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Uniqueness of solutions to the coagulation–fragmentation equation with singular kernel

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Abstract

The existence of a solution to an important *singular* coagulation equation with a multiple fragmentation kernel has been recently proved in Jpn J Ind Appl Math 35(3):1283–1302, 2018. This paper proves the uniqueness of the solution to the same problem in the function space $\Omega_{\cdot,r_2}(T) = \bigcup_{\lambda>0} \Omega_{\lambda,r_2}(T)$, where $\Omega_{\lambda,r_2}(T)$ is the space of all continuous functions f such that

$$\|f\|_{\lambda,r_2} := \sup_{0\le t\le T} \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^{r_2}}\right)|f(x,t)|dx \;<\; \infty$$

and $0 < r_2 < 1$.

Keywords Coagulation–fragmentation equation · Singular coagulation kernel · Multiple fragmentation kernel · Uniqueness result

Mathematics Subject Classification 35A01 · 34A12

1 Introduction

This paper continues the analysis on the solutions of a *singular* coagulation equation, with a multiple fragmentation, that is considered in [10]. In [10], the existence of a solution to the problem is proved. In this article, we prove the uniqueness of the solutions to the problem. The analysis of solutions to the problem under consideration is important due to the appearance of such problem in the practical fields. It is noteworthy that the problem that we consider to study

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includes a class of practical kernels, for instances, Smoluchowski kernel (1917) (in Brownian diffusion kernel) [20], Kapur kernel (1972) (in granulation) [14], Hounslow equipartition kinetic energy kernel (1998) (in granulation) [13], Shiloh et al. (1973) (in nonlinear velocity profile) [19], Friedlander kernels (2000) (in aerosol dynamics), Peglow kernel (2005) (in granulation kernel) [18], Ding et al. kernel (2006) (in activated sludge flocculation) [6], etc. Broadly, coagulation–fragmentation process appears in many natural science and engineering problems, for examples, astrophysics, rock fracture, degradation of large polymer chains, DNA fragmentation, evolution of phytoplank to aggregates, liquid droplet break up or break up of solid drugs in organisms, etc.

Coagulation-and-fragmentation process is a particulate process which describes the time evolution of a system in which clusters react to coagulate or break. This process has been first studied by Smoluchowski [20]. It concerns about Brownian motion. The model involves an infinite set of nonlinear differential equation. After that, Muller [16] introduced its continuous version. Melzak [15] derived the coagulation–fragmentation equation (C–F equation) which is formulated as follows:

$$\begin{aligned}\frac{\partial f(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K(x-y,y)\,f(x-y,t)f(y,t)\,dy \\ &\quad - f(x,t)\int_0^\infty K(x,y)\,f(y,t)\,dy \\ &\quad - f(x,t)\int_0^x \frac{y}{x}\Gamma(x,y)dy + \int_x^\infty \Gamma(y,x)f(y,t)\;dy\end{aligned} \tag{1.1}$$

with the initial condition $f(x,0)=f_0(x)\geq 0$. In this equation, the function $S(x)$, known as the selection function, gives the rate of breaking of x-size particles. The precise interrelations between the fragmentation kernel $\Gamma(x,y)$, the breakage function $b(x,y)$ and $S(x)$ is given by

$$\Gamma(x,y)=b(x,y)\,S(y) \quad \text{and} \quad S(x)=\int_0^x \frac{y}{x}\Gamma(x,y)\,dy.$$

With the help of these two relations, the Eq. (1.1) reduces to

$$\left.\begin{aligned}\frac{\partial f(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,dy \\ &\quad - f(x,t)\int_0^\infty K(x,y)f(y,t)\,dy \\ &\quad + \int_x^\infty b(x,y)S(y)f(y,t)\,dy - S(x)f(x,t) \\ &\text{with the initial data } f(x,0)=f_0(x).\end{aligned}\right\} \tag{1.2}$$

In this paper, we consider to investigate the *uniqueness of the solution* to the Eq. (1.2). For a detailed interpretation of the terms in (1.2), we refer to [10].

1.1 Literature survey

In this section, we report the existing studies on the existence and uniqueness of the solutions to the equations those are analogous or a variant of the Eq. (1.2) with various types of kernels.

Costa [5] derived the existence and uniqueness result for the *discrete* C–F equation by considering the kernel in the following form:

$$a_{j,k} \leq K_a(jk)^{\alpha},\ \alpha \in \left(\frac{1}{2}, 1\right] \quad \text{and} \quad \sum_{j=1}^{h(r)} j^{\mu} b_{j,r-j} \geq K_f(\mu) r^{\gamma+\mu}.$$

In the simplest case of the *continuous* C–F equation (1.2), i.e., for *constant* coagulation and fragmentation kernels, Aizenman and Bak [1] showed the uniqueness of the solution to (1.2), the so-called Boltzman equation, in the space $\{f : [0, \infty) \to \mathbb{R} : |||f||| < \infty\}$, where $|||f||| = \int_0^{\infty}(1+x)|f(x)|dx$.

Melzak [15] has considered C–F equation and have shown that the unique solution exist under the following restrictions:

(i) $0 \leq K(x, y) < \tau_0$, a constant, and
(ii) $0 \leq F(x, y) < \tau_1$, a constant, $\int_0^x yF(x, y) \leq x$, and $\int_0^x F(x, y)dy < \infty$.

The global existence and uniqueness of the solution to the binary coagulation and fragmentation equation with *linear* coagulation kernel and unbounded fragmentation kernel has been given in [7].

Norris [17] has shown existence of a unique solution to the problem (1.2) when the coagulation kernel satisfies $K(x, y) \leq \psi(x)\psi(y)$ for all $x > 0, y > 0$ and continuous *sublinear* function $\psi : (0, \infty) \to (0, \infty)$. In [2], Banasiak studied a fragmentation model, i.e., in the absence of the coagulation terms in (1.2) and described that the existence of multiple solutions.

Galkin and Dubovskiĭ [9] reported the existence and uniqueness of solutions to a coagulation equation with a symmetric nonnegative coagulation kernel $K(x, y)$ that satisfies

(i) $\sup_{0\leq x,y<\infty} K(x, y)(1 + x + y)^{-1} < \infty$,
(ii) for each $\alpha \in [0, 1)$ and $y \in [0, \infty)$, the function $\lim_{x\to\infty} \phi(x, y)x^{-\alpha}$ is bounded on each finite segment of the change of y, and
(iii) $K(x, y) \leq C(1 + x^{\alpha}y^{\alpha})$, where C is a nonnegative constant.

In [12], the uniqueness of C–F equation has been studied for a strong fragmentation kernel and the coagulation kernel that satisfies

$$K(x, y) \leq C\left[(1 + x)^{\alpha}(1 + y)^{\beta} + (1 + x)^{\beta}(1 + y)^{\alpha}\right],$$

where $0 \leq \alpha \leq \beta \leq 1$.

In [3], a proof for the unique global time-dependent solution has been established for the problem with a polynomially bounded fragmentation process and a bounded coagulation rate.

The existence and uniqueness of a weak solutions to (1.2) are found in [11]. In [11], the coagulation kernel is taken in such a way that $K(x,y) \le \phi(x)\phi(y)$, where ϕ satisfies

$$\phi(x) \le k_1(1 + x^{\mu}), \quad 0 \le \mu < 1,$$

and the selection function follows

$$S(x) \le k_2(1 + x)^{\nu}, \quad 0 \le \nu < 1.$$

Ernst et al. [8] analyzed the gelation property (formation of an infinite cluster after a finite time) of the Smoluchowski's coagulation system that has the coagulation kernel in the form $(xy)^{\alpha}$, $\frac{1}{2} < \alpha \le 1$. For a special kernel, Ernst et al. [8] also provided an explicit form of the solution.

From the existing literature, one can notice that there is a minimal study on the analysis of the C–F equation with singular kernels. Recently, Ghosh and Kumar [10] and Camejo et al. [4] proved the existence of the solutions for C–F equation with a particular type of singular kernels. In this article, we explore the uniqueness of solutions to the singular C–F equation that is considered by Ghosh and Kumar [10]. We prove the uniqueness of the solution in the function space that is defined in [10].

The rest of the paper is presented as follows. Section 2 states the existence theory of the solutions. In Sect. 3, we obtain the uniqueness of the solutions. Finally, we conclude the entire analysis in Sect. 4 and give a future scope of studies.

2 Existence theorem

Here we recall the function space $\Omega_{\lambda,r_2}(T)$ which is defined in [10]. The notation $\Omega_{\lambda,r_2}(T)$ denotes the set all continuous functions f's those are bounded with respect to the norm

$$\|f\|_{\lambda,r_2} := \sup_{0\le t\le T} \int_0^{\infty} \left(\exp(\lambda x) + \frac{1}{x^{r_2}}\right)|f(x,t)|dx,$$

where $0 < r_2 < 1$. We denote

$$\Omega_{\cdot,r_2}(T) = \bigcup_{\lambda>0} \Omega_{\lambda,r_2}(T).$$

Further, $\Omega^+_{\lambda,r_2}(T)$ and $\Omega^+_{\cdot,r_2}(T)$ denote the cone of nonnegative functions from $\Omega_{\lambda,r_2}(T)$ and $\Omega_{\cdot,r_2}(T)$, respectively.

In this paper, the proposed uniqueness result is analyzed in the space $\Omega_{\lambda,r_2}(T)$. Before providing the uniqueness result, we state the result on existence of a solution to the C–F equation (1.2) in the space $\Omega^+_{\cdot,r_2}(T)$.

Theorem 1 (Existence result [10]). *Suppose the coagulation kernel and the breakage functions, $K(x, y)$ and $b(x, y)$, respectively, be continuous and nonnegative in* $(0,\infty)\times(0,\infty)$. *Further, we assume that $K(x, y)$ be symmetric in* $(0,\infty)\times(0,\infty)$ *and $S(x)$ be a nonnegative and continuous function on* $(0,\infty)$. *Moreover, let there exist four positive constants k, S_1, β and n_0 such that*

(i) $K(x,y) \le k\,\frac{(1+x^{\theta}+y^{\theta})}{(xy)^{\mu}}$ *for all* $x, y \in (0,\infty)$, where $\mu \in \left[0, \frac{1}{2}\right]$ *and* $\theta - \mu \in [0,1]$,
(ii) $S(x) \le S_1\, x^{\beta}$ *for all* $x > 0$,
(iii) *for some* $\gamma \in (0,1)$, $\int_0^y \frac{1}{x^{\gamma}} b(x,y)\,dx \le \frac{n_0}{y^{\gamma}}$, *and*
(iv) $\lim_{y\to\infty} \sup_{x\in[x_1,x_2]} b(x,y) < \infty$ *for* $0 < x_1 < x_2 < \infty$.

If the initial data f_0 lies in $\Omega^{+}_{\cdot,r_2}(0)$, then the problem (1.2) has a solution in $\Omega^{+}_{\cdot,r_2}(T)$.

3 Uniqueness theorem

In Theorem 1, the existence of a solution to (1.2) is given. In this section, we prove the uniqueness of the solutions to (1.2), in the space of functions $\Omega_{\cdot,r_2}(T)$, under the assumptions stated in Theorem 2 below. To prove the uniqueness result we use the following lemma.

Lemma 1 (See [7]). *Suppose $v(\lambda,t)$ and its partial derivatives v_λ and $v_{\lambda\lambda}$ are continuous on $D = \{(\lambda,t) : 0 \le \lambda \le \lambda_0,\ 0 \le t \le T\}$. Assume that $\alpha(\lambda)$, $\beta(\lambda,t)$, $\gamma(\lambda,t)$ and $\theta(\lambda,t)$ are real-valued and continuous on D, and their partial derivatives with respect to λ are continuous on D. Moreover, suppose that the functions $v, v_\lambda, \beta, \gamma$ are nonnegative. Let the following inequalities also hold on D :*

$$v(\lambda,t) \le \alpha(\lambda) + \int_0^t (\beta(\lambda,s)v_\lambda(\lambda,s) + \gamma(\lambda,s)v(\lambda,s) + \theta(\lambda,s))ds$$

and

$$v_\lambda(\lambda,t) \le \alpha_\lambda(\lambda) + \int_0^t \frac{\partial}{\partial\lambda}(\beta(\lambda,s)v_\lambda(\lambda,s) + \gamma(\lambda,s)v(\lambda,s) + \theta(\lambda,s))ds.$$

We denote $C_0 = \sup_{0\le\lambda\le\lambda_0} \alpha(\lambda)$, $C_1 = \sup_D \beta$ and $C_3 = \sup_D \theta$. Then,

$$v(\lambda,t) \le C_0 \exp(C_2 t) + \tfrac{C_3}{C_2}(\exp(C_2 t) - 1)$$

in a region as follows:

$$R = \{(\lambda,t) : 0 \le t \le t' < T';\ \lambda_1 - C_1 t \le \lambda \le \lambda_0 - C_1 t,\ 0 < \lambda_1 < \lambda_0\} \subset D,$$

where $T' = \min\left\{\frac{\lambda_1}{C_1}, T\right\}$.

Theorem 2 (*Uniqueness result*). *Let the kernels*$K(x, y)$ *and* $b(x, y)$ *in* (1.2) *be nonnegative and continuous in* $(0, \infty) \times (0, \infty)$. *Further, let* $K(x, y)$ *be symmetric on* $(0, \infty) \times (0, \infty)$. *Suppose also that* $S(x)$ *be continuous and nonnegative in* $(0, \infty)$. *Furthermore, let there exist five positive constants* k, S_1, β, n_0 *and* $\bar{b}$ *such that*

(i) $K(x, y) \leq k\, \frac{(1+x^{\theta}+y^{\theta})}{(xy)^{\mu}}$ *for all* $x, y \in (0, \infty)$, where $\mu \in \left[0, \frac{1}{2}\right]$ *and* $\theta - \mu \in [0, 1]$,
(ii) $S(x) \leq S_1\, x^{\beta}$ *for all* $x > 0$,
(iii) *for some* $\gamma \in (0, 1)$, $\int_0^y \frac{1}{x^{\gamma}} b(x, y)\, dx \leq \frac{n_0}{y^{\gamma}}$, *and*
(iv) $\lim_{y\to\infty} \sup_{x\in[x_1,x_2]} b(x, y) \leq \bar{b}$ for all x_1, x_2 *such that* $0 < x_1 < x_2$.

Then, the solution to the initial value problem (1.2) *is unique in the space* $\Omega_{\cdot,r_2}(T)$.

Proof Suppose that there are two solutions c and g, in $\Omega_{\cdot,r_2}(T)$, to the initial value problem (1.2). We prove that $c = g$.

Let $u(x, t) = c(x, t) - g(x, t)$ and $\psi(x, t) = c(x, t) + g(x, t)$. Since $c, g \in \Omega_{\cdot,r_2}(T)$, there exists a $\hat{\lambda} > 0$ such that

$$\left.\begin{aligned} &\int_0^{\infty} \left(\exp(\hat{\lambda}x) + \frac{1}{x^{\nu}}\right) u(x, t)\, dx < \infty \\ \text{and } &\int_0^{\infty} \left(\exp(\hat{\lambda}x) + \frac{1}{x^{\nu}}\right) \psi(x, t)\, dx < \infty \end{aligned}\right\} \tag{3.1}$$

uniformly with respect to t, $0 \leq t \leq T$.

Let $0 \leq \lambda < \hat{\lambda}$. Then, by the definition of u, we obtain

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} = &\ \frac{1}{2} \int_0^x K(x - y, y)\{c(x - y, t)c(y, t) - g(x - y, t)g(y, t)\}\, dy \\ &- \int_0^{\infty} K(x, y)\{c(x, t)c(y, t) - g(x, t)g(y, t)\}\, dy \\ &+ \int_x^{\infty} b(x, y)S(y)\{c(y, t) - g(y, t)\}\, dy - S(x)\{c(x, t) - g(x, t)\}. \end{aligned} \tag{3.2}$$

Here we recall that

$$\operatorname{sgn}(t) = \begin{cases} 1, & \text{when } t > 0, \\ 0, & \text{when } t = 0, \\ -1, & \text{when } t < 0 \end{cases}$$

and

$$\frac{d|P(t)|}{dt} = \mathrm{sgn}(P(t))\frac{d}{dt}\,P(t).$$

We define

$$U(\lambda, t) = \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)|u(x,t)|\,dx \tag{3.3}$$

and

$$\Psi(\lambda, t) = \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)|\psi(x,t)|\,dx, \tag{3.4}$$

where ν is so chosen that $0 < \nu \le r_2 - \mu$.

Multiplying both the sides of (3.2) by $(\exp(\lambda x) + \frac{1}{x^\nu})$ and then integrating we get

$$\begin{aligned}
&U(\lambda, t)\\
&= \int_0^t \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))\\
&\quad\left[\frac{1}{2}\int_0^x K(x-y,y)\{c(x-y,s)c(y,s) - g(x-y,s)g(y,s)\}\,dy\right.\\
&\quad - \int_0^\infty K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\\
&\quad\left. + \int_x^\infty b(x,y)S(y)\{c(y,s) - g(y,s)\}\,dy - S(x)\{c(x,s) - g(x,s)\}\right]dx\,ds\\
&= \int_0^t (I_1 + I_2 + I_3)ds,
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
I_1 = \frac{1}{2}\int_0^\infty &\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))\\
&\int_0^x K(x-y,y)\{c(x-y,s)c(y,s) - g(x-y,s)g(y,s)\}\,dy\,dx,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
I_2 = \int_0^\infty &\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))\\
&\int_0^\infty K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\,dx,
\end{aligned} \tag{3.7}$$

and

$$I_3 = \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\operatorname{sgn}(u(x,s)) \left[\int_x^\infty b(x,y)S(y)\{c(y,s) - g(y,s)\}\,dy - S(x)\{c(x,s) - g(x,s)\}\right] dx. \tag{3.8}$$

By changing the order of integration of I_1 and then substituting $x - y = x'$, $y = y'$ and re-changing the order of integration, we obtain

$$I_1 = \frac{1}{2}\int_0^\infty \int_0^\infty \left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)\operatorname{sgn}(u(x+y,s))K(x,y) \{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\,dx.$$

Putting this relation in (3.5), we get

$$U(\lambda,t) = \int_0^t \int_0^\infty \int_0^\infty \left[\frac{1}{2}\left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)\operatorname{sgn}(u(x+y,s)) - \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\operatorname{sgn}(u(x,s))\right] K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\,dx\,ds \tag{3.9}$$

$$+ \int_0^t \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\operatorname{sgn}(u(x,s)) \left[\int_x^\infty b(x,y)S(y)\{c(y,s) - g(y,s)\}\,dy - S(x)\{c(x,s) - g(x,s)\}\right] dx\,ds. \tag{3.10}$$

From (3.9), we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left[\frac{1}{2}\left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)\operatorname{sgn}(u(x+y,s)) - \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\operatorname{sgn}(u(x,s))\right] K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\,dx \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \left[\left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)\operatorname{sgn}(u(x+y,s)) - \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\operatorname{sgn}(u(x,s)) - \left(\exp(\lambda y) + \frac{1}{y^\nu}\right)\operatorname{sgn}(u(y,s))\right] \\
&\quad K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\,dx.
\end{aligned} \tag{3.11}$$

We note that $c(x,s)c(y,s) - g(x,s)g(y,s) = u(x,s)c(y,s) + g(x,s)u(y,s)$.

Thus, (3.11) gives

$$
\begin{aligned}
&\Bigg[\ \left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)\operatorname{sgn}(u(x+y,s))\\
&\quad-\left(\exp(\lambda x)+\frac{1}{x^{\nu}}\right)\operatorname{sgn}(u(x,s))-\left(\exp(\lambda y)+\frac{1}{y^{\nu}}\right)\operatorname{sgn}(u(y,s))\Bigg]\\
&\quad K(x,y)\ u(x,s)\ c(y,s)\\
&\quad=\ \Bigg[\ \left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)\operatorname{sgn}(u(x+y,s))\\
&\quad-\left(\exp(\lambda x)+\frac{1}{x^{\nu}}\right)\operatorname{sgn}(u(x,s))-\left(\exp(\lambda y)+\frac{1}{y^{\nu}}\right)\operatorname{sgn}(u(y,s))\Bigg]\\
&\quad K(x,y)\ \operatorname{sgn}(u(x,s))\ |u(x,s)|\ c(y,s)\quad \text{since } z=\operatorname{sgn}(z)\ |z|\\
&\quad=\Bigg[\ \left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)\operatorname{sgn}(u(x+y,s))\ \operatorname{sgn}(u(x,s))\\
&\quad-\ \left(\exp(\lambda x)+\frac{1}{x^{\nu}}\right)(\operatorname{sgn}(u(x,s))^{2}\\
&\quad-\left(\exp(\lambda y)+\frac{1}{y^{\nu}}\right)\operatorname{sgn}(u(y,s))\ \operatorname{sgn}(u(x,s))\Bigg]\\
&\quad K(x,y)\ |u(x,s)|\ c(y,s)\\
&\quad\le\left[\left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)+\left(\exp(\lambda y)+\frac{1}{y^{\nu}}\right)\right]K(x,y)|u(x,s)|c(y,s)\\
&\quad(\text{because } \operatorname{sgn}(u(x+y,s))\operatorname{sgn}(u(x,s))\le 1,\ \operatorname{sgn}(u(x,s))\operatorname{sgn}(u(y,s))\le 1,\\
&\quad\text{and } \left(\exp(\lambda x)+\frac{1}{x^{\nu}}\right)(\operatorname{sgn}(u(x,s))^{2},\ c(y,s) \text{ and } K(x,y) \text{ are nonnegative})\\
&\quad\le\ \left[\left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)+\left(\exp(\lambda y)+\frac{1}{y^{\nu}}\right)\right]K(x,y)|u(x,s)|\psi(y,s).
\end{aligned}
\tag{3.12}
$$

By a similar approach, we get

$$
\begin{aligned}
&\left[\left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)\operatorname{sgn}(u(x+y,s))\right.\\
&\quad\left.-\left(\exp(\lambda x)+\frac{1}{x^{\nu}}\right)\operatorname{sgn}(u(x,s))-\left(\exp(\lambda y)+\frac{1}{y^{\nu}}\right)\operatorname{sgn}(u(y,s))\right]\\
&\quad K(x,y)g(x,s)u(y,s)\\
&\quad\le\ \left[\left(\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\right)+\left(\exp(\lambda x)+\frac{1}{x^{\nu}}\right)\right]K(x,y)|u(y,s)|\psi(x,s).
\end{aligned}
\tag{3.13}
$$

With the help of (3.12) of (3.13), from (3.11), we get

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \left[\frac{1}{2}\left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)\mathrm{sgn}(u(x+y,s))\right. \\
&\quad \left. -\left(\exp(\lambda x) + \frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))\right] \\
&\quad K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\, dy\, dx \\
&\le \int_0^\infty \int_0^\infty \left[\left(\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right) + \left(\exp(\lambda y) + \frac{1}{y^\nu}\right)\right] \\
&\quad K(x,y)|u(x,s)|\psi(y,s)\, dy\, dx \\
&\le \int_0^\infty \int_0^\infty 2\exp(\lambda x)\left[\exp(\lambda y) + \frac{1}{y^\nu}\right]\frac{\left(1 + x^\theta + y^\theta\right)}{(xy)^\mu}|u(x,s)|\psi(y,s)\, dy\, dx.
\end{aligned} \tag{3.14}
$$

In the next, we evaluate a few inequalities to proceed for the proof.

$$
\begin{aligned}
\text{(i)}\quad &\int_0^\infty \exp(\lambda x)x^{-\mu}|u(x,t)|dx \\
&= \int_0^1 \exp(\lambda x)x^{-\mu}|u(x,t)|dx + \int_1^\infty \exp(\lambda x)x^{-\mu}|u(x,t)|dx \\
&\le \exp(\lambda)\int_0^1 x^{-\mu}|u(x,t)|dx + \int_1^\infty \exp(\lambda x)|u(x,t)|dx \\
&\le \exp(\lambda)\int_0^1 (\exp(\lambda x) + x^{-\nu})|u(x,t)|dx \\
&\quad + \int_1^\infty (\exp(\lambda x) + x^{-\nu})|u(x,t)|dx \\
&= U(\lambda,t)(1 + \exp(\lambda)).
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
\text{(ii)}\quad &\int_0^\infty \exp(\lambda x)x^{\theta-\mu}|u(x,t)|dx \\
&= \int_0^1 \exp(\lambda x)x^{\theta-\mu}|u(x,t)|dx + \int_1^\infty \exp(\lambda x)x^{\theta-\mu}|u(x,t)|dx \\
&\le \exp(\lambda)\int_0^1 |u(x,t)|dx + \int_1^\infty x\exp(\lambda x)|u(x,t)|dx \\
&\le \exp(\lambda)\int_0^1 (\exp(\lambda x) + x^{-\nu})|u(x,t)|dx \\
&\quad + \int_1^\infty (x\exp(\lambda x) + x^{-\nu})|u(x,t)|dx \\
&\le \chi_0 U(\lambda,t) + U_\lambda(\lambda,t), \text{ where } \chi_0 = 1 + \exp(\lambda).
\end{aligned} \tag{3.16}
$$

$$\text{(iii)}\quad \int_0^\infty (\exp(\lambda y) + y^{-\nu})y^{-\mu}c(y,t)dy$$
$$= \int_0^1 (\exp(\lambda y) + y^{-\nu})y^{-\mu}c(y,t)dy + \int_1^\infty (\exp(\lambda y) + y^{-\nu})y^{-\mu}c(y,t)dy$$
$$\le (\exp(\lambda) + 1)\int_0^1 y^{-\nu-\mu}c(y,t)dy + \int_1^\infty (\exp(\lambda y) + y^{-\nu})c(y,t)dy$$
$$\le \chi_0 \bar{N}_{-\nu-\mu} + \Psi(\lambda,t),$$
$$= \chi_0\chi_1 + \Psi,\ \text{where}\ \chi_1 = \bar{N}_{-\nu-\mu}$$
$$= \Gamma_1,\ \text{say}. \tag{3.17}$$

$$\text{(iv)}\quad \int_0^\infty (\exp(\lambda y) + y^{-\nu})y^{\theta-\mu}c(y,t)dy$$
$$= \int_0^1 (\exp(\lambda y) + y^{-\nu})y^{\theta-\mu}c(y,t)dy + \int_1^\infty (\exp(\lambda y) + y^{-\nu})y^{\theta-\mu}c(y,t)dy$$
$$\le \int_0^1 (\exp(\lambda y) + y^{-\nu})c(y,t)dy + \int_1^\infty (\exp(\lambda y) + 1)yc(y,t)dy$$
$$\le \|c\|_{\lambda,r_2} + \int_1^\infty (y\exp(\lambda y) + y^{-\nu})c(y,t)dy + \int_1^\infty yc(y,t)dy$$
$$\le \Psi + \Psi_\lambda + M$$
$$= \Gamma_2,\ \text{say}. \tag{3.18}$$

For the expression (3.8) we have

$$I_3 = \int_0^t \int_0^\infty \left(\exp(\lambda x) + \frac{1}{x^\nu}\right)$$
$$\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,dy - S(x)|U(x,s)|\right] dx\,ds$$
$$= \int_0^t \int_0^\infty \exp(\lambda x)\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,dy - S(x)|U(x,s)|\right] dx\,ds$$
$$+ \int_0^t \int_0^\infty \frac{1}{x^\nu}\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,dy - S(x)|U(x,s)|\right] dx\,ds. \tag{3.19}$$

By a similar analysis of the expression (2.5) in [10], we note that

$$\int_0^\infty x^k\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,dy - S(x)|U(x,s)|\right] dx \le 0,\quad \text{for } k = 1,2,3,\ldots.$$

With the help of the hypothesis (iii) of Theorem 2 we get from the Eq. (3.19) that

$$\int_0^t \int_0^\infty \left(1+\frac{1}{x^\nu}\right)\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,dy - S(x)|U(x,s)|\right] dx\,ds$$
$$\leq (n_0-1)\int_0^t \int_0^\infty \left(1+\frac{1}{y^\nu}\right)S(y)|U(y,s)|\,dy\,ds$$
$$\leq (n_0-1)S_1\int_0^t \int_0^\infty \left(1+\frac{1}{y^\nu}\right)y^\beta|U(y,s)|\,dy\,ds$$
$$\leq (n_0-1)S_1\int_0^t \left[\int_0^1 \left(1+\frac{1}{y^\nu}\right)y^\beta|U(y,s)|\,dy\,ds + \int_1^\infty \left(1+\frac{1}{y^\nu}\right)y^\beta|U(y,s)|\,dy\,ds\right] \tag{3.20}$$

$$\leq\ (n_0-1)S_1\int_0^t \left[\int_0^1 (1+\frac{1}{y^\nu})|U(y,s)|\,dy\,ds + \int_1^\infty (1+1)y^\beta|U(y,s)|\,dy\,ds\right]$$
$$\leq\ \Gamma_3\int_0^t U(\lambda,s)ds. \tag{3.21}$$

By the inequalities (3.15)–(3.18) we obtain from (3.9) and (3.10) that

$$U(\lambda,t) \leq 2\int_0^t [\Gamma_1 U(1+\exp(\lambda)) + \Gamma_2 U(1+\exp(\lambda)) + \Gamma_1(U+U_\lambda) + \Gamma_3 U]ds$$
$$= 2\int_0^t [U((\Gamma_1+\Gamma_2)(1+\exp(\lambda)) + \Gamma_1+\Gamma_3) + \Gamma_1 U_\lambda]ds$$
$$= 2\int_0^t [U((2\chi_0^2\chi_1 + M\chi_0 + \Gamma_3) + \Psi(3\chi_0) + \chi_0\Psi_\lambda) + U_\lambda(\chi_0\chi_1 + \Psi)]ds. \tag{3.22}$$

Till now we have attempted to obtain a bound for the function $U(\lambda,t)$. In the next, we endeavour to obtain an upper bound of the partial derivative function $U_\lambda(\lambda,t)$.

We recall from (3.3) that

$$U_\lambda(\lambda,t) = \int_0^\infty \left(x\exp(\lambda x) + \frac{1}{x^\nu}\right)|u(x,t)|dx,$$

where we choose ν to be $0<\nu\leq r_2-\mu$. Multiplying both sides of (3.2) by $(x\exp(\lambda x)+\frac{1}{x^\nu})$ and integrating with respect to $x\in(0,\infty)$ we get

$$U_\lambda(\lambda,t)$$
$$= \int_0^t\int_0^\infty\int_0^\infty \left[\frac{1}{2}\left((x+y)\exp(\lambda(x+y)) + \frac{1}{(x+y)^\nu}\right)\mathrm{sgn}(u(x+y,s))\right.$$
$$\left. -\left(x\exp(\lambda x)+\frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))\right]$$
$$K(x,y)\{c(x,s)c(y,s) - g(x,s)g(y,s)\}\,dy\,dx\,ds \tag{3.23}$$

$$+\int_0^t \int_0^\infty \left(x\exp(\lambda x)+\frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))$$
$$\left[\int_x^\infty b(x,y)S(y)\{c(y,s)-g(y,s)\}\,dy - S(x)\{c(x,s)-g(x,s)\}\right]dx\,ds. \tag{3.24}$$

From (3.23), we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \left[\frac{1}{2}\left((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)\mathrm{sgn}(u(x+y,s))\right.\\
&\quad\left.-\left(x\exp(\lambda x)+\frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))\right]\\
&\quad K(x,y)\{c(x,s)c(y,s)-g(x,s)g(y,s)\}\,dy\,dx\\
&=\frac{1}{2}\int_0^\infty \int_0^\infty \left[\left((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)\mathrm{sgn}(u(x+y,s))\right.\\
&\quad\left.-\left(x\exp(\lambda x)+\frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))-\left(y\exp(\lambda y)+\frac{1}{y^\nu}\right)\mathrm{sgn}(u(y,s))\right]\\
&\quad K(x,y)\{c(x,s)c(y,s)-g(x,s)g(y,s)\}\,dy\,dx.
\end{aligned} \tag{3.25}$$

We note that

$$c(x,s)c(y,s)-g(x,s)g(y,s)=u(x,s)c(y,s)+g(x,s)u(y,s).$$

Thus, from (3.25), in the lines of obtaining (3.12), we have

$$\begin{aligned}
&\left[\left((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)\mathrm{sgn}(u(x+y,s))\right.\\
&\quad\left.-\left(x\exp(\lambda x)+\frac{1}{x^\nu}\right)\mathrm{sgn}(u(x,s))-\left(y\exp(\lambda y)+\frac{1}{y^\nu}\right)\mathrm{sgn}(u(y,s))\right]\\
&\quad K(x,y)u(x,s)c(y,s)\\
&\le\left[\left((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)\right.\\
&\quad\left.+\left(y\exp(\lambda y)+\frac{1}{y^\nu}\right)\right]K(x,y)\;|u(x,s)|\;c(y,s)\\
&\le\left[\left((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^\nu}\right)+\left(y\exp(\lambda y)+\frac{1}{y^\nu}\right)\right]\\
&\quad K(x,y)\;|u(x,s)|\;\psi(y,s).
\end{aligned} \tag{3.26}$$

Similarly,

$$\begin{aligned}&\Bigg[\Bigg((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\Bigg)\mathrm{sgn}(u(x+y,s))\\&\quad-\Big(x\exp(\lambda x)+\frac{1}{x^{\nu}}\Big)\mathrm{sgn}(u(x,s))-\Big(y\exp(\lambda y)+\frac{1}{y^{\nu}}\Big)\mathrm{sgn}(u(y,s))\Bigg]\\&\quad K(x,y)g(x,s)u(y,s)\\&\leq\Bigg[\Bigg((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\Bigg)+\Big(x\exp(\lambda x)+\frac{1}{x^{\nu}}\Big)\Bigg]\\&\quad K(x,y)\,|u(y,s)|\,\psi(x,s).\end{aligned}\tag{3.27}$$

With the help of (3.26) and (3.27), from (3.25), we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty\Bigg[\frac{1}{2}\Bigg((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\Bigg)\mathrm{sgn}(u(x+y,s))\\&\quad-\Big(x\exp(\lambda x)+\frac{1}{x^{\nu}}\Big)\mathrm{sgn}(u(x,s))\Bigg]\\&\quad K(x,y)\{c(x,s)c(y,s)-g(x,s)g(y,s)\}\,dy\,dx\\&\leq\int_0^\infty\int_0^\infty\Bigg[\Bigg((x+y)\exp(\lambda(x+y))+\frac{1}{(x+y)^{\nu}}\Bigg)+\Big(y\exp(\lambda y)+\frac{1}{y^{\nu}}\Big)\Bigg]\\&\quad K(x,y)|u(x,s)|\psi(y,s)\,dy\,dx\\&\leq\int_0^\infty\int_0^\infty\Bigg[(x\exp(\lambda(x+y))+y\exp(\lambda(x+y))+2\Big(y\exp(\lambda y)+\frac{1}{y^{\nu}}\Big)\Bigg]\\&\quad K(x,y)|u(x,s)|\psi(y,s)\,dy\,dx.\end{aligned}\tag{3.28}$$

Next, we proceed after executing the following inequalities.

$$\begin{aligned}\text{(i)}\qquad&\int_0^\infty x\exp(\lambda x)x^{\theta-\mu}|u(x,s)|dx\\&=\int_0^1 x\exp(\lambda x)x^{\theta-\mu}|u(x,s)|dx+\int_1^\infty x\exp(\lambda x)x^{\theta-\mu}|u(x,s)|dx\\&\leq\int_0^1 x\exp(\lambda x)|u(x,s)|dx+\int_1^\infty x^2\exp(\lambda x)|u(x,s)|dx\\&\leq U_\lambda+U_{\lambda\lambda}.\end{aligned}\tag{3.29}$$

(ii) $$\int_0^\infty \exp(\lambda y)y^{-\mu}c(y,s)dy$$
$$= \int_0^1 \exp(\lambda y)y^{-\mu}c(y,s)dy + \int_1^\infty \exp(\lambda y)y^{-\mu}c(y,s)dy$$
$$\leq \int_0^1 \exp(\lambda)y^{-\mu}c(y,s)dy + \int_1^\infty \exp(\lambda y)c(y,s)dy$$
$$\leq [\exp(\lambda)+1]\Psi(\lambda,s). \tag{3.30}$$

(iii) $$\int_0^\infty x\exp(\lambda x)x^{-\mu}|u(x,s)|dx$$
$$= \int_0^1 x\exp(\lambda x)x^{-\mu}|u(x,s)|dx + \int_1^\infty x\exp(\lambda x)x^{-\mu}|u(x,s)|dx$$
$$\leq \int_0^1 x^{-\nu}\exp(\lambda)|u(x,s)|dx + \int_1^\infty x\exp(\lambda x)|u(x,s)|dx$$
$$\leq \exp(\lambda)\ U_\lambda. \tag{3.31}$$

(iv) $$\int_0^\infty (y\exp(\lambda y)+y^{-\nu})y^{-\mu}c(y,s)dy$$
$$= \int_0^\infty (y^{1-\mu}\exp(\lambda y)+y^{-\mu-\nu})c(y,s)dy$$
$$= \int_0^1 y^{1-\mu}\exp(\lambda y)c(y,s)dy + \int_1^\infty y^{1-\mu}\exp(\lambda y)c(y,s)dy + N_{-\mu-\nu}$$
$$\leq \Psi + \Psi_\lambda + N_{-\mu-\nu}. \tag{3.32}$$

(v) $$\int_0^\infty y^{\theta-\mu}(y\exp(\lambda y)+y^{-\nu})c(y,s)dy \leq \Psi + \Psi_{\lambda\lambda} + N_{\theta-\mu-\nu}. \tag{3.33}$$

Similarly, we obtain

$$\left.\begin{aligned}
&\int_0^\infty \exp(\lambda x)x^{-\mu}|u(x,s)|dx \leq U(1+\exp(\lambda)),\\
&\int_0^\infty \exp(\lambda x)x^{\theta-\mu}|u(x,s)|dx \leq U + U_\lambda,\\
&\int_0^\infty x\exp(\lambda x)x^{-\mu}|u(x,s)|dx \leq U_\lambda(1+\exp(\lambda)),\\
&\text{and}\quad \int_0^\infty x\exp(\lambda x)x^{\theta-\mu}|u(x,s)|dx \leq U_\lambda + U_{\lambda\lambda}.
\end{aligned}\right\} \tag{3.34}$$

Further,

$$\left.\begin{aligned}&\int_0^\infty \exp(\lambda x)x^{-\mu}c(x,s)dx \le (1+\exp(\lambda))\Psi,\\&\int_0^\infty \exp(\lambda x)x^{\theta-\mu}c(x,s)dx \le (1+\exp(\lambda))\Psi_\lambda,\\&\int_0^\infty xx^{-\mu}\exp(\lambda x)g(x,s)dx \le \Psi,\\&\text{and}\quad \int_0^\infty xx^{\theta-\mu}\exp(\lambda x)g(x,s)dx \le \exp(\lambda)\Psi_\lambda + \Psi_{\lambda\lambda}.\end{aligned}\right\} \tag{3.35}$$

With the help of (3.34) and (3.35), the inequalities in (3.23) and (3.24) yields

$$\begin{aligned}&U_\lambda(\lambda,t)\\&\le 2\int_0^t [U_\lambda(1+\exp(\lambda))^2\Psi + (U_\lambda+U_{\lambda\lambda})(1+\exp(\lambda))\Psi + U_\lambda(1+\exp(\lambda))^2\Psi_\lambda\\&\quad + U(1+\exp(\lambda))\Psi + (U+U_\lambda)\Psi + U(1+\exp(\lambda))(\exp(\lambda)\Psi_\lambda+\Psi_{\lambda\lambda})\\&\quad + 4U(1+\exp(\lambda))(\Psi+\Psi_\lambda+N_{-\mu-\nu}) + (U+U_\lambda)(\Psi+\Psi_\lambda+N_{-\mu-\nu})\\&\quad + (\Psi+\Psi_{\lambda\lambda}+N_{\theta-\mu-\nu})U(1+\exp(\lambda)) + \Gamma_3 U]ds\\&= 2\int_0^t [U(\Psi(6\chi_0+2)+\Psi_\lambda(\chi_0^2+4\chi_0+1)+\Psi_{\lambda\lambda}(2\chi_0)\\&\quad + (4\chi_0\chi_1+\chi_1+\chi_0\chi_2)+\Gamma_3)\\&\quad + U_\lambda(\Psi(\chi_0^2+\chi_0+2)+\Psi_\lambda(\chi_0^2+1)+\chi_1)+\chi_0\Psi U_{\lambda\lambda}]ds,\end{aligned} \tag{3.36}$$

where $\chi_0 = 1+\exp(\lambda)$, $\chi_1 = N_{-\nu-\mu}$ and $\chi_2 = N_{\theta-\mu-\nu}$.

The functions U and Ψ, defined in (3.3) and (3.4), respectively, are analytic in $0<\lambda<\hat{\lambda}$ and for any fixed $t\in[0,T]$.

Let us choose λ that satisfies

$$0 \le \lambda \le \lambda_0 < \hat{\lambda}. \tag{3.37}$$

Then, the inequality (3.1) ensures that for any integer $i \ge 1$,

$$\sup_{0\le t\le T,\ 0\le\lambda\le\lambda_0}\left\{\frac{\partial^i}{\partial\lambda^i}U(\lambda,t),\ \frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t)\right\} < \infty. \tag{3.38}$$

Since $u(x,t)$ and $\psi(x,t)$ are continuous on $\Pi = \{(x,t) : x\in(0,\infty),\ t\in[0,T]\}$ and they satisfy the inequalities in (3.1), corresponding to a given $\epsilon>0$ there exist numbers $\delta(\epsilon)>0$ and $\delta_i(\epsilon)>0$ such that for $i\ge 1$,

$$\left.\begin{aligned} &\sup_{0\le\lambda\le\lambda_0}\left\{|U(\lambda,t')-U(\lambda,t)|,|\Psi(\lambda,t')-\Psi(\lambda,t)|\right\}<\epsilon\\ &\text{and}\ \sup_{0\le\lambda\le\lambda_0}\left\{\left|\frac{\partial^i}{\partial\lambda^i}U(\lambda,t')-\frac{\partial^i}{\partial\lambda^i}U(\lambda,t)\right|,\left|\frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t')-\frac{\partial^i}{\partial\lambda^i}\Psi(\lambda,t)\right|\right\}<\epsilon\end{aligned}\right\} \tag{3.39}$$

for $|t'-t|<\delta$ with $t\ge 0$ and $t'\le T$.

To show the inequalities in (3.39), we note from (3.1) that for a sufficiently large $\xi>0$,

$$\begin{aligned}\left|U(\lambda,t')-U(\lambda,t)\right| &\le \int_0^\infty\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)|u(x,t')-u(x,t)|\,dx\\ &=\int_0^\xi\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)|u(x,t')-u(x,t)|\,dx\\ &\quad+\int_\xi^\infty\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)|u(x,t')-u(x,t)|\,dx,\end{aligned}$$

which is finite, by (3.1). Hence, the term $\int_\xi^\infty\left(\exp(\lambda x)+\frac{1}{x^\nu}\right)|u(x,t')-u(x,t)|\,dx$ can be made arbitrarily small. Thus, there exists a $\delta_1(\epsilon)>0$ such that

$$\sup_{0\le\lambda\le\lambda_0}|U(\lambda,t')-U(\lambda,t)|<\epsilon$$

for $|t'-t|<\delta$ with $t\ge 0$ and $t'\le T$. Similarly, the other three terms in (3.39) involving Ψ, $\frac{\partial^i}{\partial\lambda^i}U$ and $\frac{\partial^i}{\partial\lambda^i}\Psi$ are arbitrarily small in a range of t.

It follows from (3.38) and (3.39) that U and Ψ and their partial derivatives are continuous on λ in $D=\{(\lambda,t)\mid 0\le\lambda\le\lambda_0, 0\le t\le T\}$. From inequalities (3.22) and (3.36) we get the following functions:

$$c_1(\lambda,t)=(a_1\lambda+a_2+b_1)\Psi+(2(\chi_0^2+1)+a_3)\Psi_\lambda+(a_4\lambda+a_5+2\chi_1), \tag{3.40}$$

$$\text{and } c_2(\lambda,t)=2(\chi_0+1)\Psi+2\chi_0\chi_1, \tag{3.41}$$

where $a_1=2(6\chi_0+2)$, $b_1=2(3\chi_0+2+(\chi_0^2+\chi_0+2))$, $a_2=2(\chi_0^2+4\chi_0+1)$, $a_3=6\chi_0$, $a_4=2(4\chi_0\chi_1+\chi_1+\chi_0\chi_2+\Gamma_3)$, $a_5=2(\chi_0^2\chi_1+M\chi_0+\Gamma_3)$, $\chi_0=1+\exp(\lambda)$ and $\chi_1=N_{-\nu-\mu}$ such that

$$\left.\begin{aligned} U(\lambda,t)&\le\int_0^t\{c_1(\lambda,s)U(\lambda,s)+c_2(\lambda,s)U_\lambda(\lambda,s)\}ds\\ \text{and } U_\lambda(\lambda,t)&\le\int_0^t\frac{\partial}{\partial\lambda}\{c_1(\lambda,s)U(\lambda,s)+c_2(\lambda,s)U_\lambda(\lambda,s)\}ds,\end{aligned}\right\} \tag{3.42}$$

and U, Ψ and their partial derivatives with respect to λ are nonnegative in D. Then, by applying Lemma 1 in D, we obtain $U(\lambda,t)=0$ in R as defined in Lemma 1.

Since $u(x,t)$ is continuous, $u(x,t)=0$ for $0 \le t \le t',\ 0 < x < \infty$. Thus, $U(\lambda,t)=0$ is not only true on R, but also for $0 \le \lambda \le \lambda_0, 0 \le t \le t'$.

Applying an analogous argument on the interval $[t',\ 2t']$, we see that $u(x,t)=0$ for $0 \le t \le 2t',\quad 0 < x < \infty$. Continuing this process, we establish that $u(x,t)=0$ on Π, i.e., $c=g$ on Π. This completes the proof. □

4 Conclusion

In this paper, we have proved the uniqueness of mass conserving solution for a continuous coagulation–fragmentation equation. The considered class of coagulation kernels has a singularity at the origin. In the next step of this study, we will attempt to investigate the *gelation phenomenon* and *asymptotic behavior* of the time-dependent solution for fragmentation and coagulation models with a singular kernel. In the future, one can also attempt to explore an explicit solution to the problem that we have considered here. The *self-similar approach* might be an appealing procedure to find the explicit solution.

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