Tutorial Sheet 5: Strain and Strain Transformation

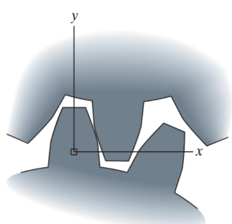
1. The components of a displacement field are:

$$u_x = (x^2 + 20) \times 10^{-4},$$
$$u_y = 2yz \times 10^{-3},$$
$$u_z = (z^2 - xy) \times 10^{-3}.$$

Determine the different components of strain at $(2, -1, 3)$.

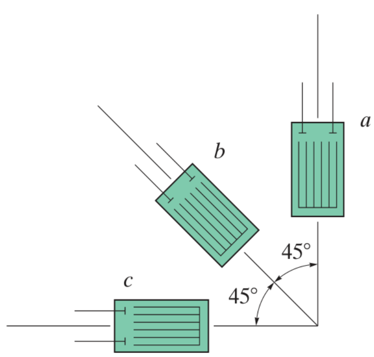
[$\varepsilon_{11} = 4 \times 10^{-4}$, $\varepsilon_{22} = 0.006$, $\varepsilon_{33} = 0.006$, $\varepsilon_{12} = 0$, $\varepsilon_{23} = -0.002$, $\varepsilon_{31} = 5 \times 10^{-4}$]

2. The state of strain at the point on the gear tooth has components $\varepsilon_{xx} = 850 \times 10^{-6}$, $\varepsilon_{yy} = 480 \times 10^{-6}$, $\gamma_{xy} = 650 \times 10^{-6}$. Determine (a) the in-plane principal strains, (b) the maximum in-plane shear strain, and (c) the average normal strain. In each case specify the orientation of the element.



[(a) 1039×10^{-6}, $30.2°$; 291×10^{-6}, $120.2°$
(b) 748×10^{-6}, $-14.8°$
(c) 665×10^{-6}]

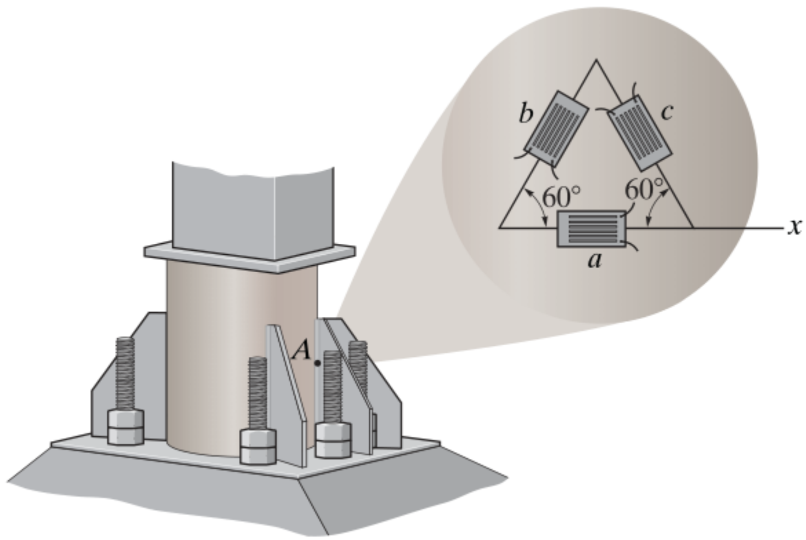
3. The strain rosette shown in the figure is mounted on a machine element. The readings from the strain gauges are: $\varepsilon_a = 650 \times 10^{-6}$, $\varepsilon_b = -300 \times 10^{-6}$, and $\varepsilon_c = 480 \times 10^{-6}$. Determine (a) the in-plane principal strains, and (b) the maximum in-plane shear strain and (c) the average normal strain associated with the maximum in-plane shear strain.



[(a) 1434×10^{-6}, -304×10^{-6},
(b) 1738×10^{-6},
(c) 565×10^{-6}]

4. The strain rosette is attached to point A on the surface of the support. The readings from the strain gauges are: $\varepsilon_a = 300\mu$, $\varepsilon_b = -150\mu$, and $\varepsilon_c = -450\mu$. Determine (a) the in-plane principal strains, and (b) the maximum in-plane shear strain and (c) the average normal strain associated with the maximum in-plane shear strain. Specify the orientation of each element that has these states of strain with respect to the x-axis.*

[(a) 336μ, $11.7°$; -536μ, $101.7°$, (b) 872μ, $-33.3°$ (c) -100μ]



5. For the case of plane stress, show that the following can be obtained from the generalized Hooke's law:

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}),$$
$$\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}),$$

where E and ν are the Young's modulus and Poisson's ratio.

6. For a material that behaves according to the generalized Hooke's law:

(a) Considering the case of plane stress (xy-plane), derive the strain transformation equations from the stress transformation equations.

(b) How does the strain component ε_{zz} transform in part (a)?

(c) Considering the case of plane strain (xy-plane), derive the stress transformation equations from the strain transformation equations.

(d) How does the stress component σ_{zz} transform in part (c)?

*Note that the strains have been written in terms of the symbol for micron, μ, i.e. $1\mu = 10^{-6}$.

7. Given a general 3D state of strain:

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix},$$

the strain along any direction represented by the unit vector, $[\hat{\mathbf{N}}] = [N_x \;\; N_y \;\; N_z]^{\mathsf{T}}$ is

$$\varepsilon_N = [\hat{\mathbf{N}}]^{\mathsf{T}}[\boldsymbol{\varepsilon}][\hat{\mathbf{N}}].$$

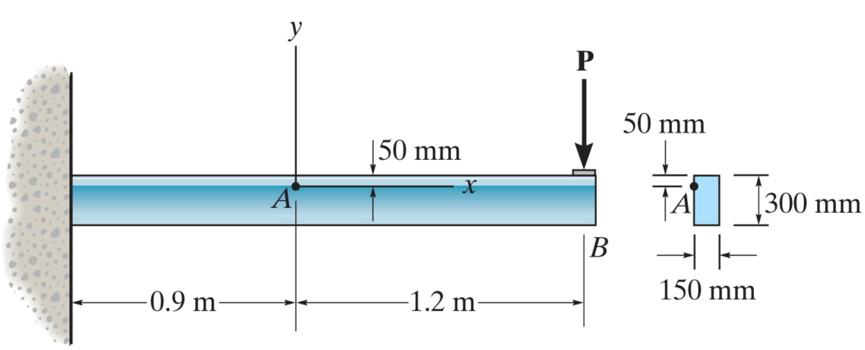
Consider the problem of extremizing ε_N with respect to a changing $\hat{\mathbf{N}}$ subject to the constraint that $N_x^2 + N_y^2 + N_z^2 = 1$. This *constrained extremization* problem can be addressed using the method of Lagrange multiplier.† By proceeding along this method, show that the following eigenvalue problem can be reached:

$$\begin{bmatrix} \varepsilon_{xx} - \lambda & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} - \lambda & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} - \lambda \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where λ is the Lagrange multiplier. Note that it is this Lagrange multiplier which give us the extremum values, i.e. the principal strains.

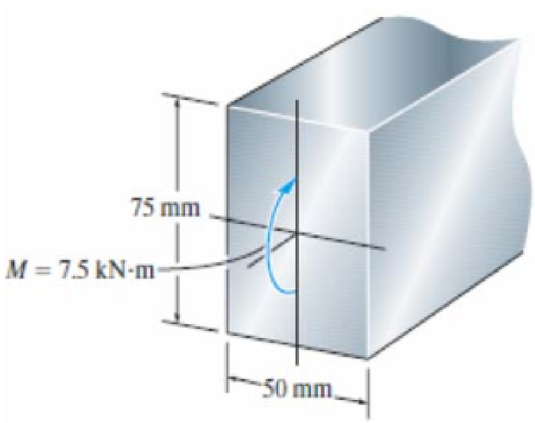
8. Deduce that in the case of plane strain (xy-plane) for a body made of a material that follows the generalized Hooke's law, the stress component σ_{zz} itself is a principal stress.

9. The strain the x-direction at point A on the structural steel beam ($E = 203$ GPa and $G = 76$ GPa) is measured and found to be $\varepsilon_{xx} = 100\mu$. Determine the applied load P. What is the shear strain γ_{xy} at point A? [$P = 57$ kN; $\gamma_{xy} = -13.91\mu$]



† Refer to your notes from Advanced Calculus (MA11003)

10. The aluminium beam has the rectangular cross-section shown. If it is subjected to a bending moment of $M = 7.5$ kN·m, determine the increase in the 50 mm dimension at the top of the cross section and the decrease in this dimension at the bottom. For the aluminium, $E = 70$ GPa and $\nu = 0.3$. *Think* about the implications of your answers in terms of the assumptions of the Euler-Bernoulli beam hypothesis (plane sections in a beam do not change dimensions). [top: 0.03429 mm; bottom: −0.03429 mm]



11. The transformation equations for both stress and strain can be written in the form:

$$\zeta_{x'x'} = \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}\zeta_{xx} & \zeta_{xy}\\ \zeta_{xy} & \zeta_{yy}\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix},$$

$$\zeta_{y'y'} = \begin{bmatrix}\cos\left(\theta+\frac{\pi}{2}\right) & \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix}\begin{bmatrix}\zeta_{xx} & \zeta_{xy}\\ \zeta_{xy} & \zeta_{yy}\end{bmatrix}\begin{bmatrix}\cos\left(\theta+\frac{\pi}{2}\right)\\ \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix},$$

$$\zeta_{x'y'} = \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}\zeta_{xx} & \zeta_{xy}\\ \zeta_{xy} & \zeta_{yy}\end{bmatrix}\begin{bmatrix}\cos\left(\theta+\frac{\pi}{2}\right)\\ \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix},$$

where ζ can be either σ or ε. Verify that the above relations can be written in the following compact form:

$$\begin{bmatrix}\zeta_{x'x'} & \zeta_{x'y'}\\ \zeta_{x'y'} & \zeta_{x'y'}\end{bmatrix} = [Q]\begin{bmatrix}\zeta_{xx} & \zeta_{xy}\\ \zeta_{xy} & \zeta_{yy}\end{bmatrix}[Q]^{\mathsf{T}},$$

where $[Q] = \begin{bmatrix}\cos\theta & \sin\theta\\ \cos\left(\theta+\frac{\pi}{2}\right) & \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix}$ is the rotation matrix. Write a computer program utilizing these relations to quickly compute the required values in different problems.

12. For a general vector $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$, verify by actually taking components that the transformation rule when going from a xy-coordinate system to a $x'y'$-coordinate system is:

$$\begin{bmatrix}v_{x'}\\ v_{y'}\end{bmatrix} = [Q]\begin{bmatrix}v_x\\ v_y\end{bmatrix},$$

where $[Q]$ is the same rotation matrix described in the previous problem.

13. (*This problem is primarily intended for students interested in higher studies. And, it will be much easier to do it using symbolic maths in Python or MATLAB.*)
The strain-displacement relations in polar coordinates (r, θ) may be derived from those in the Cartesian coordinates (x, y) by utilizing the strain transformation rules. However some extra care is required. First of all note that since $x = r\cos\theta$, $y = r\sin\theta$, from which $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$, we have the following (verify yourself):

$$\frac{\partial r}{\partial x} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \sin\theta,$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Thus, for a generic *scalar* function $f(r, \theta)$, we have:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r} \tag{a}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}. \tag{b}$$

Now, carry out the following steps:

(i) Consider the displacement vector $[\mathbf{u}_{\text{pol}}] = [u(r,\theta), \quad v(r,\theta)]^\mathsf{T}$ in the polar coordinate system. Transform it to the Cartesian system using $[\mathbf{u}_{\text{rec}}] = [Q]^\mathsf{T}[\mathbf{u}_{\text{pol}}]$. (Note that $[\mathbf{u}_{\text{pol}}] = [Q][\mathbf{u}_{\text{rec}}]$)

(ii) Obtain the strain-displacement relations in the Cartesian system using the definition $\boldsymbol{\varepsilon}_{\text{rec}} = \frac{1}{2}\left(\nabla\mathbf{u}_{\text{rec}} + (\nabla\mathbf{u}_{\text{rec}})^\mathsf{T}\right)$ and the above relations (a) and (b), treating each of $u(r,\theta)$ and $v(r,\theta)$ as a scalar function.

(iii) Finally, carry out the Cartesian to polar transformation using $[\boldsymbol{\varepsilon}_{\text{pol}}] = [Q][\boldsymbol{\varepsilon}_{\text{rec}}][Q]^\mathsf{T}$.

14. In polar coordinates, the strain-displacement relations are given by

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) + \frac{u}{r}, \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right).$$

Using these relations, determine the two-dimensional strains for the following displacement fields (here A, B, C are arbitrary constants):

(a) $u = \frac{A}{r}$, $v = B\cos\theta$

(b) $u = Ar^2$, $v = Br\sin\theta$

[(a) $[\boldsymbol{\varepsilon}] = \begin{bmatrix} -\frac{A}{r^2} & -\frac{B}{2r}\cos(\theta) \\ -\frac{B}{2r}\cos(\theta) & \frac{1}{r^2}(A - Br\sin(\theta)) \end{bmatrix}$ (b) $[\boldsymbol{\varepsilon}] = \begin{bmatrix} 2Ar & 0 \\ 0 & Ar + B\cos(\theta) \end{bmatrix}$]