# Mathematical Induction

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**Abstract**

This document discusses about Mathematical Induction as a method of proof.

# 1 Introduction

## 1.1 What is Mathematical Induction?

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers.
It is done by proving that the first statement in the infinite sequence of statements is true, and then proving that if any one statement in the infinite sequence of statements is true, then so is the next one. It is a form of deductive reasoning.

## 1.2 A stepwise approach to the process

Suppose the statement to be proved is $\forall n \geq n_0, P(n)$

The proof consists of two steps:

1. The **Basis Step**: showing that the statement holds when $n = n_0$.
2. The **Induction Step**: showing that **if** the statement holds for some n, **then** the statement also holds when $n + 1$ is substituted for $n$.

# 2 Steps Elaborated

## 2.1 The Basis Step

The basis step is generally very easy to prove.
Prove it directly by substituting the value of $n_0$ in $P(n)$ and checking the validity of the statement.

## 2.2 The Induction Step

In the next step,prove that,

$$P(k) \Rightarrow P(k+1)$$

is a tautology for any choice of $k \geq n_0$.

In other words, $P(0)$ being true along with $(P(k) \Rightarrow P(k+1) \ \forall \ k \geq n_0)$ being true,together imply that $P(n)$ is true for all natural n.

A proof by induction is then a proof that these two conditions hold, thus implying the required conclusion.

# 3 Strong Induction

In a variant of Mathematical induction,called Strong Form of Mathematical Induction or Strong Induction,
The Induction step is to show that,

$$\mathbf{P(n_0) \wedge P(n_0+1) \wedge P(n_0+2) \wedge \ldots \wedge P(k) \Rightarrow P(k+1)}$$

is a tautology.Then we conclude that if p($j$) is true $\forall j \in (n_0, \ldots, k)$ then P($k$+1) is true.

# 4 Examples

## 4.1 Sum of natural numbers less than or equal to $n$

Mathematical induction can be used to prove that the statement:

$$\mathbf{1+2+3+\ldots+n=\frac{n(n+1)}{2}} \tag{1}$$

holds for all natural numbers $n$.
It gives a formula for the sum of the natural numbers less than or equal to number $n$.
The proof that the statement is true for all natural numbers $n$ proceeds as follows.
Call this statement P($n$).

**Basis Step**:
Show that the statement holds for $n$=1.

$$\mathbf{P(1)=\frac{1.(1+1)}{2}=1} \tag{2}$$

which is true.Thus P(1) holds.
**Induction Step:**
Show that if P($k$) holds, then also P($k$ + 1) holds. This can be done as follows.
Assume P($k$) holds (for some unspecified value of $k$).
It must be shown that then P($k$ + 1) holds.
Using the induction hypothesis that P($k$) holds, the left-hand side of Eq. (1) can be rewritten:

$$\mathbf{1+2+3+\ldots+(k+1)=(1+2+3+....+k)+(k+1)=\frac{k(k+1)}{2}+(k+1)} \tag{3}$$

$$\mathbf{\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)((k+1)+1)}{2}} \tag{4}$$

Therefore P($k$+1) holds.Since both the basis and the induction step have been proved, it has now been proved by mathematical induction that P($n$) holds for all natural $n$

## 4.2 Extended De Morgan's Law

Let $A_1, A_2, A_3, \ldots, A_n$ be any $n$ sets.Prove by Mathematical induction that,

$$\overline{\left(\bigcup_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{A_i}\right)} = \bigcap_{\mathbf{i=1}}^{\mathbf{n}} \overline{\mathbf{A_i}}$$

Let the given predicate that the equality holds for n sets be denoted by P($n$).then,

**Basis Step:** P(1) is the statement $\overline{\mathbf{A_1}} = \overline{\mathbf{A_1}}$ ,which is obviously true.

**Induction Step:** Let P($k$) be true.Then L.H.S of P($k$+1) is,

$$\begin{aligned}
\overline{\left(\bigcup_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{A_i}\right)} &= \overline{\mathbf{A_1} \cup \mathbf{A_2} \cup \ldots \cup \mathbf{A_k} \cup \mathbf{A_{k+1}}} \\
&= \overline{(\mathbf{A_1} \cup \mathbf{A_2} \cup \ldots \cup \mathbf{A_k}) \cup \mathbf{A_{k+1}}} && \text{associative propety of } \cup \\
&= \overline{(\mathbf{A_1} \cup \mathbf{A_2} \cup \ldots \cup \mathbf{A_k})} \cap \overline{\mathbf{A_{k+1}}} && \text{De Morgan's law for two sets} \\
&= \left(\bigcap_{\mathbf{i=1}}^{\mathbf{k}} \overline{\mathbf{A_i}}\right) \cap \overline{\mathbf{A_{k+1}}} && \text{using P(k)} \\
&= \bigcap_{\mathbf{i=1}}^{\mathbf{k+1}} \overline{\mathbf{A_i}}
\end{aligned}$$

Thus P($k$) $\Rightarrow$ P($k$+1),is a tautology and therefore by the principle of Mathematical Induction P($n$) is true for all $n \geq 1$.

## 4.3 Writing a positive number uniquely as a product of primes

Prove that every positive integer $n > 1$ can be written uniquely as $p_1^{a_1}p_2^{a_2}\ldots p_s^{a_s}$, where the $p_i$ are primes and $p_1 < p_2 < \ldots < p_s$.

**Proof (by Strong Induction):**

**Basis Step:**
Here $n_0$=2. $\therefore$P(2) is true.

**Induction Step:**
$k$+1 can be written uniquely as $p_1^{a_1}p_2^{a_2}\ldots p_s^{a_s}$,where $p_i$ are primes and $p_1 < p_2 < \ldots < p_s$. Two cases arise.They are,

1. If $k$+1 is a prime then P($k$+1) is true.
2. If $k$+1 is not prime,then $k + 1 = lm$.where,$l$ and $m$ can be uniquely written as product of primes,using P($l$) and P($m$).$(2 \leq l \leq k, 2 \leq m \leq k)$
   But since the factorisation of $l$ and $m$ are unique,so is the factorisation of $k$+1.

Thus the staement is true $\forall n > 1$.

# 5 References

**Books:**
Discrete Mathematical Structures,5th Edition by Kolman,Busby and Ross - Prentice-Hall India

**Links:**
Mathematical Induction-Wikipedia