad \delta F_z = -\frac{\partial p}{\partial z}\delta x \delta y \delta z$$

$$\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k} = \nabla \mathrm{p}$$

This begs for vector notation!

$$\delta F_s = -\left(\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}\right)\delta x \delta y \delta z$$

$$\delta F_s = -\nabla \mathrm{p}\ \delta x \delta y \delta z$$

Surface Forces acting on element of fluid due to pressure gradient

Apply Newton's Second Law

$$-\delta W \boldsymbol{k} = -\gamma \delta x \delta y \delta z \; \boldsymbol{k}$$

Since the z axis is vertical , the weight of the element is

$$\sum \delta F = \delta F_s - \delta W \boldsymbol{k} = \delta \mathrm{m} \, \mathbf{a}$$

Newton's second law

$$\delta m = \gamma \delta x \delta y \delta z$$

Mass of element of fluid

$$-\nabla \mathrm{p} \; \delta x \delta y \delta z - \gamma \; \delta x \delta y \delta z \; \mathbf{k} = \rho \; \delta x \delta y \delta z \; \mathbf{a}$$

Substitute into Newton's 2^{nd} Law

$$-\nabla \mathrm{p} - \gamma \; \mathbf{k} = \rho \; \mathbf{a}$$

General Equation of motion for fluid with no shearing stress

Fluid at Rest

$$\nabla \mathbf{p} + \gamma\, \mathbf{k} = 0$$

For fluid at rest **a** =0

Or in component form

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial y} = 0 \qquad \frac{\partial p}{\partial z} = -\gamma$$

In horizontal plane the pressure does not change and varies only with depth as written by ordinary differential equation

$$\frac{dp}{dz} = -\gamma$$

γ may or may not be constant

Pressure Variation When the Specific Weight is Constant

- What are the two things that could make specific weight (γ) vary in a fluid?

$\underline{\gamma = \rho g}$

Changing density

Changing gravity

$$dp = -\gamma dz$$

Constant specific weight! (Incompressible Fluid)

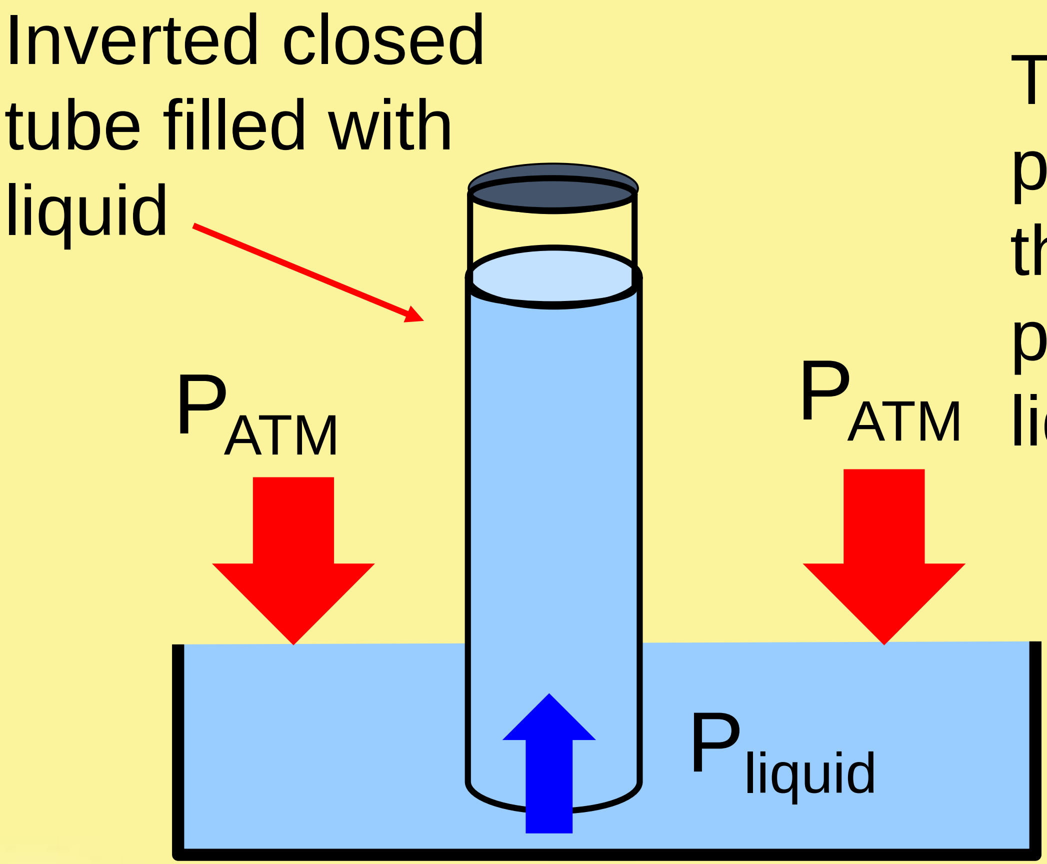
$$\int_{p1}^{p2} dp = -\gamma \int_{z1}^{z2} dz$$

Piezometric head is constant in a static incompressible fluid

$$p_2 - p_1 = -\gamma(z_2 - z_1)$$

$$\overbrace{\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma}} + z_2$$

Measuring atmospheric pressure - Barometers



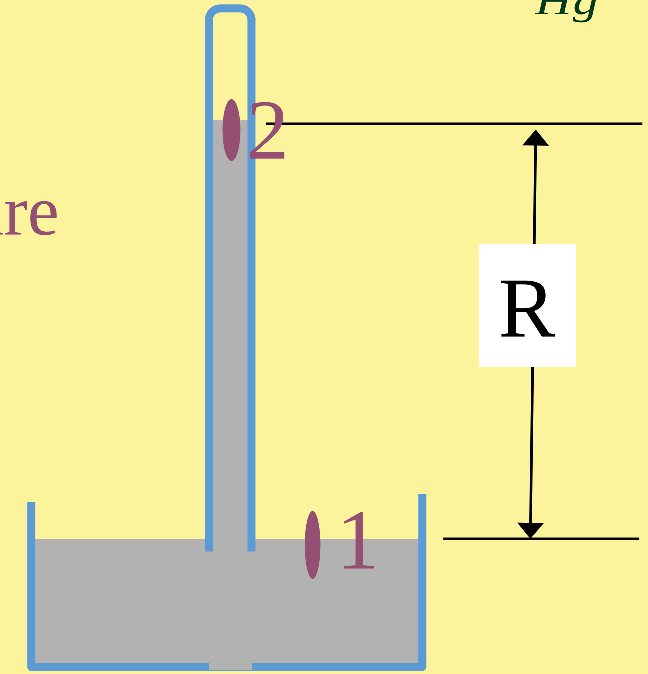
The column of liquid is held up by the pressure of the liquid in the tank. Near the surface this pressure is atmospheric pressure, so the atmosphere holds the liquid up.

Mercury Barometer (Question)

What is the local atmospheric pressure (in kPa) when R is 750 mm Hg?

$$S_{Hg} = 13.6 = \frac{\rho_{Hg}}{\rho_{water}}$$

P_2 = Hg vapor pressure



Assume constant ρ

Mercury Barometer (Question)

What is the local atmospheric pressure (in kPa) when R is 750 mm Hg?

$$S_{Hg} = 13.6 = \frac{r_{Hg}}{r_{water}}$$

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2$$

$$p_1 = \cancel{p_2} + \gamma_{Hg}(z_2 - z_1)$$

$$p_1 = \gamma_{Hg} R$$

$$\gamma_{Hg} = S_{Hg}\gamma_{water}$$

$$p_1 = S_{Hg}\gamma R$$

$$p_1 = 13.6(9806 N/m^3)0.75m = 100{,}000 Pa$$

Pressure Variation in a Compressible Fluid

- Perfect gas at constant temperature (Isothermal)
- Perfect gas with constant temperature gradient – You should try this at home.

Perfect Gas at Constant Temperature (Isothermal)

$$dp = -\gamma dz \qquad \gamma = \rho g$$

ρ is function of p

$$p\forall = nRT \qquad \rho = \frac{nM_{gas}}{\forall} = \frac{pM_{gas}}{RT}$$

M_{gas} is molecular mass

$$dp = -\frac{pM_{gas}g}{RT}dz$$

$$\int_{p_1}^{p_2}\frac{dp}{p} = \int_{z_1}^{z_2} -\frac{M_{gas}g}{RT}dz$$

Integrate…

$$\ln\frac{p_2}{p_1} = -\frac{M_{gas}g}{RT}(z_2 - z_1)$$

$$p_2 = p_1 e^{\left[-\frac{M_{gas}g}{RT}(z_2 - z_1)\right]}$$

Pressure Measurement

- Barometers — Measure atmospheric pressure
- Manometers
  - Standard — Pressure relative to atm.
  - Differential — Pressure difference between 2 pts.
- Pressure Transducers (Read yourself)

Standard Manometers

What is the gauge pressure at A given h?

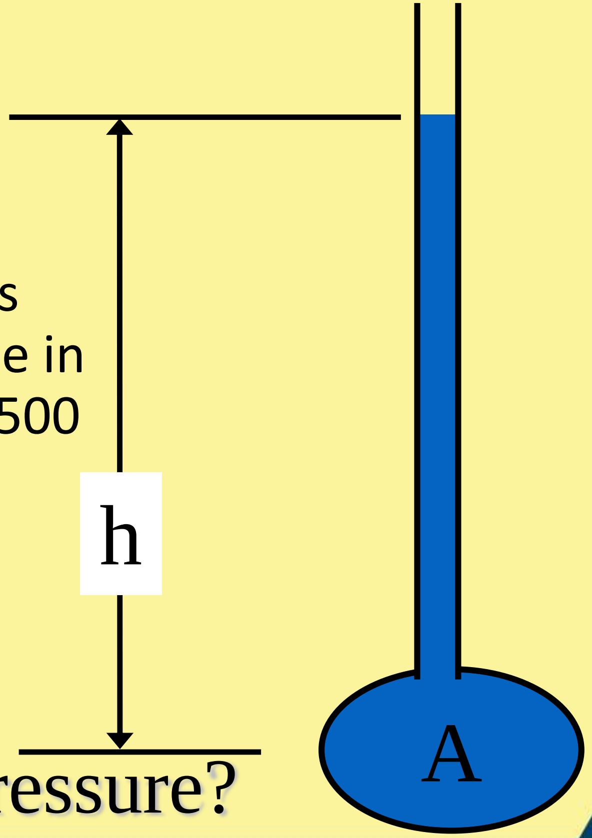
$p = \gamma h$

Pressure in water distribution systems commonly varies between 175 to 700 kPa. How high would the water rise in a manometer connected to a pipe containing water at 500 kPa?

$h = p/\gamma$

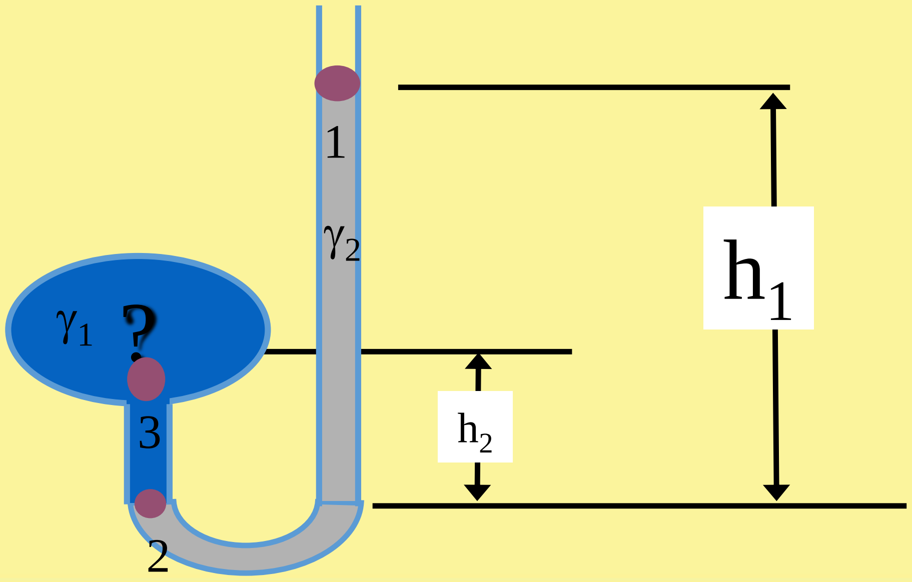
$h = 500{,}000\ \text{Pa}/(9800\ \text{N/m}^3)$

$h = 51\ \text{m}$ Why is this a reasonable pressure?



Manometers for High Pressures

Find the **gauge** pressure in the center of the sphere. The sphere contains fluid with γ_1 and the manometer contains fluid with γ_2.



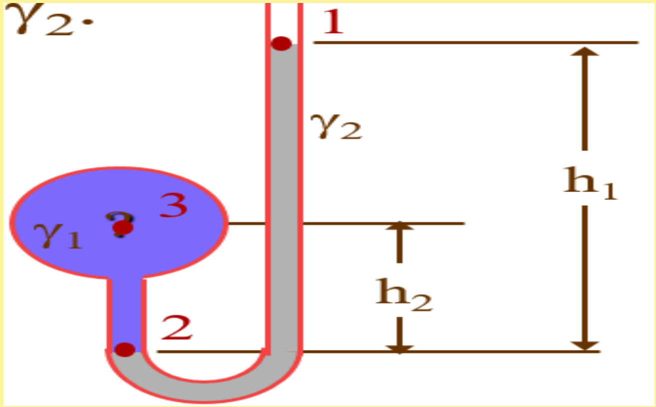
Manometers for High Pressures

What do you know? $\underline{P_1 = 0}$

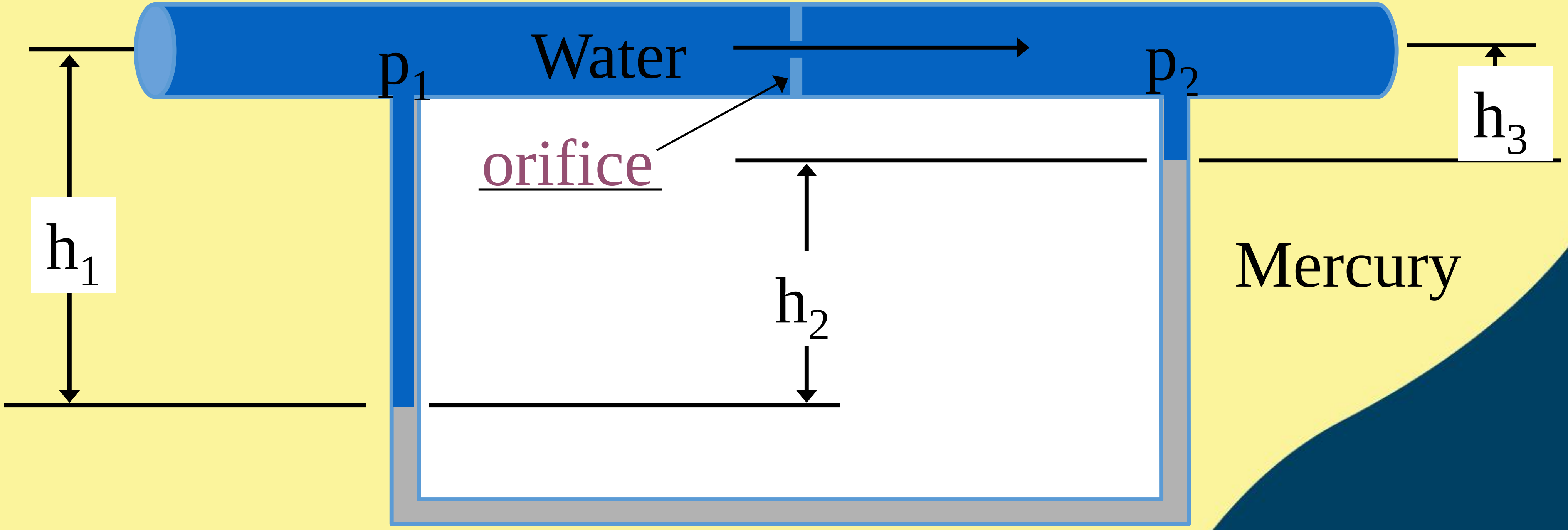
Use statics to find other pressures.

$P_1 + h_1\gamma_2 - h_2\gamma_1 = P_3$

For small h_1 use fluid with high density. Mercury!

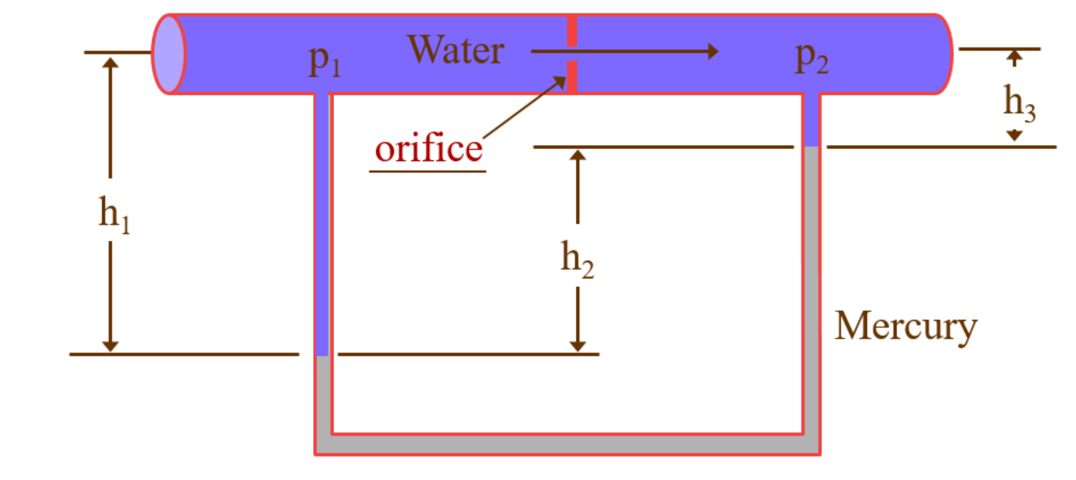


Differential Manometers



Find the drop in pressure between point 1 and point 2.

Differential Manometers



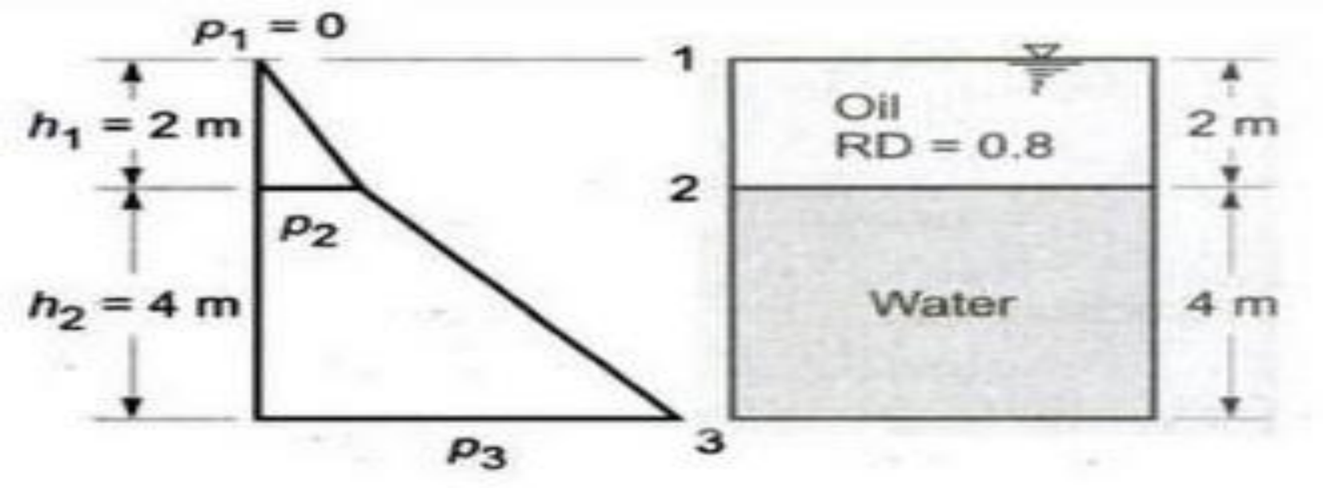
$$p_1 + h_1\gamma_w - h_2\gamma_{Hg} - h_3\gamma_w = p_2$$

$$p_1 - p_2 = (h_3 - h_1)\gamma_w + h_2\gamma_{Hg}$$

$$p_1 - p_2 = h_2(\gamma_{Hg} - \gamma_w)$$

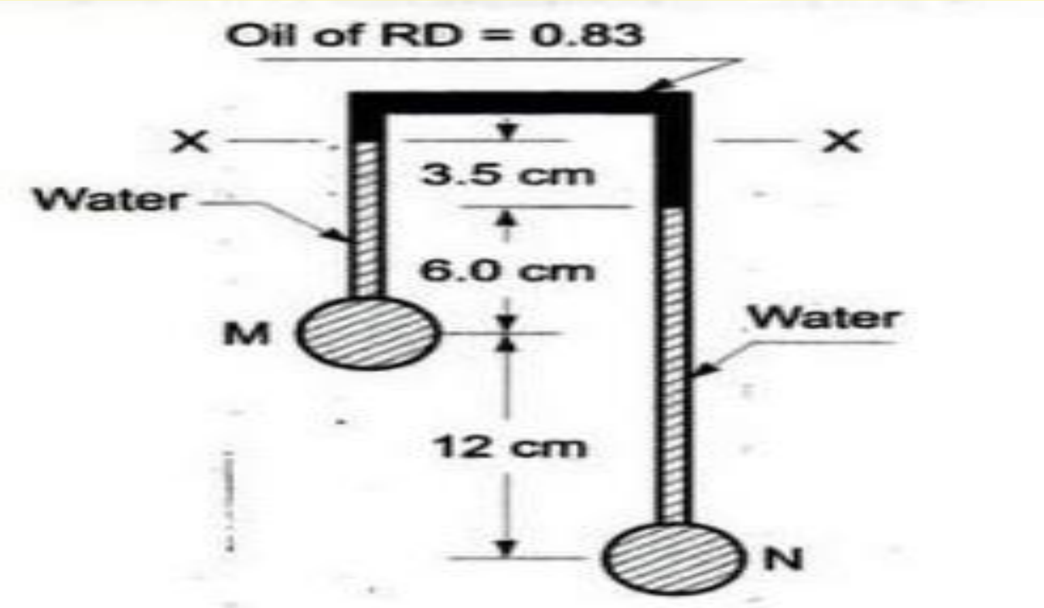
Practice Problem

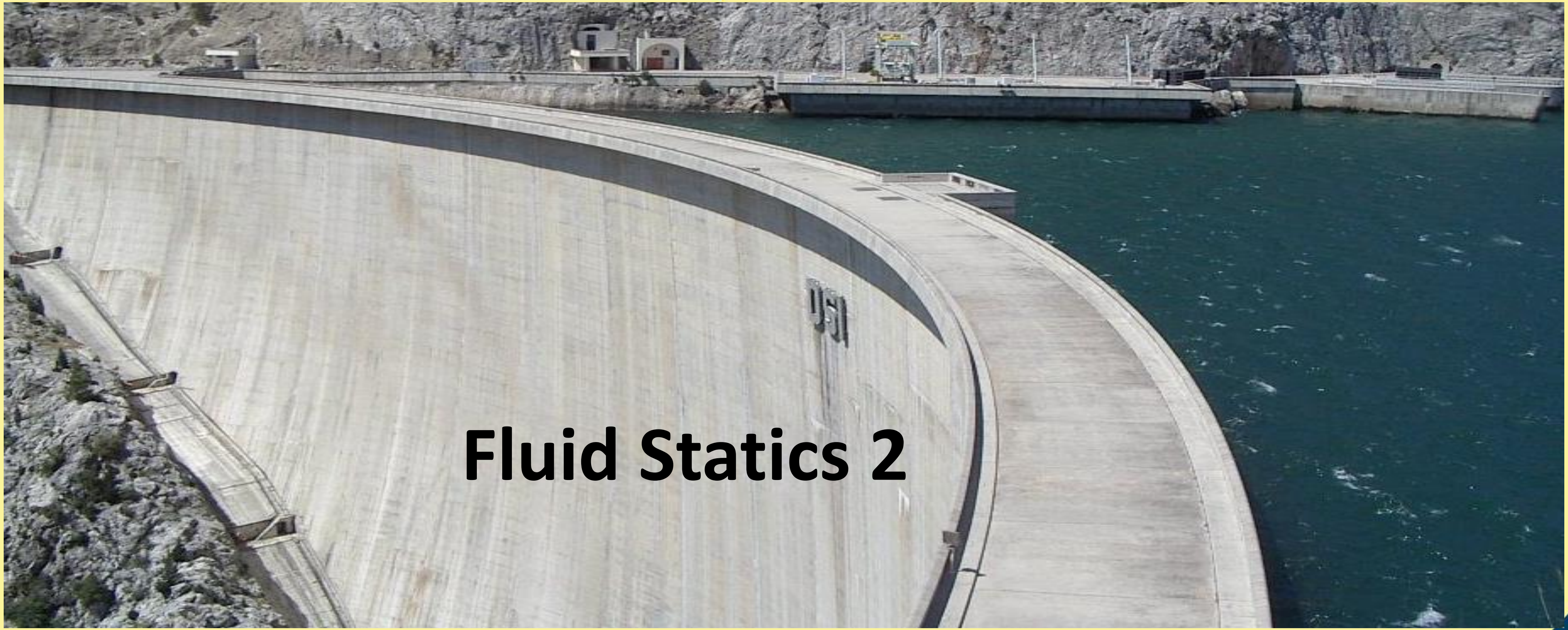
A 6 m deep tank contains 4 m of water and 2 m of oil of relative density 0.88. Determine the pressure at bottom of the tank.



Practice Problem

For the manometer shown in Fig below. calculate the pressure difference between points M and N.





Fluid Statics 2

Hydraulic Engineering

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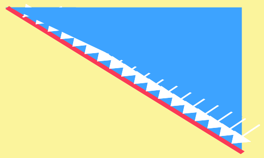
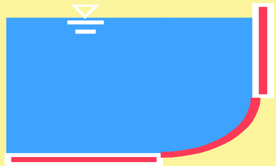
Static Surface Forces

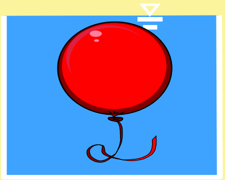
- Forces on plane areas



- Forces on curved surfaces



- Buoyant force

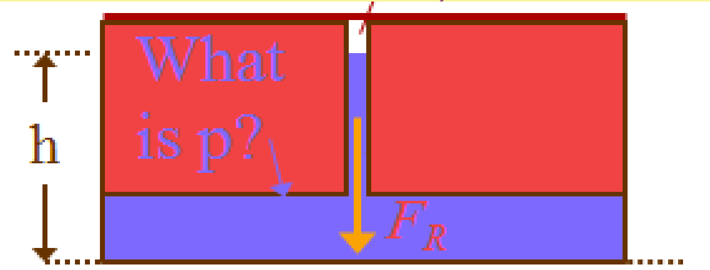


Forces on Plane Areas: Horizontal surfaces

net

P = 500 kPa

What is the force on the bottom of this tank of water?



h = Vertical distance to free surface

gauge

$$F_R = \int p\,dA = p\int dA = pA \qquad p = \rho g h$$

$$F_R = \rho g\, hA$$

= volume

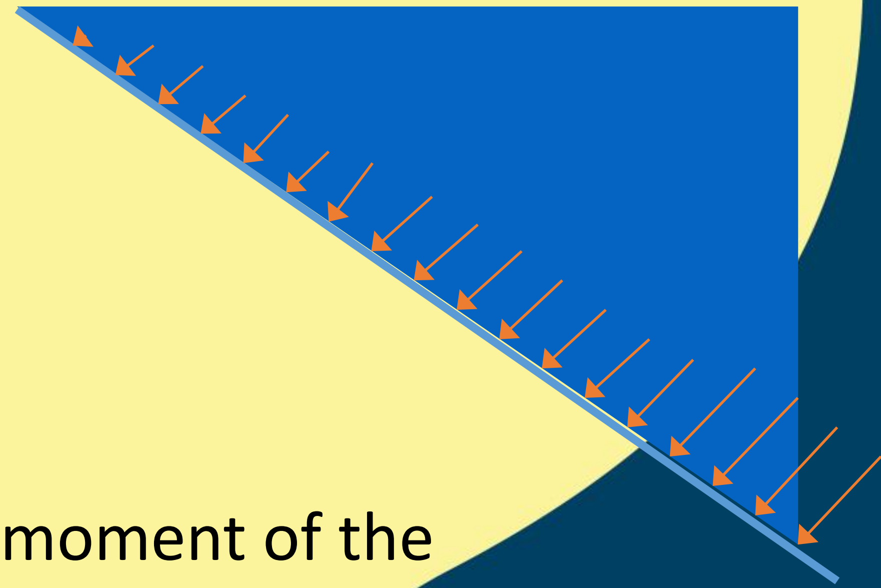
F_R = weight of overlying fluid!

F is normal to the surface and towards the surface if p is positive.

F passes through the centroid of the area.

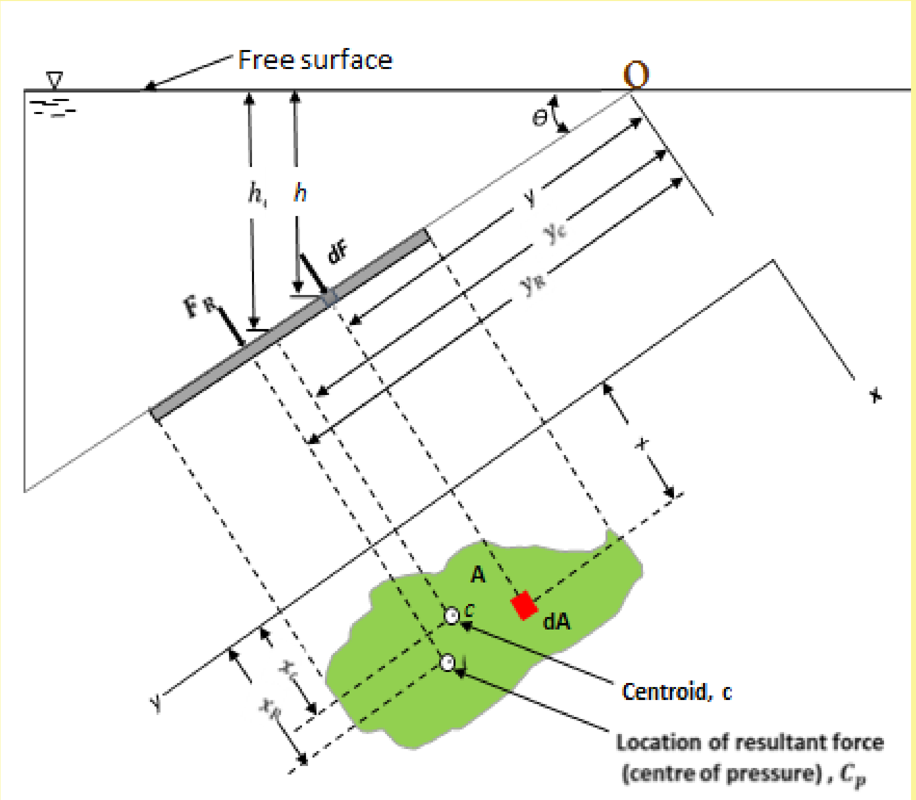
$$-\nabla p = \rho \boldsymbol{a} \qquad -\frac{\partial p}{\partial x} = \rho a_x = 0$$

Forces on Plane Areas: Inclined Surfaces



- Direction of force Normal to the plane
- Magnitude of force
  - integrate the pressure over the area
  - pressure is no longer constant!
- Line of action
  - Moment of the resultant force must equal the moment of the distributed pressure force

Forces on Plane Areas: Inclined Surfaces



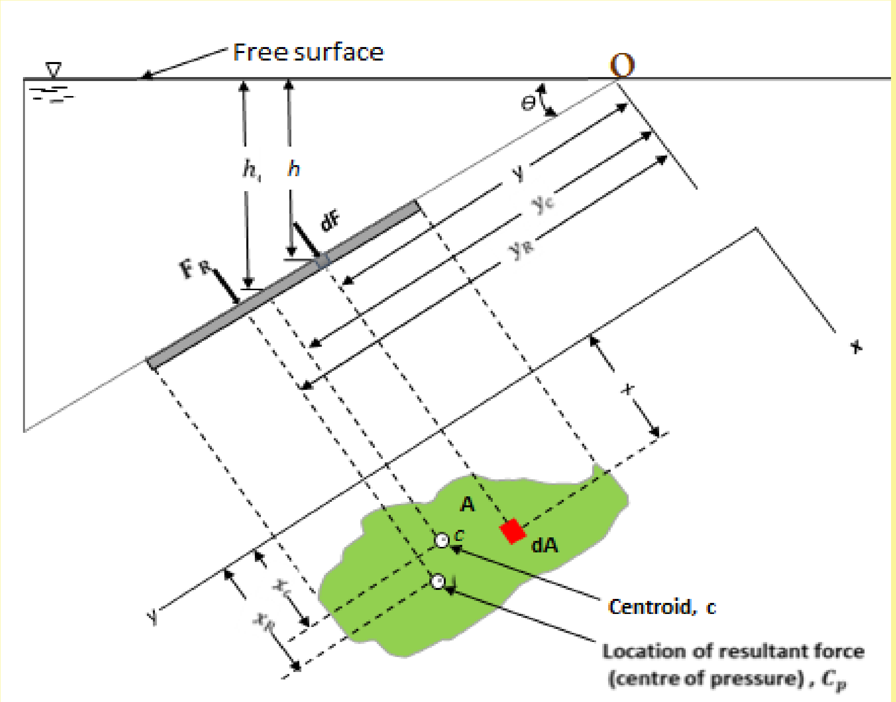
Determine location, direction and magnitude of the Resultant force acting on one side of this area due to the liquid in contact with water

Let the plane in which the surface lies intersect the free surface at point O

Let this make an angle θ with the surface

The x-y coordinate system is defined such that O is the origin and x axis is directed along the surface as shown

Forces on Plane Areas: Inclined Surfaces



At any given depth h,   Force acting on dA

dF = γh dA   (perpendicular to surface)

$$F_R = \int_A \gamma h dA = \int_A \gamma y sin\theta dA$$

$$F_R = \gamma sin\theta \int_A y dA$$

First moment of inertia

$$\int_A y dA = y_c A$$

y_c is coordinate of centroid of area A measured from x axis which passes through O

$$F_R = \gamma A y_c sin\theta$$

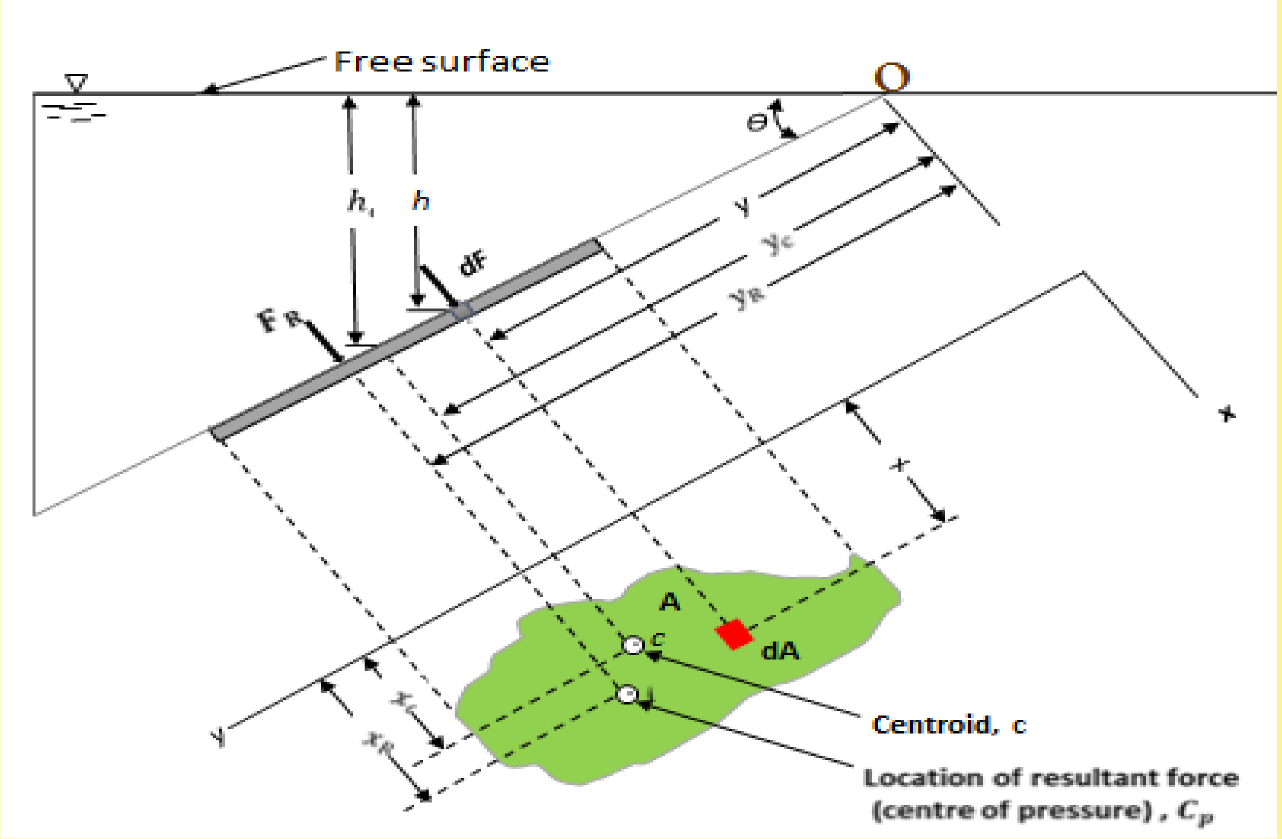
$$F_R = \gamma A h_c$$

h_c vertical distance from fluid surface to centroid of area

Forces on Plane Areas: Important Results

- Equation of F_R suggests that the magnitude of resultant force is equal to pressure at the centroid multiplied by the total area.
- F_R must be perpendicular to the surface
  - Since all differential forces were perpendicular
- The point through which resultant force acts is called center of pressure
- The center of pressure is not at the centroid (because pressure is increasing with depth)

Center of Pressure: y_R



Coordinate y_r can be Determined by summation of moment around x-axis

Center of Pressure: y_R

$y_R F_R = \int_A y\, dF = \int_A \gamma \sin\theta\, y^2 dA$ Sum of the moments

$F_R = \gamma A y_c \sin\theta$

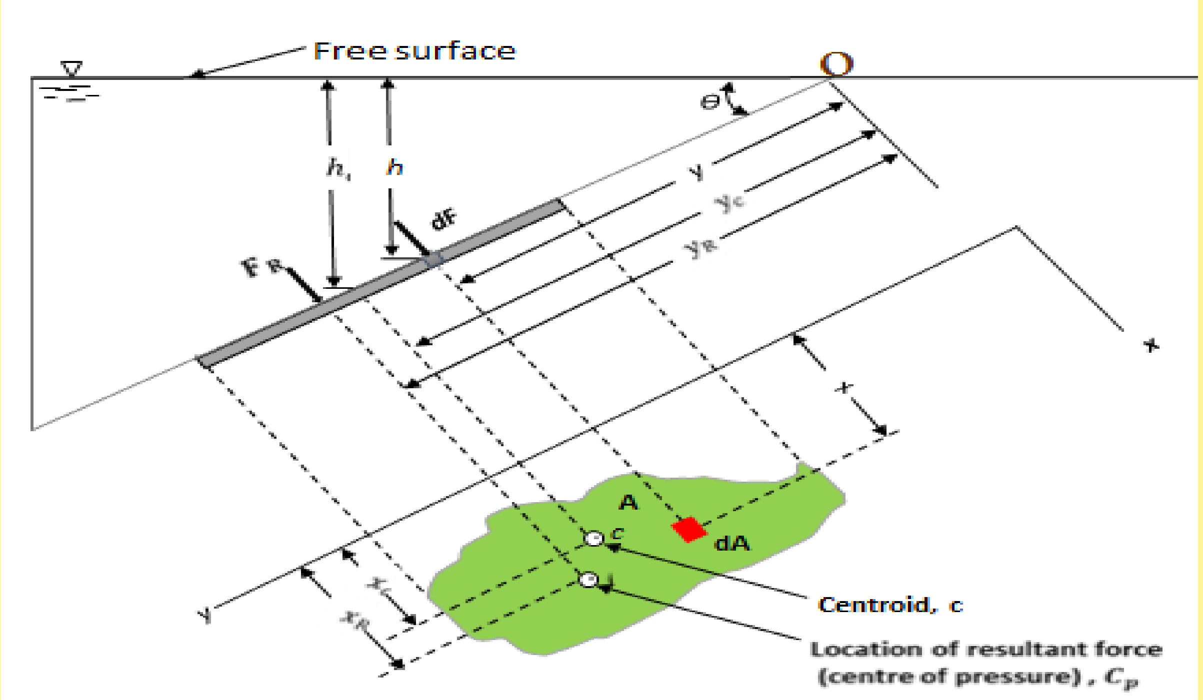
$y_R = \dfrac{\int_A y^2 dA}{y_c A}$ Second moment of Inertia I_x wrt x axis $y_R = \dfrac{I_x}{y_c A}$

Using parallel axis theorem $I_x = I_{xc} + A y_c^2$

I_{xc} is the second moment of area wrt an axis passing through centroid and parallel to x axis

$$y_R = y_c + \frac{I_{xc}}{y_c A}$$

Center of Pressure: x_R



Coordinate x_r can be determined by summation of moment around y-axis

Center of Pressure: x_R

$$x_R F_R = \int_A x\, dF = \int_A \gamma \sin\theta\, xy\, dA$$ Sum of the moments

$$F_R = \gamma A y_c \sin\theta$$

$$x_R = \frac{\int_A xy\, dA}{y_c A}$$

Product of Inertia I_{xy} wrt x and y axis $x_R = \frac{I_{xy}}{y_c A}$

Using parallel axis theorem $I_{xy} = I_{xyc} + A y_c x_c$

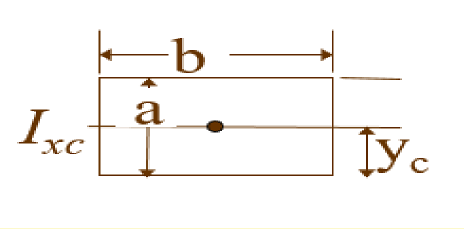
I_{xyc} is the product of inertia wrt an orthogonal coordinate System passing through centroid

$$x_R = x_c + \frac{I_{xyc}}{y_c A}$$

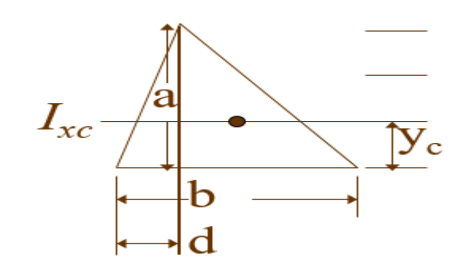
Center of Pressure

- If the submerged area is symmetrical wrt to axis passing through centroid and parallel to either x or y axis, the resultant force must pass lie along $x=x_c$ since I_{xyc} is identically zero.
- As y_c increases, center of pressure moves closer to the centroid of the area.
- Since $y_c= h_c/\sin\theta$, y_c will increase if depth of submergence h_c increases or for a given depth the area is rotated such that the angle θ decreases.

Properties of Areas

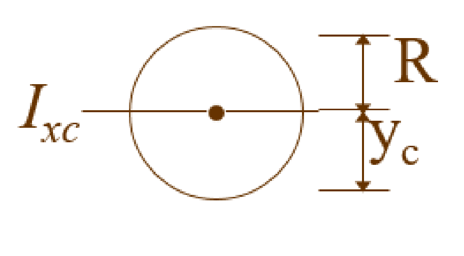


$$A = ab \qquad y_c = \frac{a}{2} \qquad I_{xc} = \frac{ba^3}{12} \qquad I_{xyc} = 0 \qquad \frac{I_{xc}}{A} = \frac{a^2}{12}$$



$$A = \frac{ab}{2} \qquad y_c = \frac{a}{3} \qquad I_{xc} = \frac{ba^3}{36} \qquad I_{xyc} = \frac{ba^2}{72}(b - 2d)$$

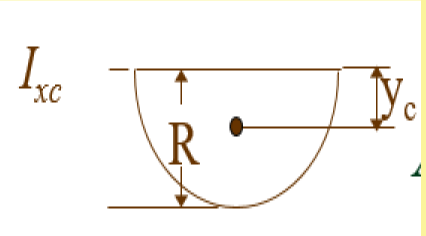
$$x_c = \frac{b+d}{3} \qquad \frac{I_{xc}}{A} = \frac{a^2}{18}$$



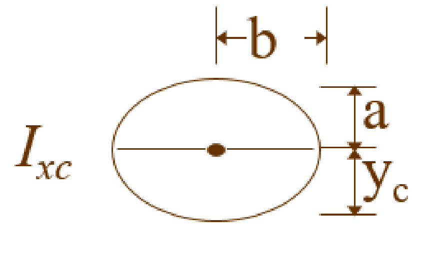
$$A = \pi R^2 \qquad y_c = R \qquad I_{xc} = \frac{\pi R^4}{4}$$

$$I_{xyc} = 0 \qquad \frac{I_{xc}}{A} = \frac{R^2}{4}$$

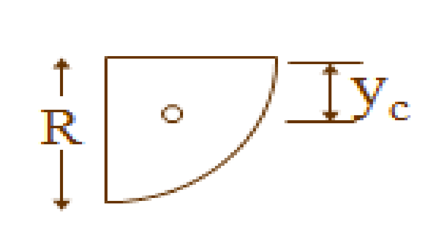
Properties of Areas



$$A = \frac{\pi R^2}{2} \qquad y_c = \frac{4R}{3\pi} \qquad I_{xc} = \frac{\pi R^4}{8} \qquad I_{xyc} = 0 \qquad \frac{I_{xc}}{A} = \frac{R^2}{4}$$



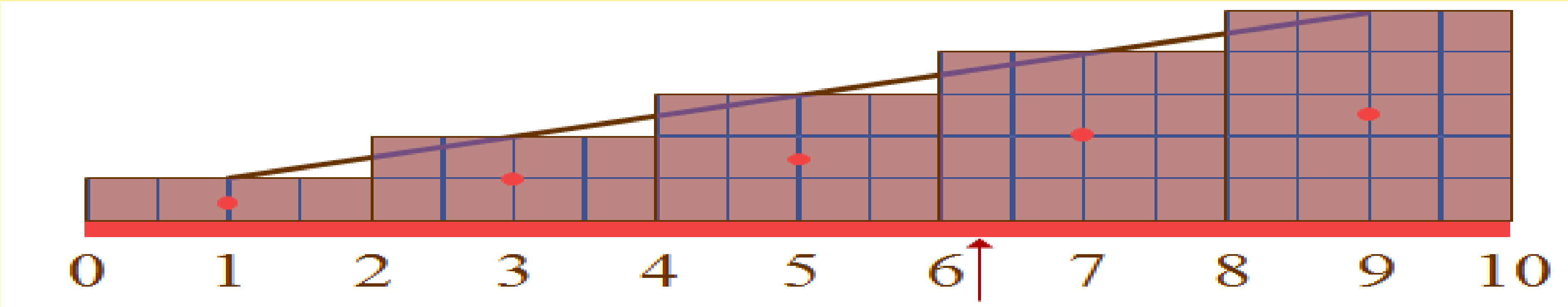
$$A = \pi ab \qquad y_c = a \qquad I_{xc} = \frac{\pi b a^3}{4} \qquad I_{xyc} = 0 \qquad \frac{I_{xc}}{A} = \frac{a^2}{4}$$



$$A = \frac{\pi R^2}{4} \qquad y_c = \frac{4R}{3\pi} \qquad I_{xc} = \frac{\pi R^4}{16}$$

$$\frac{I_{xc}}{A} = \frac{R^2}{4}$$

Location of average pressure vs. line of action



What is the average depth of blocks? 3 blocks

Where does that average occur? 5

Where is the resultant? Use moments

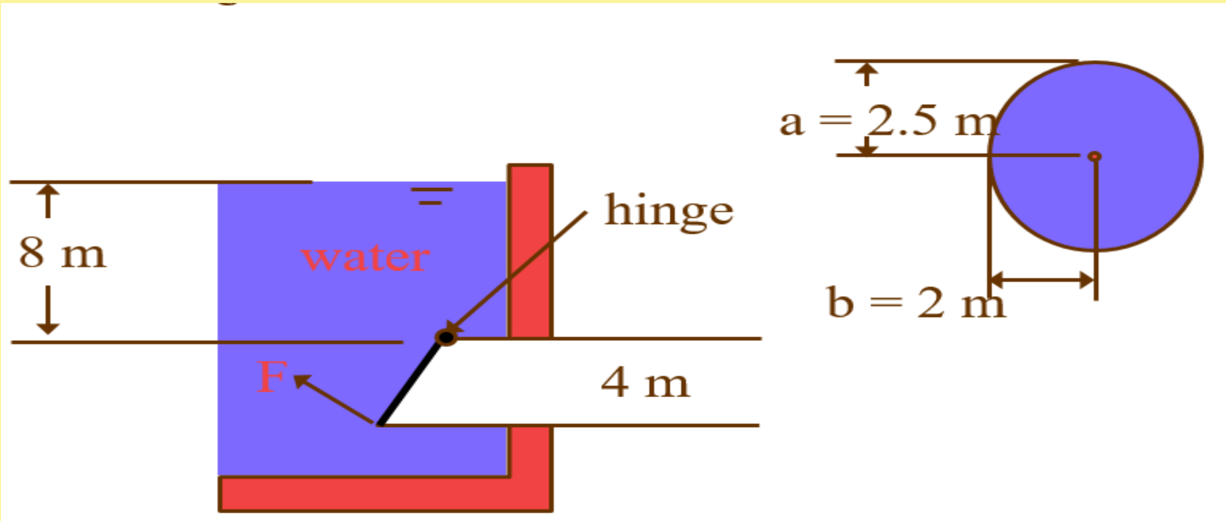
$$y_R F_R = 1m \cdot 4blocks + 3m \cdot 8blocks + 5m \cdot 12blocks + 7m \cdot 16blocks + 9m \cdot 20blocks$$

$$y_R F_R = 380m \cdot blocks$$

$$y_R = \frac{380m \cdot blocks}{60blocks} = 6.333m$$

Example using Moments

An elliptical gate covers the end of a pipe 4 m in diameter. If the gate is hinged at the top, what normal force F applied at the bottom of the gate is required to open the gate when water is 8 m deep above the top of the pipe and the pipe is open to the atmosphere on the other side? Neglect the weight of the gate.



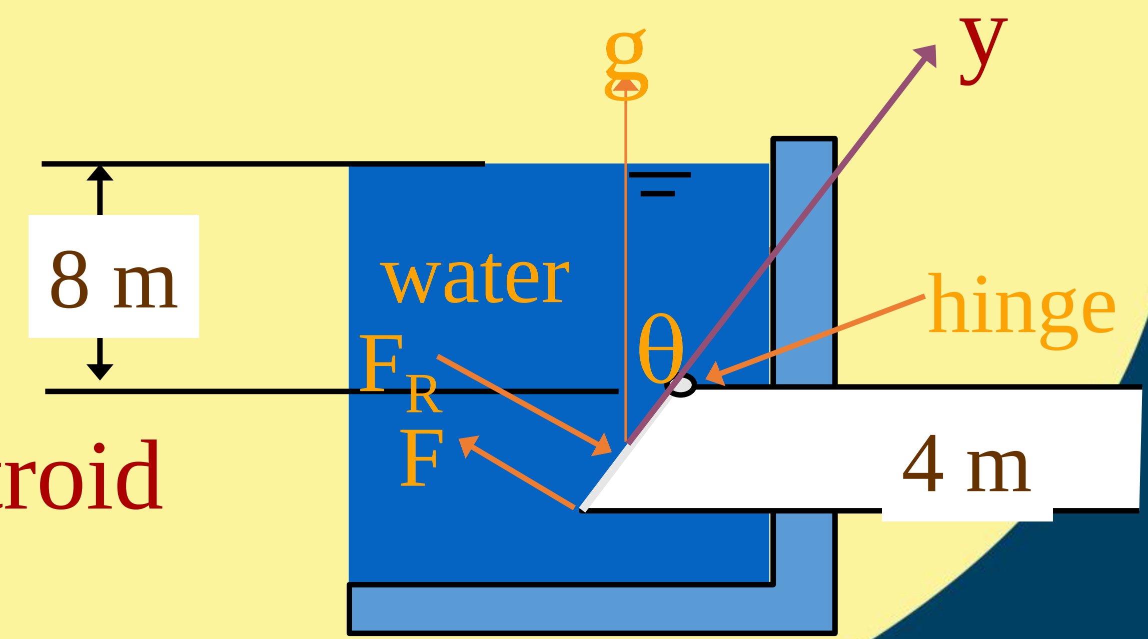
Magnitude of the Force

Pressure datum? atm Y axis?

$$F_R = p_c A$$

$$A = \pi ab$$

h_c = 10 m Depth to the centroid



$$F_R = \rho g h_c \pi ab$$

$$F_R = \left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\left(9.8\frac{\text{m}}{\text{s}^2}\right)(10\ \text{m})\pi(2.5\ \text{m})(2\ \text{m})$$

F_R= =1.54 MN

Location of Resultant Force

$$y_R = y_c + \frac{I_{xc}}{y_c A}$$

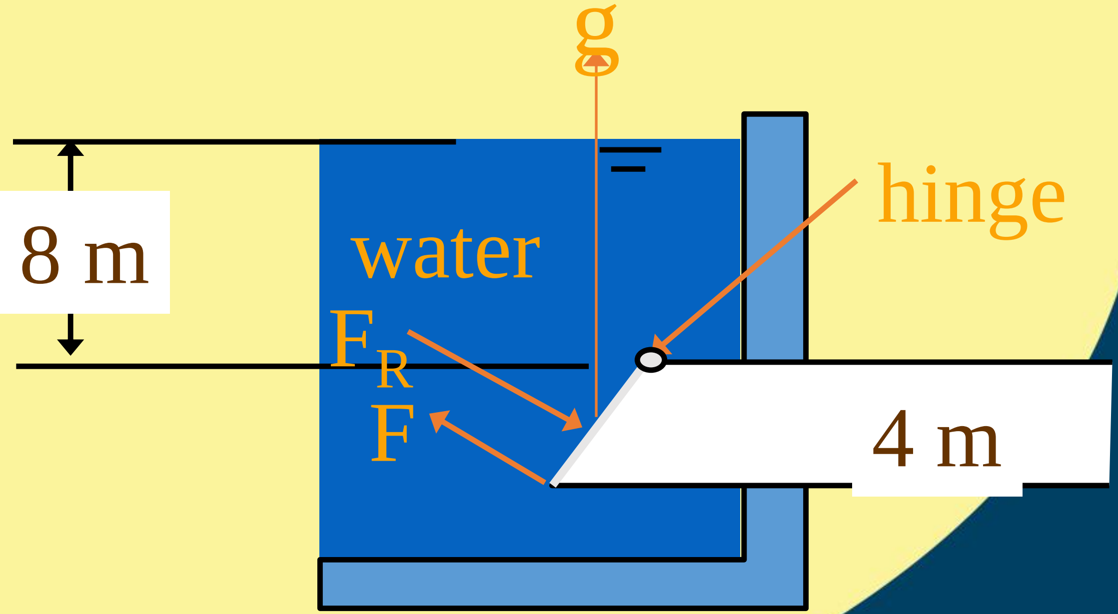
$$\frac{I_{xc}}{A} = \frac{a^2}{4}$$

?

$$y_c = \frac{h_c}{\sin\theta} \qquad \sin\theta = 4/5$$

$$y_R = y_c + \frac{a^2}{4h_c}\sin\theta$$

$$y_R = y_c + \frac{a^2}{5h_c} = \underline{0.125\text{ m}} \qquad x_R = \underline{0}$$



Force Required to Open Gate

How do we find the required force?

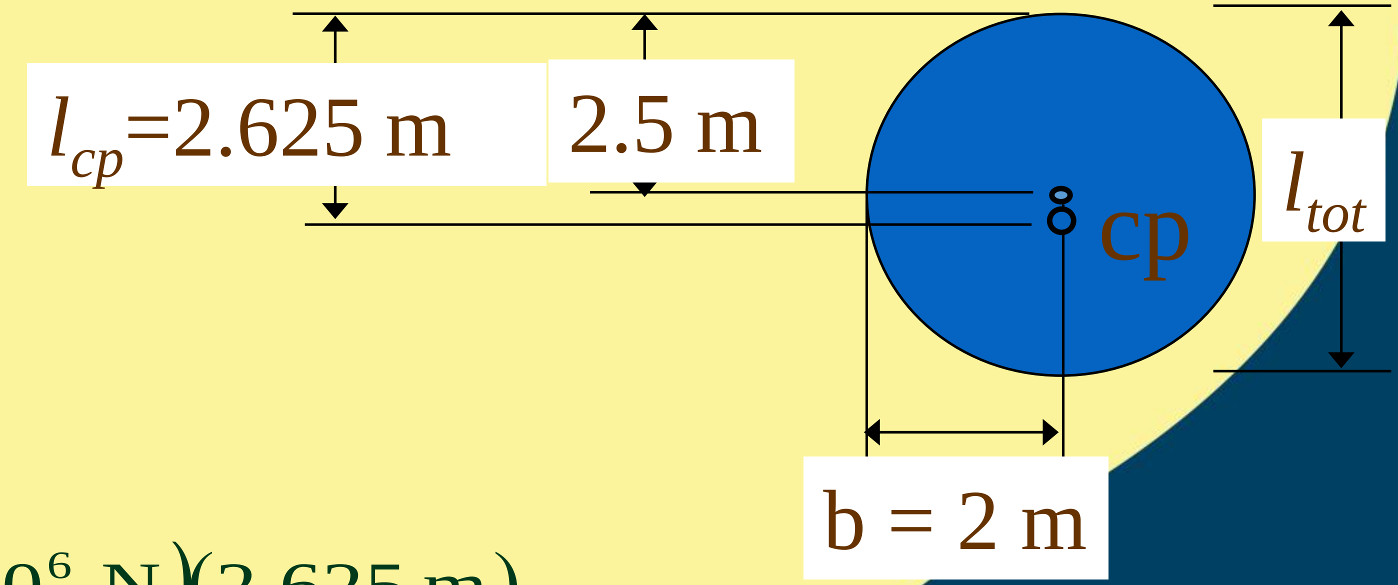
Moments about the hinge

$$\sum M_{hinge} = 0 = Fl_{tot} - F_R l_{cp}$$

$$F = \frac{F_R l_{cp}}{l_{tot}}$$

$$F = \frac{(1.54 \times 10^6\ \mathrm{N})(2.625\ \mathrm{m})}{(5\ \mathrm{m})}$$

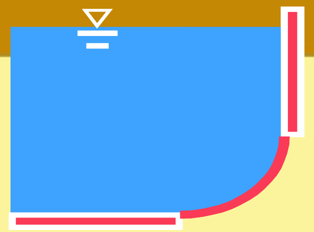
F = 809 kN



Forces on Plane Surfaces Review

- The average magnitude of the pressure force is the pressure at the centroid
- The horizontal location of the pressure force was at x_c (WHY?)
  The gate was symmetrical about at least one of the centroidal axes.
- The vertical location of the pressure force is below the centroid. (WHY?)
  Pressure increases with depth.

Forces on Curved Surfaces



- Horizontal component
- Vertical component

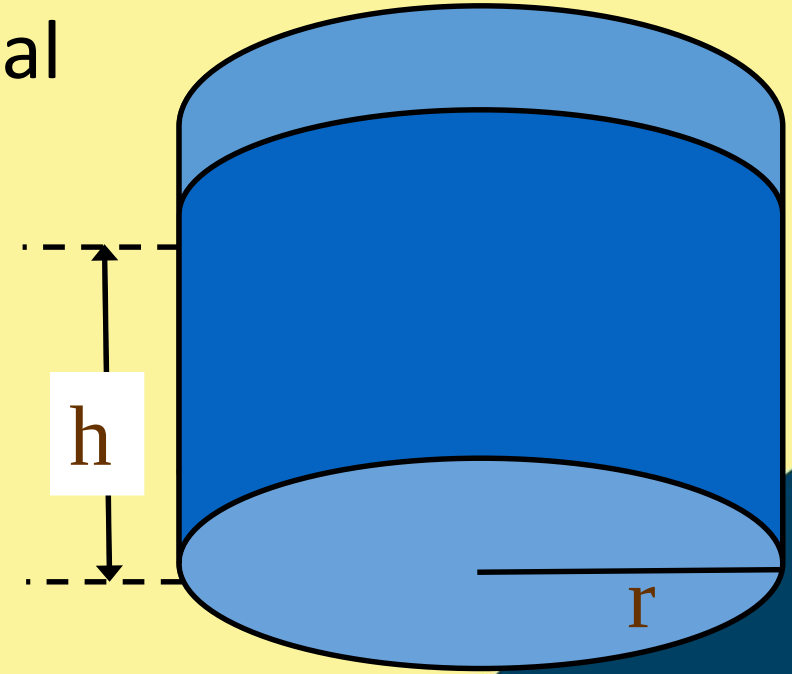
Forces on Curved Surfaces: Vertical Component

- What is the magnitude of the vertical component of force on the cup?

$$F = pA$$

$$p = \rho g h$$

$$F = \rho g h \pi r^2 \ = W!$$



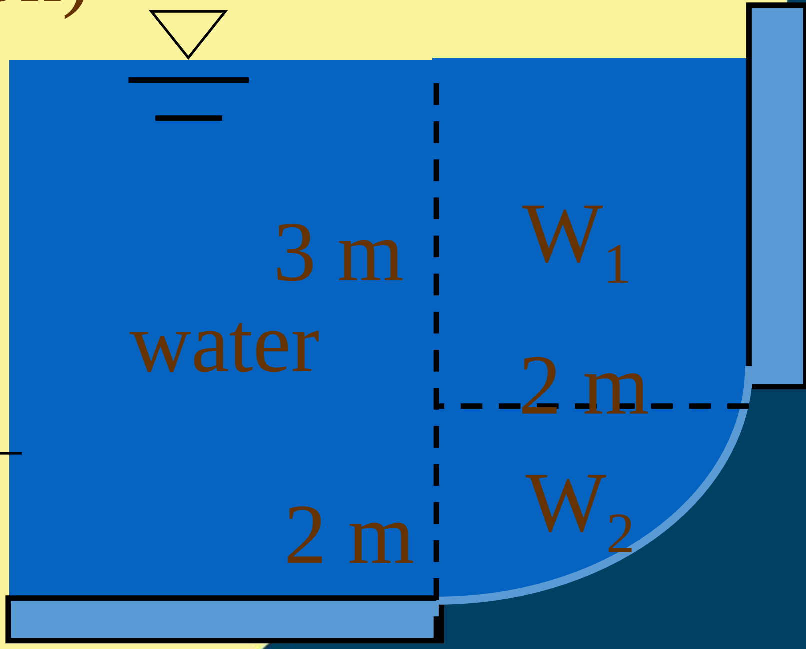
Forces on Curved Surfaces: Vertical Component

The vertical component of pressure force on a curved surface is equal to the weight of liquid vertically above the curved surface and extending up to the

surface where the pressure is equal to the reference pressure.

Example: Forces on Curved Surfaces

Find the resultant force (magnitude and location) on a 1 m wide section of the circular arc.



$$F_V = \underline{W_1 + W_2}$$

$$= \underline{(3\text{ m})(2\text{ m})(1\text{ m})\gamma + \pi/4(2\text{ m})^2(1\text{ m})\gamma}$$

$$= \underline{58.9\text{ kN} + 30.8\text{ kN}}$$

$$= \underline{89.7\text{ kN}}$$

$$F_H = p_c A = \underline{\gamma(4\text{ m})(2\text{ m})(1\text{ m})} = \underline{78.5\text{ kN}}$$

Example: Forces on Curved Surfaces

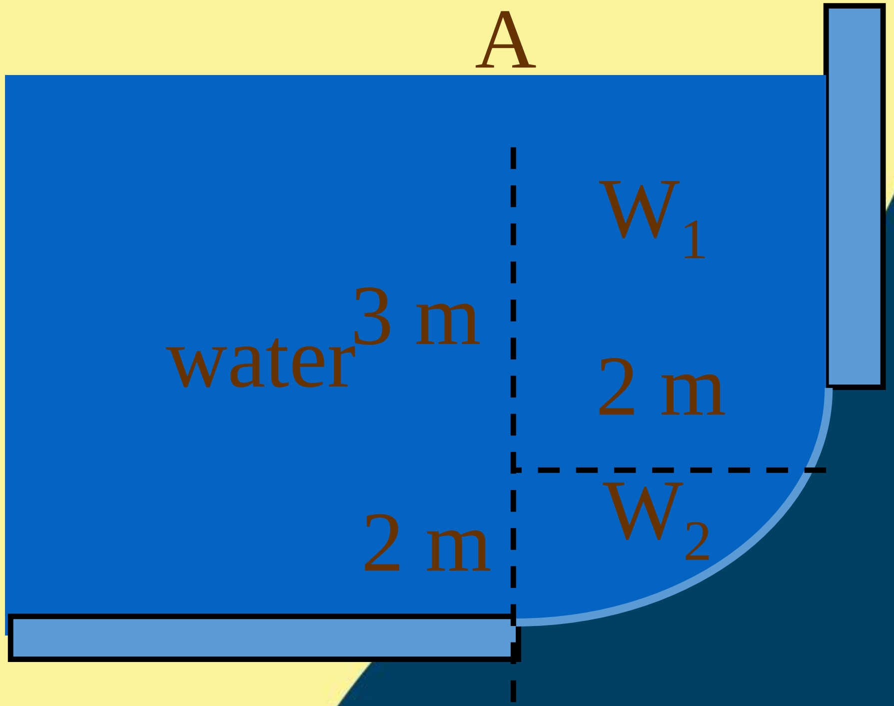
The vertical component line of action goes through the centroid of the volume of water above the surface. Expectation???

Take moments about a vertical axis through A.

$$x_c F_V = (1\text{ m})W_1 + \frac{4(2\text{ m})}{3\pi}W_2 \qquad \boxed{\frac{4R}{3\pi}}$$

$$x_c = \frac{(1\text{ m})(58.9\text{ kN}) + \frac{4(2\text{ m})}{3\pi}(30.8\text{ kN})}{(89.7\text{ kN})}$$

= 0.948 m (measured from A) with magnitude of 89.7 kN

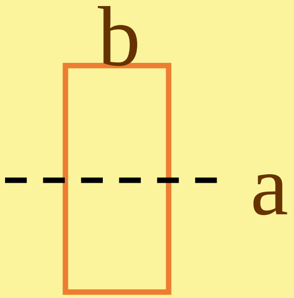


Example: Forces on Curved Surfaces

The location of the line of action of the horizontal component is given by

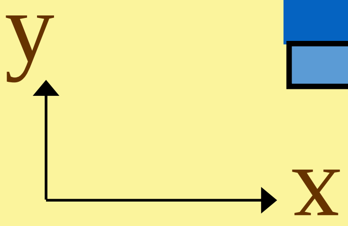
$$y_R = y_c + \frac{I_{xc}}{y_c A} \qquad Here, hc = y_c$$

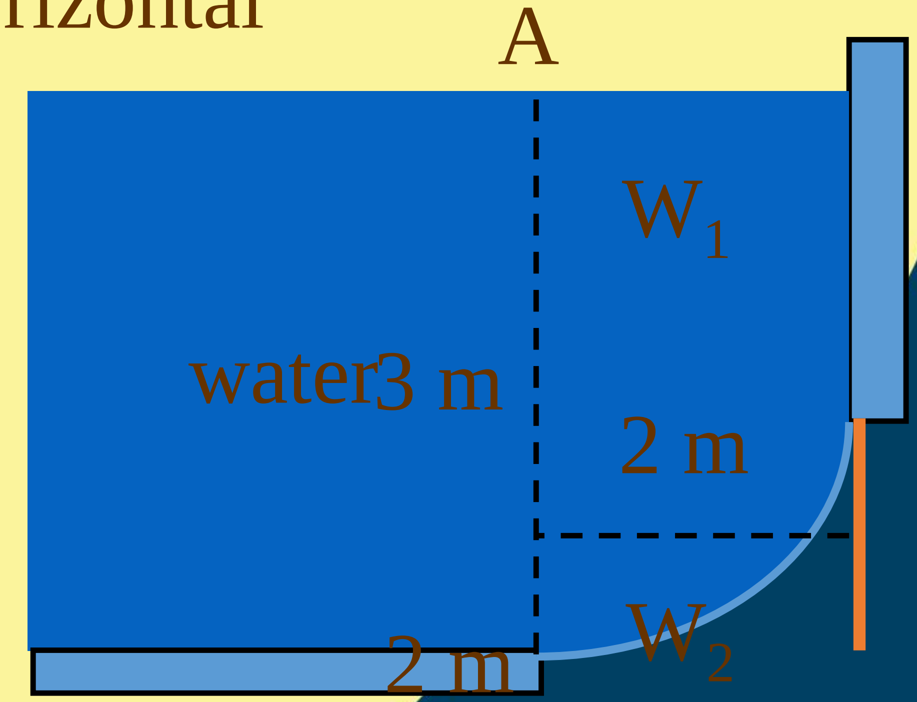
$$y_R = y_c + \frac{a^2}{12h_c} \qquad \frac{I_{xc}}{A} = \frac{a^2}{12}$$



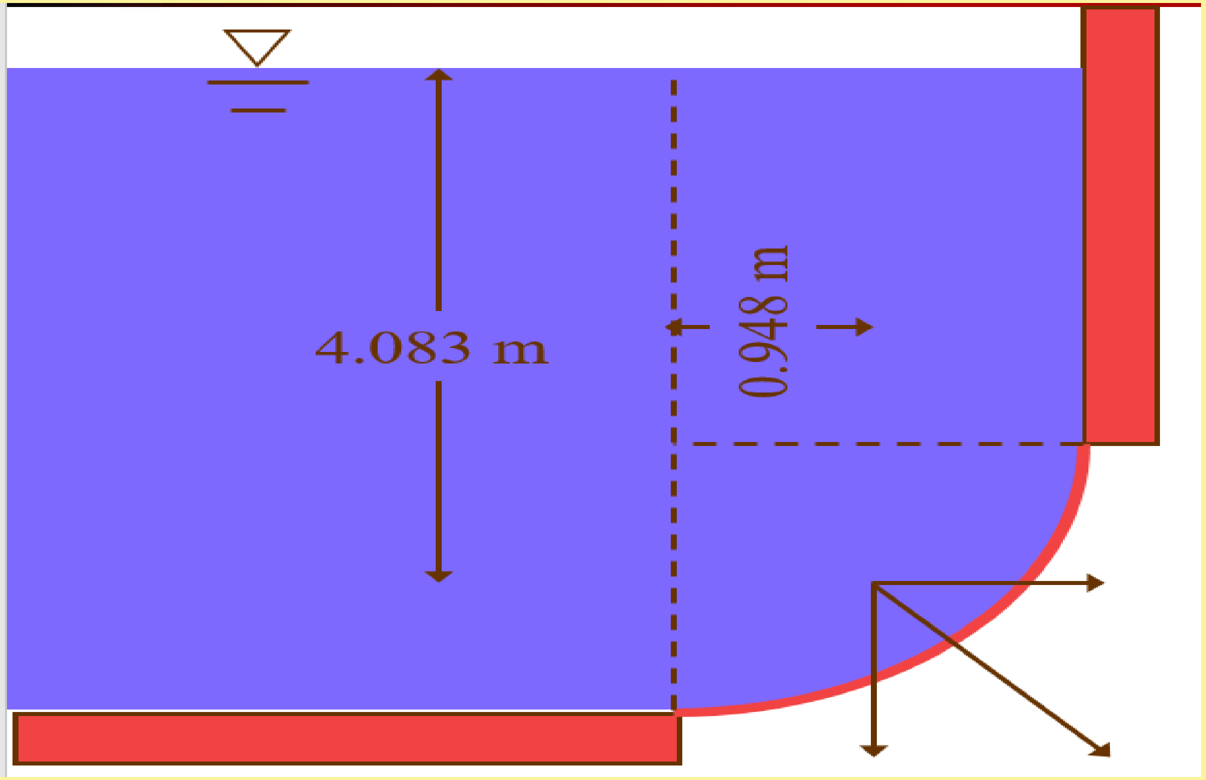
$h_c =$ 4 m

$$y_R = y_c + 0.083$$





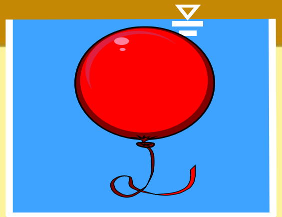
Example: Forces on Curved Surfaces



78.5 kN horizontal

89.7 kN vertical

119.2 kN resultant



Buoyant Force

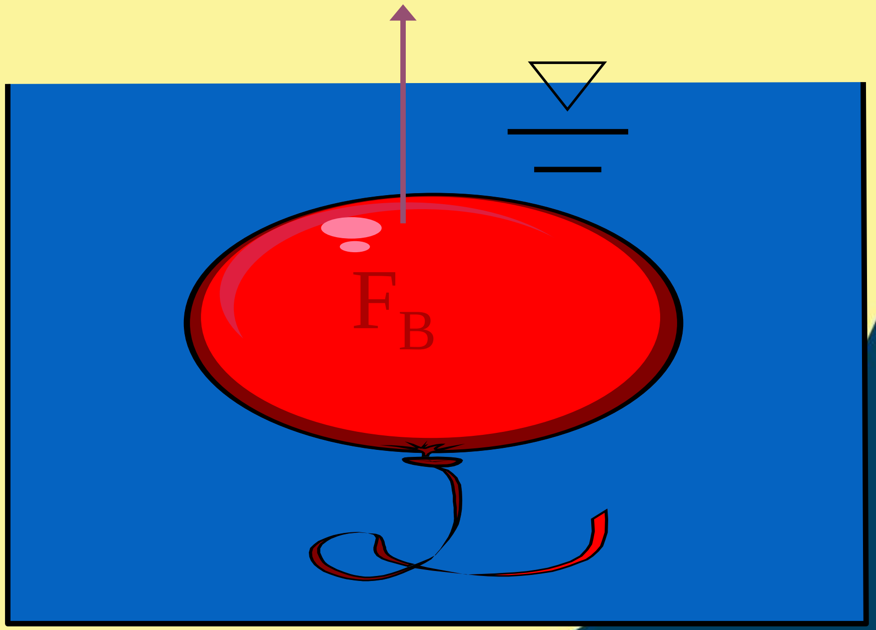
- The resultant force exerted on a body by a ***static fluid*** in which it is fully or partially submerged
  - The projection of the body on a vertical plane is always zero

    (Two surfaces cancel, net horizontal force is zero.)
  - The vertical components of pressure on the top and bottom surfaces are different

Buoyant Force: Thought Experiment

Place a thin wall balloon filled with water in a tank of water.

- What is the net force on the balloon? Zero
- Does the shape of the balloon matter? No
- What is the buoyant force on the balloon? Weight of water displaced

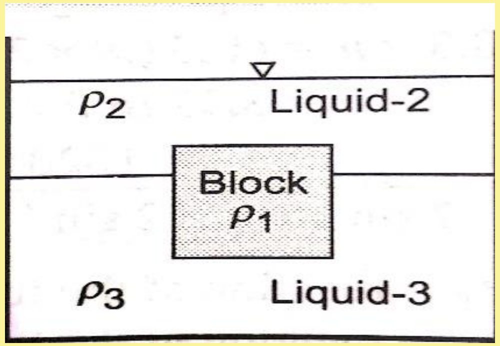


$$F_B = \rho g \forall$$

Problem on Buoyancy

- A block of wood of density $\rho_1$floating completely submertged at the interface of two liquid $\rho_2\ and\ \rho_2$.if V_2 is the volume of the block in the upper liquid and V_1 is the total volume of the block, Show That

$$\frac{V_3}{V_1} = \frac{(\rho_1 - \rho_2)}{(\rho_3 - \rho_2)}$$



Problem on Buoyancy

- A body weighs 20N and 10N when weighed under submerged conditions in liquids of relative densities 0.8 and 1.2 respectively. Determine its volume and weight in air.

References

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