Tutorial Sheet 7: Miscellaneous Applications - II

1. *(This problem is similar in spirit to Q13 in Tutorial Sheet 5. Use symbolic mathematics in Python or MATLAB to implement the steps.)*

   The stress equilibrium equations in the cylindrical coordinate system (r, θ, z) can be derived from those in the Cartesian system (x, y, z). Carry out the transformation using the following steps:

   (i) Consider the stress matrix in the cylindrical coordinate system:

   $$[\boldsymbol{\sigma}_{\text{cyl}}] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{zr} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix}.$$

   Each of the six independent stress components in this stress matrix is a function of r, θ, and z. Transform this stress matrix into that in the Cartesian system using $[\boldsymbol{\sigma}_{\text{Cart}}] = [\mathbf{Q}]^{\mathsf{T}}[\boldsymbol{\sigma}_{\text{cyl}}][\mathbf{Q}]$, where $[\mathbf{Q}]$ is the rotation matrix:

   $$[\mathbf{Q}] = \begin{bmatrix} = \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (ii) Extract the six independent stress components in the Cartesian system from $\boldsymbol{\sigma}_{\text{Cart}}$, and set up the stress equilibrium equations from $\nabla \cdot \boldsymbol{\sigma}_{\text{Cart}} = 0$, to obtain:

   $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0,$$
   $$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0,$$
   $$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0.$$

   It is these equations which need to be transformed to the cylindrical system. The important thing to note here is that these three equations can be viewed as the three components of a vector equation. The left hand side, i.e. $\nabla \cdot \boldsymbol{\sigma}_{\text{Cart}}$ is indeed a vector, and it can be transformed to the cylindrical system using:

   $$[\nabla \cdot \boldsymbol{\sigma}_{\text{cyl}}] = [\mathbf{Q}][\nabla \cdot \boldsymbol{\sigma}_{\text{Cart}}].$$

The solution to this question, implemented in Symbolic Python, has been posted in the GitHub repository for this course and can be directly viewed in this link.

2. Consider a thick-walled pressure vessel with external pressure zero.
   (a) Show that the radial and hoop stresses, respectively, are:
   $$\sigma_{rr} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right),$$
   $$\sigma_{\theta\theta} = \frac{p_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$
   (b) What is the nature of the radial stress (compressive or tensile), and where is it a maximum?
   (c) What is the nature of the hoop stress (compressive or tensile), and where is it a maximum?
   (d) What is the maximum shear stress and where does it occur?
3. Consider a thick-walled pressure vessel with internal pressure zero.
   (a) Show that the radial and hoop stresses, respectively, are:
   $$\sigma_{rr} = -\frac{p_o r_o^2}{r_o^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right),$$
   $$\sigma_{\theta\theta} = -\frac{p_o r_o^2}{r_o^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right).$$
   (b) What is the maximum compressive stress and where does it occur?
4. Make a comparison of the radial and hoop stresses obtained using the thick-walled pressure vessel formulae (external pressure zero) with the radial and hoop stresses obtained using the thin-walled pressure vessel formulae for the two cases: $r_o = 1.1r_i$ and $r_o = 5r_i$. Plot the variation of σ_{rr}/p_i and $\sigma_{\theta\theta}/p_i$ with r/r_i for the $r_o = 5r_i$ case.
5. For a thick-walled pressure vessel subjected only to internal pressure, show that no matter how large the external radius is, the maximum hoop stress is never less than p_i.
6. The problem of a thin annular disk subjected to both internal and external pressure can be solved by following the same kind of steps as the problem of the thick-walled pressure vessel. However, there is an important difference in that the thin annular disk problem is a case of *plane stress*. Following all the steps show that the radial and hoop stresses for this problem are the same as those in the thick-walled pressure vessel problem.
7. Consider the thin annular disk problem again but with the inner periphery attached to a rigid rod (whose radius is equal to the inner radius of the disk). The attachment, however, only constrains the radial displacement with *slippage allowed* in the axial direction. Determine the radial and hoop stresses in this disk.

8. Consider a thick-walled *spherical* pressure vessel subjected to an internal pressure p_i and an external pressure p_o. Here, we can neither assume plane strain nor plane stress; rather we assume perfect spherical symmetry. In this situation, referring to the spherical coordinate system (r, θ, ϕ), there is no difference between the θ and the ϕ directions. Furthermore, there is no variation along the θ (or, ϕ) direction. The only variation is along the radial direction. Under these simplified conditions, the stress equilibrium equation in the radial direction for the spherical pressure vessel is*:

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \qquad r \in [r_i, \;\; r_o]$$

which, except for the factor 2, is identical to the corresponding equation for the cylindrical pressure vessel case. Consider the material to follow generalized Hooke's law. For the spherically symmetric case, we have $\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi}$, and from the strain-displacement relations, we have: $\varepsilon_{rr} = \dfrac{\mathrm{d}u}{\mathrm{d}r}$ and $\varepsilon_{\theta\theta} = \dfrac{u}{r}$, where $u \equiv u(r)$ is the displacement in the radial direction. Determine the radial and circumferential stresses.

9. Consider a thick-walled cylindrical pressure vessel, subjected to internal pressure p_i and external pressure p_o, whose wall is made of two materials: Material 1 (E_1, ν_1) between r_i and r_m and Material 2 (E_2, ν_2) between r_m and r_o with $r_i < r_m < r_o$. Determine the radial and circumferential stresses in both materials. Assume that the materials are perfectly attached at $r = r_m$, in which case the radial displacements as well as the radial stresses of both materials must be equal at $r = r_m$.

10. *(Question for thinking)* A blood vessel may be considered to be some sort of a thick-walled pressure vessel. However, the equations which you have learnt and derived till now are not directly applicable. What are the challenges involved in modelling such a blood vessel as a thick-walled pressure vessel? Some pointers for your thoughts:

    - Is the blood (fluid pressure) uniform along the length of the blood vessel? Compare with the typical pressure- vessel case you have been studying.
    - Can the flow of blood through the vessel be modelled directly as a Poiseuille flow case †)? Think about the change in the radius of the vessel.

*You are not required to derive it.

†Poiseuille flow is one of the typical "exact solution" examples of the Navier-Stokes equations taught in Fluid Mechanics/Thermo-Fluid Science.